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A REFINEMENT OF AN INEQUALITY OF JOHNSON, LOEWY AND LONDON ON NONNEGATIVE MATRICES AND SOME APPLICATIONS.*

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Dedicated to Hans Schneider on the occasion of his seventieth birthday.

Abstract. Let A be an entrywise nonnegative $n \times n$ matrix and let $s_k := \text{trace}\,(A^k)$ $(k = 1, 2, \ldots)$. It is shown that if n is odd and $s_1 = 0$, then $(n-1)s_4 \geq s_2^2$. The result is applied to show that $(3, \frac{1}{2}(1 \pm \sqrt{17}), -2, -2)$ is not the spectrum of a nonnegative 5×5 matrix while $(3, \frac{1}{2}(1 \pm \sqrt{17}), -2, -2, 0)$ is the spectrum of a nonnegative symmetric 6×6 matrix.

Key words. nonnegative matrix, spectrum, eigenvalues, Johnson-Loewy-London inequalities, symmetric, triangular graph, copositive matrix.

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1. Introduction. Let A be an entrywise nonnegative $n \times n$ matrix and let

$$s_k := \text{trace}\,(A^k), \qquad (k = 1, 2, \ldots).$$

The JLL-inequalities discovered independently by Loewy and London [10] and Johnson [5] state that

$$(JLL) \qquad n^{m-1} s_{km} \geq s_k^m$$

for all positive integers k, m.

Equality can occur in (JLL) for various k, m; for example, if A^k is a scalar matrix, then equality holds for all m.

In this paper we obtain a refinement of one of the inequalities in a special case. We prove MAIN THEOREM. *Let A be an entrywise nonnegative $n \times n$ matrix with $trace(A) = 0$. Then, if n is odd,*

$$(n-1)s_4 \geq s_2^2,$$

that is,

$$(n-1)\,trace(A^4) \geq \big(trace(A^2)\big)^2.$$

This inequality is best possible—for example if A is the matrix

$$\text{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \text{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (0)$$

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then equality occurs. It also fails in general for n even, for example

$$A = \operatorname{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \operatorname{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

provides a counterexample. In the case $n = 5$, this inequality was proved in a number of special cases by Reams [11], [12].

Though the refinement in the JLL inequality is small, it has some important applications. It enables one to show that there exists a list $(\lambda_1, \ldots, \lambda_5)$ of real numbers which is not the spectrum of a nonnegative 5×5 matrix but the list with 0 adjoined, $(\lambda_1, \ldots, \lambda_5, 0)$, is the spectrum of a nonnegative symmetric 6×6 matrix. In [6], Johnson, Loewy and the first author presented a list $(\lambda_1, \ldots, \lambda_r)$ of real numbers which forms the spectrum of a nonnegative $r \times r$ matrix but not the spectrum of a nonnegative symmetric $r \times r$ matrix; see also [8]. This was placed in the context of Boyle-Handelman theory [2], [3] and showed that in realizing a given list satisfying the obvious necessary conditions as the nonzero part of the spectrum of a nonnegative matrix A, one cannot in general place restrictions on the rank of the zero eigenvalue part of the Jordan form of A. However, if A is real symmetric, then its rank is the same as the number of nonzero elements in its spectrum, so the example presented here is significant.

The paper is organized as follows. In Section 1 we consider a quadratic form associated with the triangular graphs. In Section 2 we show how the desired inequality can be formulated in terms of the copositivity of the form. This copositivity is established in Section 3. In Section 4, the result is applied to determine the realizability of certain spectra.

2. The triangular graph. Let $n \geq 5$ be an integer. Let S be the set of all 2-subsets of $\{1, 2, \ldots, n\}$. Define a graph Γ_n as follows:
Γ_n has vertex set S and for $\alpha, \beta \in S$, $\alpha\beta = \beta\alpha$ is an edge of Γ_n if and only if $|\alpha \cap \beta| = 1$. The graph Γ_n is known as the triangular graph on $\{1, 2, \ldots, n\}$. It is an example of a strongly regular graph (see van Lint and Wilson [9, p. 231]) and its parameters are

$$\left(\binom{n}{2}, 2(n-2), n-2, 4\right).$$

This means that it has $\binom{n}{2}$ vertices, each vertex has degree $2(n-2)$ and if α, β are distinct vertices, then if $\alpha\beta$ is an edge, there are $n-2$ vertices γ incident with both α and β while if $\alpha\beta$ is not an edge, there are 4 vertices incident with α and β. Note also that Γ_n is the line graph $L(K_n)$ of the complete graph K_n on n vertices; see Cvetkovič, Doob, Sachs [4, p. 169].

