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SIGN-CONSISTENCY AND SOLVABILITY OF CONSTRAINED LINEAR SYSTEMS*

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Abstract. Sign-solvable linear systems were introduced in modelling economic and physical systems where only qualitative information is known. Often economic and physical constraints require the entries of a solution to be nonnegative. Yet, to date the assumption of nonnegativity has been omitted in the study of sign-solvable linear systems. In this paper, the notions of sign-consistency and sign-solvability of a constrained linear system $Ax = b,\ x \gneq 0$, are introduced. These notions give rise to new classes of sign patterns. The structure and the complexity of the recognition problem for each of these classes are studied. A qualitative analog of Farkas' Lemma is proven, and it is used to establish necessary and sufficient conditions for the constrained linear system $Ax = b,\ x \gneq 0$ to be sign-consistent. Also, necessary and sufficient conditions for the constrained linear system $Ax = b,\ x \gneq 0$ to be sign-solvable are determined, and these are used to establish a polynomial-time recognition algorithm. It is worth noting that the recognition problem for (unconstrained) sign-solvable linear systems is known to be NP-complete.

Key words. Linear systems, Qualitative Matrix Theory, Sign-solvable.

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1. Introduction.

Consider the following physics problem.

> **Problem 1.** Is it possible to apply three forces in the plane to a point at the origin so that the resulting force makes an (counter-clockwise) angle of $4\pi/3$ with the positive x-axis, if the angle between the three forces and the positive x-axis are $3\pi/4$, $7\pi/6$ and $11\pi/6$, respectively?

The answer to the problem is yes if and only if there exists a (entrywise) nonnegative solution to the linear system

$$\begin{bmatrix} -\sqrt{2}/2 & -\sqrt{3}/2 & \sqrt{3}/2 \\ \sqrt{2}/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}. \qquad (1)$$

Since the coefficient matrix of this system has full rank, the linear system has a solution. This is not enough to guarantee a nonnegative solution. However, $(x_1, x_2, x_3)^T = (0, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$ is a nonnegative solution to the system in (1), and hence the answer to Problem 1 is yes.

Now consider a related physics problem in which less specific information is known.

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> **Problem 2.** Three directions in the plane, one in each of quadrants 2, 3 and 4, are specified. Is it always possible to assign magnitudes to forces in these directions, so that the resultant force has a prescribed direction in the third quadrant?

Here the exact angles of the force vectors are not known. However, we do know that one vector has negative x component and positive y component, another vector has negative x component and negative y component, etc. This leads to a "qualitative linear system" of the form

$$\begin{bmatrix} - & - & + \\ + & - & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}. \tag{2}$$

The question now becomes, does each linear system of the form (2) have a nonnegative solution? Thus, we are led to the topic of this paper: understanding the behavior of nonnegative solutions to a linear system in which only qualitative information is known.

In many applications a linear system is used to model a physical system. Often certain natural constraints make a solution with negative entries meaningless. Thus, one is only concerned with the behavior of the nonnegative solutions to the linear system. In this paper we study the question of when is it possible to determine the qualitative behavior of the (entrywise) nonnegative solutions of $Ax = b$ given only qualitative information concerning A and b. To make this more precise we introduce the following definitions.

The *sign* of a real number a is defined to be

$$\operatorname{sign} a = \begin{cases} +1 & \text{if } a > 0, \\ \;\;0 & \text{if } a = 0, \text{ and} \\ -1 & \text{if } a < 0. \end{cases}$$

The *sign pattern* of a real matrix A is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. The *zero pattern* of A is the $(0, 1)$-matrix obtained from A by replacing each nonzero entry by 1. We write $A \geq 0$ if each of A's entries is nonnegative, and $A \gneq 0$ if $A \geq 0$ and $A \neq 0$. We say A is *nonnegative* if $A \geq 0$, and A is *nonpositive* if $-A \geq 0$. We say A is a *positive*, respectively *negative*, matrix if each of its entries is positive, respectively negative. We call a $(0, 1, -1)$-matrix a *sign pattern*. A sign pattern, B, determines a *qualitative class*, $\mathcal{Q}(B)$, consisting of all matrices with sign pattern B.

Consider a linear system of m equations in n unknowns given by

$$Ax = b,$$

where A is an m by n real matrix, and b is an m by 1 column vector. We call the system

$$Ax = b, \; x \gneq 0 \tag{3}$$

a *constrained* linear system. Thus, a constrained linear system is a system of linear equations along with the additional inequalities $x \gneq 0$. The constrained linear system (3) is *consistent* provided there exists a nonzero nonnegative vector z such that $Az = b$.

Farkas' Lemma (see Lemma 3.4 on p. 1657 of [9]) asserts that there exists a nonnegative vector x such that $Ax = b$ if and only if $y^T b \geq 0$ holds for each vector $y \in \mathbb{R}^m$ such that $y^T A \geq 0$. Thus when $b \neq 0$, the constrained system (3) is consistent if and only if $y^T b \geq 0$ holds for each vector $y \in \mathbb{R}^m$ such that $y^T A \geq 0$.