Label the vertices of Γ_n by the symbols $1, 2, \ldots, N := \binom{n}{2}$, and let P_n be its adjacency matrix. By van Lint and Wilson [9, p. 231]

$$P_n^2 + \big(4 - (n-2)\big)P_n + \big(4 - 2(n-2)\big)I_N = 4J_N,$$

where I_N is the $N \times N$ identity matrix and J_N the $N \times N$ matrix with all its entries equal to 1. The vector with all components equal to 1 is an eigenvector for both P_n

and J_N and the corresponding eigenvalue of P_n is $2(n-2)$. The other eigenvalues of J_N are all 0 and it follows that the remaining eigenvalues of P_n satisfy the equation

$$x^2 + \bigl(4-(n-2)\bigr)x + 4 - 2(n-2) = 0.$$

So they are $n-4$ and -2 and the corresponding multiplicities are $(n-1)$ and $\binom{n-1}{2}-1$ (as determined from the equation trace$(P_n) = 0$). [The fact that the eigenvalues of P_n are integers is related to the fact that the **Z**-span of I, J_N, P_n is a ring (the Bose-Messner algebra of , $_n$ [9, pp. 234–235])].

Note also that since , $_n$ is the line graph $L(K_n)$,

$$P_n = R_n^T R_n - 2I_N$$

where R_n is the incidence matrix of the graph K_n (where the edges of K_n are labeled consistently with our labeling of the vertices of P_n).

Consider the quadratic form

$$\begin{aligned} Q := (x_1 \cdots x_N) P_n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} &= (x_1 \cdots x_N) R^T R \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} - 2\sum_{i=1}^{N} x_i^2 \\ &= \Omega_1^2 + \cdots + \Omega_n^2 - 2\sum_{i=1}^{N} x_i^2, \end{aligned}$$

where $\Omega_i = x_{i_{j_i(1)}} + \cdots + x_{i_{j_i(n-1)}}$ and $j_i(1), \ldots, j_i(n-1)$ are the edges of K_n containing the vertex i $(i = 1, 2, \ldots, n)$. Thus for $i \neq k$, Ω_i and Ω_k involve exactly one common symbol. We show in the next section that the problem of minimizing the form

$$(x_1 \cdots x_N)(P_n + I) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \Omega_1^2 + \cdots + \Omega_n^2 - \sum_{i=1}^{N} x_i^2$$

over $\mathcal{D} := \bigl\{(x_1, \ldots, x_N) | x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{N} x_i = 1\bigr\}$ arises in the proof of the main theorem.

3. Nonnegative matrices of trace 0. Let $A = (a_{ij})$ be a nonnegative $n \times n$ matrix of trace 0 and let $s_k :=$ trace(A^k) for $k = 1, 2, \ldots, n$. Note that

$$s_2 = \Omega_1 + \cdots + \Omega_n$$

where

$$\Omega_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} a_{ji} \qquad (i = 1, 2, \ldots, n).$$

Write $y_1 = a_{12}a_{21}$, $y_2 = a_{13}a_{31}, \ldots$, $y_{n-1} = a_{1n}a_{n1}$, $y_n = a_{23}a_{32}$, $y_{n+1} = a_{24}a_{42}, \ldots$, $y_N = a_{n-1\,n}a_{n\,n-1}$ where $N := \binom{n}{2}$.

We now consider $s_4 = \text{trace}\,(A^4)$. The diagonal terms of A^2 contribute $\Omega_1^2 + \cdots \Omega_n^2$ to s_4. We observe that this contribution is a quadratic form in $y_1, y_2, \ldots, y_N$. Thus we are led to consider that part of s_4 which can be expressed as a quadratic form in $y_1, \ldots, y_N$. Since $\text{trace}\,(A) = 0$, all diagonal entries of A are 0 and thus s_4 is a sum of terms of the form $a_{r_1r_2}a_{r_2r_3}a_{r_3r_4}a_{r_4r_1}$, where either all the r_i are distinct or it equals a term of the form $a_{pq}a_{qp}a_{rs}a_{sr}$.

The 4-cycle type $a_{r_1r_2}a_{r_2r_3}a_{r_3r_4}a_{r_4r_1}$ with r_1, r_2, r_3, r_4 distinct does not give a quadratic form in $y_1, \ldots, y_N$.

Consider a term of the form

$$a_{pq}a_{qp}a_{rs}a_{sr}$$

where $p \neq q$, $r \neq s$. If $\{r, s\} = \{p, q\}$, this term only arises in the component $\Omega_1^2 + \cdots + \Omega_n^2$. Suppose $\{r, s\} \neq \{p, q\}$. If $\{r, s\} \cap \{p, q\}$ is empty, this term cannot occur in s_4 at all. Hence we are led to consider the occurrence of terms $a_{pq}a_{qp}a_{rs}a_{sr}$ in which

$$\big|\{p, q\} \cap \{r, s\}\big| = 1$$

(where $|\cdot|$ denotes cardinality).

Suppose $p = r$. Then such a term occurs in $s_4 - (\Omega_1^2 + \cdots + \Omega_n^2)$ in the following ways: in the contribution of the (q, s) term of A^2 and the (s, q) term of A^2 to the (q, q) element of A^4 and also to the (s, s) element of A^4 and in no other way, so its coefficient in $s_4 - (\Omega_1^2 + \cdots + \Omega_n^2)$ is 2.

If $p = s$, then the term arises in the (q, q) and (r, r) terms contribution to $s_4 - (\Omega_1^2 + \cdots + \Omega_n^2)$ and not in any other term.

Similarly the term occurs with coefficient 2 in $s_4 - (\Omega_1^2 + \cdots + \Omega_n^2)$ if $\{q\} = \{p, q\} \cap \{r, s\}$.

Thus we find that

$$s_4 \geq \Omega_1^2 + \cdots + \Omega_n^2 + 2\sum y_l y_m$$

where the sum is over all $\{l, m\}$ such that if $y_l = a_{pq}a_{qp}$ and $y_m = a_{rs}a_{sr}$, then $\big|\{p, q, r, s\}\big| = 3$.

Thus in the notation of Section 1,

$$s_4 \geq \Omega_1^2 + \cdots + \Omega_n^2 + (y_1 \cdots y_N)P_n \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

But note that

$$\Omega_1^2 + \cdots + \Omega_n^2 = (y_1 \cdots y_N)R^T R \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

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Since $P_n = R^T R - 2I_N$, we have

$$s_4 \geq 2(\Omega_1^2 + \cdots + \Omega_n^2) - 2\sum_{i=1}^{N} y_i^2.$$

To prove the theorem, it suffices to show that

$$(E) \qquad 2(n-1)(\Omega_1^2 + \cdots + \Omega_n^2) - 2(n-1)\sum_{i=1}^{N} y_i^2 \geq s_2^2 = 4(y_1 + \cdots + y_N)^2.$$

4. An optimization problem. Motivated by Section 2, we formulate the following question. Let $\mathcal{D} = \{(y_1, \ldots, y_N) | y_i \geq 0$ for all i and $\sum_{i=1}^N y_i = 1\}$ where n is an odd integer ≥ 3 and $N = \binom{n}{2}$. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be a collection of subsets of $\{y_1, y_2, \ldots, y_N\}$ satisfying the following properties.
1. Each Λ_i contains $n-1$ y_is
2. $|\Lambda_i \cap \Lambda_j| = 1$ for all $1 \leq i \neq j \leq n$
3. Each y_i belongs to exactly two of the Λ_js.
Let Ω_i be the sum of the elements of Λ_i and let

$$\Phi(y_1, \ldots, y_N) := \Omega_1^2 + \cdots + \Omega_n^2 - \sum_{i=1}^{N} y_i^2.$$

The problem is to determine the global minimum of $\Phi(y_1, \ldots, y_N)$ as y runs through $\mathcal{D}$. Since Φ is a continuous function on the compact set $\mathcal{D}$ the global minimum exists. We now suppose $y_1, \ldots, y_N$ have been chosen so that
(1) $\Phi(y_1, \ldots, y_N)$ is the global minimum of Φ on $\mathcal{D}$
(2) The number of nonzero y_i is least possible subject to condition (1).