The constrained linear system (3) is *sign-consistent* provided the constrained system $\widetilde{A}x = \widetilde{b},\ x \gneq 0$ is consistent for all $\widetilde{A} \in \mathcal{Q}(A)$ and $\widetilde{b} \in \mathcal{Q}(b)$. Thus, Problem 2 is equivalent to the question: Is the constrained linear system $Ax = b,\ x \gneq 0$, where

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

sign-consistent? The answer is yes. To see this, let $y = [y_1, y_2]^T$ be a 2 by 1 sign pattern. It is easy to verify that unless $y = [0,0]^T$ or $y = b$, then some column of

$$\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} A$$

is nonzero and nonpositive. Hence if $\tilde{y} \in \mathcal{Q}(y)$, $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$ are such that $\tilde{y}^T \widetilde{A} \geq 0$, then $\tilde{y}$ has sign pattern $[0,0]^T$ or b, and thus $\tilde{y}^T \tilde{b} \geq 0$. Therefore, by Farkas' Lemma, each system $\widetilde{A}x = \tilde{b},\ x \gneq 0$ is consistent. The constrained linear system

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

is not sign-consistent, because there is no solution to

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x \gneq 0.$$

In [3, pp. 35–38] sign patterns A and b, where each linear system $\widetilde{A}x = \widetilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A), \widetilde{b} \in \mathcal{Q}(b)$) has at least one solution, are studied. We define such a linear system $Ax = b$ to to be *sign-consistent.* Thus the notion of sign-consistent, constrained linear systems generalizes that of sign-consistent (unconstrained) linear systems. In Section 2, we introduce and study two families of sign patterns that arise in the qualitative theory of sign-consistent, constrained linear systems. In Section 3, we derive necessary and sufficient conditions for the constrained linear system $Ax = b,\ x \gneq 0$ to be sign-consistent.

The constrained linear system (3) is *sign-solvable* provided we can determine which entries of x are positive knowing only the signs of the entries of A and of b.

More precisely, (3) is *sign-solvable* provided $Ax = b,\ x \gneq 0$ is sign-consistent and the elements of the set

$$\{\tilde{x} :\ \tilde{x} \gneq 0 \text{ and there exists } \widetilde{A} \in \mathcal{Q}(A), \tilde{b} \in \mathcal{Q}(b) \text{ with } \widetilde{A}\tilde{x} = \tilde{b}\}$$

have the same zero pattern. For example, the sign-consistent system arising in Problem 2 is not sign-solvable, since both $(0,1,0)^T$ and $(3,3,1)^T$ are solutions to $Ax = \tilde{b}$ for appropriately chosen $\tilde{b} \in \mathcal{Q}(b)$. The constrained linear system

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x \gneq 0 \tag{5}$$

is clearly sign-solvable, and the zero pattern of the solution is $[1,1,0]^T$.

The above notion of sign-solvable constrained linear systems generalizes the notion of sign-solvable linear systems introduced by the economist Samuelson [8]. The linear system $Ax = b$ is *sign-solvable* provided it is sign-consistent and the vectors in

$$\{\tilde{x} :\ \text{there exists } \widetilde{A} \in \mathcal{Q}(A) \text{ and } \tilde{b} \in \mathcal{Q}(b) \text{ such that } \widetilde{A}\tilde{x} = \tilde{b}\}$$

have the same sign pattern. The (unconstrained) linear system obtained from (5) by removing the constraint that $x \gneq 0$ is not sign-solvable. For a discussion of sign-solvability see Section 1.2 of [3]. In Section 4, we introduce and study two families of sign patterns that arise in the qualitative theory of sign-solvable constrained linear systems. In Section 5, we derive necessary and sufficient conditions for the constrained system $Ax = b,\ x \gneq 0$ to be sign-solvable.

We conclude this introductory section with some necessary technical definitions. The set of all n by 1 positive vectors is denoted by $\mathbb{R}^n_+$. The *null space* of a matrix A is the set of all vectors x such that $Ax = 0$, and is denoted by $\mathrm{NS}(A)$. A vector in $\mathrm{NS}(A)$ is a *null vector* of A. A *left null vector* of A is a vector y such that $y^T A = 0$.

Let A be an m by n matrix, let α be a subset of $\{1,2,\ldots,m\}$, and let β be a subset of $\{1,2,\ldots,n\}$. Then $A[\alpha,\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. The submatrix, $A[\overline{\alpha},\overline{\beta}]$, determined by the rows whose indices are complementary to those of α and the columns whose indices are complementary to those of β is also denoted by $A(\alpha,\beta)$. If A is square and $\alpha = \beta$, then we write $A[\alpha]$ instead of $A[\alpha,\alpha]$ and $A(\alpha)$ instead of $A(\alpha,\alpha)$. If z is an m by 1 column vector then we write $z[\alpha]$ instead of $z[\alpha,\{1\}]$. A *signing* of order n is a nonzero, n by n diagonal matrix each of whose diagonal entries is in the set $\{-1,0,1\}$. A signing in which each diagonal entry is nonzero is a *strict signing*. We denote by $\mathrm{diag}(d_1,\ldots,d_n)$ the diagonal matrix of order n whose (i,i)th entry is d_i.