We first establish the following result.

Claim 1. *Assume the notation is chosen so that* $y_1 = \max_{1 \leq i \leq N} y_i$ *and that* $y_1 \in \Lambda_1 \cap \Lambda_2$. *Then* $\Omega_1 = \Omega_2 = y_1$ *(that is, all the* y_i*s occurring in* $(\Lambda_1 \cup \Lambda_2) \backslash \{y_1\}$ *are zero).*

Proof. For suppose Ω_1 contains an element $y_j (j \neq 1)$ with $y_j \neq 0$ and choose y_j least possible subject to this.

Define $y_1' = y_1 + y_j$, $y_j' = 0$.

Let $\Lambda_1 = \{y_1, y_j, y_{h_3}, \ldots, y_{h_{n-1}}\}$, $\Lambda_2 = \{y_1, y_{r_2}, \ldots, y_{r_{n-1}}\}$ and note $y_j \notin \Lambda_2$ by property 2. Put $\Lambda_1' = \{y_1', y_j', y_{h_3}, \ldots, y_{h_{n-1}}\}$, $\Lambda_2' = \{y_1', y_{r_2}, \ldots, y_{r_{n-1}}\}$. Now y_j is contained in some $\Lambda_i (i > 1)$, say $y_j \in \Lambda_3$. Define

$$\Lambda_3' = (\Lambda_3 \backslash \{y_j\}) \cup \{y_j'\}.$$

Define $\Lambda_k' = \Lambda_k$ for all $k > 3$. Consider the effect of this change on

$$\Omega_1^2 + \cdots \Omega_n^2 - \sum_{i=1}^{N} y_i^2.$$

Note that $\Omega_1' = \Omega_1$ and that $(\Omega_2')^2$ is increased (over Ω_2^2) by

$$y_j^2 + 2y_j(y_1 + y_{r_2} + \cdots + y_{r_n}) = y_j^2 + 2y_j\Omega_2.$$

Also $(\Omega_3')^2$ is reduced (over Ω_3^2) by

$$y_j^2 + 2y_j(\Omega_3 - y_j)$$

Also ${y_1'}^2 + {y_j'}^2 + \sum_{t\neq 1,j} {y_t'}^2$ is increased (over $\sum_{i=1}^N y_i^2$) by $2y_1y_j$.

Thus the overall effect on the function Φ is an increase of

$$2y_j(\Omega_2 - y_1) - 2y_j(\Omega_3 - y_j) \quad = \quad 2y_j\left[(\Omega_2 - y_1) - (\Omega_3 - y_j)\right].$$

It follows that $\Omega_2 - y_1 \geq \Omega_3 - y_j$. But if equality occurs here, then Φ is unchanged while the number of nonzero y_i's has been decreased by 1, contrary to hypothesis (2) above. Hence we have

$$(*) \qquad \Omega_2 - y_1 > \Omega_3 - y_j.$$

Consider the following swap of the original Λ_i.
Leave $\Lambda_1, \Lambda_4, \ldots, \Lambda_n$ as before.
Put

$$\Lambda_2'' = (\Lambda_3 \backslash \{y_j\}) \cup \{y_1\}$$
$$\Lambda_3'' = (\Lambda_2 \backslash \{y_1\}) \cup \{y_j\}$$

Note that the system

$$\Lambda_1, \Lambda_2'', \Lambda_3'', \Lambda_4, \ldots, \Lambda_n$$

satisfies the hypothesis and that the sum $\sum y_i^2$ is unchanged and the overall change in Φ is to add

$$2y_1(\Omega_3 - y_j) + 2y_j(\Omega_2 - y_1) = 2(y_1 - y_j)\left[(\Omega_3 - y_j) - (\Omega_2 - y_1)\right].$$

Since Φ achieves its global minimum for these y_i's, it follows that this must be nonnegative.

If $y_1 - y_j > 0$, this implies that

$$(\Omega_3 - y_j) - (\Omega_2 - y_1) \geq 0,$$

contradicting $(*)$ above. Hence $y_1 = y_j$ and $\Omega_2 > \Omega_3$. But now we can reverse the roles of y_1 and y_j and define $y_1' = 0$ and $y_j' = y_1 + y_j$. The argument then yields $\Omega_2 < \Omega_3$, thus yielding the desired contradiction. □

Claim 2. *The global minimum is achieved by an assignment in which each Λ_i has at most one nonzero y_i.*

Proof. By Claim 1, we may assume $\Lambda_1 = \Lambda_2 = \{y_1\}$, where we assume $y_1 = \max\{y_1, \ldots, y_N\}$. The Claim is now clear if $n = 3$. Suppose $n > 3$. Using our hypotheses, we may write

$$\Lambda_1 = \{y_1, y_2, \ldots, y_{n-1}\} \quad \text{and} \quad \Lambda_2 = \{y_1, y_n, y_{n+1}, \ldots, y_{2n-3}\},$$

and $y_2 = 0, \ldots, y_{n-1} = 0$, $y_n = 0, \ldots, y_{2n-3} = 0$. Note that $y_2, \ldots, y_{n-1}$ belong to distinct members of the list $\Lambda_3, \ldots, \Lambda_n$ and thus each Λ_i $(i = 3, \ldots, n)$ contains exactly one of $y_2, \ldots, y_{n-1}$. Similarly $\Lambda_3, \ldots, \Lambda_n$ contains exactly one of $y_n, \ldots, y_{2n-3}$. But now deleting those elements from $\Lambda_3, \ldots, \Lambda_n$ leads to a list $\Lambda_3', \ldots, \Lambda_n'$ of $n-2$ subsets of $\{y_{2n-2}, \ldots, y_N\}$ which satisfy our hypotheses with n replaced by $n-2$ (and $\sum_{i=1}^{N} y_i = 1$ replaced by $\sum_{i=2n-2}^{N} y_i = 1 - 2y_1$).

So now we can use induction on n to conclude that the minimum contribution from these subsets occurs for an arrangement in which each of $\Lambda_3', \ldots, \Lambda_n'$ contains at most one nonzero element. This establishes Claim 2. □

Since n is odd, Claim 2 implies that one of the Λ_ks consists of zeros since the nonzero elements pair off as $\big(\{y_1\}, \{y_1\}\big), \big(\{y_2\}, \{y_2\}\big), \ldots$ Thus the problem of minimizing Φ in this case reduces to: Minimize $2(y_1^2 + y_2^2 + \cdots + y_k^2) - (y_1^2 + \cdots + y_k^2)$ (where $k = \frac{n-1}{2}$) subject to $y_i \geq 0$ and $\sum_{i=1}^{k} y_i = 1$.

But for any real numbers a, b

$$(a-b)^2 + (a+b)^2 > 2a^2$$

if $b \neq 0$.

Hence at the global minimum, all y_i must be equal, and thus they must equal $\frac{1}{k}$ and the global minimum of Φ is

$$\frac{k}{k^2} = \frac{1}{k} = \frac{2}{n-1}.$$

Proof of (E). The proof of (E) reduces to showing that

$$2(n-1)\big[(\Omega_1^2 + \cdots + \Omega_n^2) - \sum_{i=1}^{N} y_i^2\big] \geq 4(y_1 + \cdots + y_N)^2$$

for n odd, and all $y_1 \geq 0, \ldots, y_N \geq 0$. We may assume $y_1 + \cdots + y_N = 1$, otherwise replace each y_i by $y_i \Big/ \left(\sum_{j=1}^{N} y_j\right)$.

But now by the optimization result,

$$2(n-1)\left[\Omega_1^2 + \cdots + \Omega_n^2 - \sum_{i=1}^{N} y_i^2\right] \geq \frac{2(n-1)}{k} = 4$$

since $\frac{1}{k}$ is the global minimum of

$$\Omega_1^2 + \cdots + \Omega_n^2 - \sum_{i=1}^{N} y_i^2.$$

This proves (E) and completes the proof of the Main Theorem for $n \geq 3$. Note that we have equality in the theorem when $n = 3$. □

5. Applications. 1. Consider the cubic multigraph (Cvetkovič et al [4, (4.15) p. 309]) with adjacency matrix

$$A := \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{pmatrix}$$

The spectrum of A is

$$(3, \frac{1}{2}(1+\sqrt{17}), \frac{1}{2}(1-\sqrt{17}), -2, -2, 0)$$

and $4\mathrm{trace}\,(A^4) - \left(\mathrm{trace}\,(A^2)\right)^2 = -28$ so $(3, \frac{1}{2}(1+\sqrt{17}), \frac{1}{2}(1-\sqrt{17}), -2, -2)$ is not the spectrum of a nonnegative 5×5 matrix

2. The list $(10, 8, -7, -6, -5)$ is not the spectrum of a nonnegative 5×5 matrix since

$$4s_4 - s_2^2 = -1404.$$

This list was proposed as a test spectrum by R. Loewy since it satisfied all previously known necessary conditions for realizability by a nonnegative matrix.

3. The list $(3, 3, -2, -2, -2)$ has been used as an example by several authors. It satisfies the JLL inequalities but is not realizable as the spectrum of a nonnegative matrix A. For if such an A exists, since the Perron eigenvalue 3 is repeated, A would have to be reducible under permutation similarity and thus the list would have to be partitionable into two lists which are separately realizable and this is clearly impossible. Note that its unrealizability also follows from the Main Theorem since $4s_4 - s_2^2 = -60$. One can seek the least positive number t for which

$$(3+t, 3-t, -2, -2, -2)$$

is realizable. A necessary condition is that $4s_4 - s_2^2 \geq 0$, that is

$$t^4 + 78t^2 - 15 \geq 0$$

$$t \geq t_0 := \sqrt{16\sqrt{6} - 39} = 0.43799\cdots$$

But for such a real number t, the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{15+t^2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ v & \frac{(t^4+78t^2-15)}{4} & 10+6t^2 & \frac{15+t^2}{2} & 0 \end{pmatrix}$$

where $v = -72 + 8t^2 + (15+t^2)(5+3t^2) = 3t^4 + 58t^2 + 3$ has the spectrum $(3+t, 3-t, -2, -2, -2)$. By Guo Wuwen [13], $(3+\beta, 3, -2, -2, -2)$ is therefore the spectrum of a nonnegative matrix for all $\beta \geq 2t_0 = 0.87598\cdots$

This is perhaps surprising in view of a generally held view that $\beta \geq 1$ was best possible here. The difficult question of finding the best possible bound on β will be considered elsewhere.

4. A real symmetric $n \times n$ matrix A is called copositive if $x^T Ax \geq 0$ for all vectors x with nonnegative entries; see [1], [7]. The class of copositive matrices obviously includes the class of positive semi-definite and positive definite matrices and also the class of (entrywise) nonnegative symmetric matrices. Furthermore it is clear that if $A = P + N$ when P is a positive semi-definite real symmetric matrix and N a nonnegative real symmetric matrix, then A is copositive, and a question arose as to whether conversely every copositive matrix is expressible as such a sum. A counterexample to this was constructed by A. Horn [7, (16.2)] and it is interesting to note that with a suitable ordering of the vertices, his example coincides with the leading principal 5×5 submatrix of $2(I+P_5)-J_{10}$. But, by (E) above, for nonnegative $x_1, \ldots, x_{10}$,

$$\begin{aligned}(x_1, \ldots, x_{10})^T 2(I+P_5)\begin{pmatrix} x_1 \\ \vdots \\ x_{10}\end{pmatrix} &\geq (x_1 + \cdots + x_{10})^2 \\ &= (x_1, \ldots, x_{10})^T J_{10}\begin{pmatrix} x_1 \\ \vdots \\ x_{10}\end{pmatrix}\end{aligned}$$

Hence $2(I+P_5) - J_{10}$ is copositive.

It then follows from Horn's argument that $2(I+P_5) - J_{10}$ is not of the form $X+Y$ where X is a nonnegative symmetric matrix and Y is a positive semi-definite real symmetric matrix. The cone of copositive $n \times n$ matrices is the dual of the cone of completely positive matrices; see [1], [7].

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