2. L^+- and sign-central matrices. In this section, we briefly describe certain families of sign patterns that arise in the known characterizations of (unconstrained) sign-consistent, linear systems. We then introduce and study related families of sign patterns that arise in the constrained setting.

As is customary, we use the "requires" and "allows" terminology for sign patterns. More specifically, let P be a property that a matrix can or cannot have. The sign pattern A *requires* P if each matrix in $\mathcal{Q}(A)$ has property P, and A *allows* P if there exists a matrix in $\mathcal{Q}(A)$ with property P.

Sign-consistent systems $Ax = b$ are studied in [3, pp. 35–38], where they are characterized in terms of L-matrices, and matrices that we call balanceable matrices. A sign pattern is an *L-matrix* provided it requires linearly independent rows, and is *balanceable* if it allows a left null vector with no zero entries. A square L-matrix is a *sign-nonsingular* matrix, which we abbreviate to an SNS-matrix. The following structure result is a paraphrase of Theorem 3.1.4 in [3]. In Proposition 2.2 (and later in Corollary 3.5, and Proposition 4.1) for each positive integer k we regard a 0 by k matrix as an (empty) L-matrix which is not balanceable, a k by 0 matrix as an (empty) balanceable matrix which is not an L-matrix, and a 0 by 0 matrix as both an L-matrix and a balanceable matrix. But when stating or proving results about L-matrices or balanceable matrices we implicitly assume that the matrices are not empty.

PROPOSITION 2.1. *Let A be a sign pattern. Then the rows and columns of A can be permuted so that A has the form*

$$\begin{bmatrix} A_B & O \\ * & A_L \end{bmatrix},$$

where A_L is an L-matrix, and A_B is a balanceable matrix with no zero columns. Moreover, A_L and A_B are unique, up to permutation of rows and columns.

Using Proposition 2.1, and the fact that a linear system $Ax = b$ is consistent if and only if $y^T b = 0$ for each y such that $y^T A = 0$, the following characterization (a paraphrase of Corollary 3.1.3 in [3]) can be derived.

PROPOSITION 2.2. *Let A be a sign pattern. Then the linear system $Ax = b$ is sign-consistent if and only if each nonzero entry of b lies in a row that intersects A_L.*

In Section 3, we generalize this result to constrained linear systems. We now define and study the constrained analogs of L- and balanceable matrices. Let B be an m by n matrix. We denote by $k(B)$ the cone

$$\{Bx : x \geq 0\},$$

generated by its columns, and by $k^*(B)$ its dual cone $\{y : y^T B \geq 0\}$. We define a sign pattern A to be an *L^+-matrix* if it requires the dual cone to be $\{0\}$, that is, A is an L^+-matrix if and only if $k^*(\widetilde{A}) = \{0\}$ for each $\widetilde{A} \in \mathcal{Q}(A)$. For example, $\begin{bmatrix} 1 & -1 \end{bmatrix}$ is an L^+-matrix, and $\begin{bmatrix} 1 & 1 \end{bmatrix}$ is not an L^+-matrix.

The following characterization of L-matrices follows from standard techniques; see [3, Chapter 2].

PROPOSITION 2.3. *Let A be a sign pattern. Then the following are equivalent.*
(a) A is an L-matrix.
(b) For each signing D, some column of DA is nonzero and nonpositive, or nonzero and nonnegative.

(c) For all $b \neq 0$, the linear system $Ax = b$ is sign-consistent.
(d) A requires that its columns span all of $\mathbb{R}^m$.

We now generalize this to L^+-matrices.

THEOREM 2.4. *Let A be an m by n sign pattern. Then the following are equivalent.*
(a) A is an L^+-matrix.
(b) A requires a positive null vector and A has no zero row.
(c) For each signing, D, some column of DA is nonzero and nonnegative.
(d) For each signing, D, some column of DA is nonzero and nonpositive.
(e) For all $b \neq 0$, the constrained, linear system $Ax = b$, $x \gneq 0$ is sign-consistent.
(f) A requires that the cone it generates is all of $\mathbb{R}^m$.

Proof. **(a) implies (b):** By contrapositive. If A has a zero row, then A is not an L^+-matrix. Otherwise, assume that $\widetilde{A}$ has sign pattern A and has no positive null vector. Thus,

$$\mathrm{NS}(\widetilde{A}) \cap \mathbb{R}^n_+ = \emptyset.$$

By the separation theorem for convex sets (see, for example, Chapter 11, Theorem 5 in [6]), $\mathrm{NS}(\widetilde{A})^\perp$ contains a vector y with $y \gneq 0$. But $\mathrm{NS}(\widetilde{A})^\perp$ is the row-space of $\widetilde{A}$, and hence there exists a nonzero vector z such that $z^T\widetilde{A} = y \geq 0$. Thus, $k^*(\widetilde{A}) \neq \{0\}$, and A is not an L^+-matrix.

(b) implies (c): By contrapositive. Let D be a signing such that each nonzero column of DA has a negative entry. If $DA = O$, then A has a zero row. Otherwise, by emphasizing the negative entries in DA, it is easy to construct a matrix $\widetilde{A} \in \mathcal{Q}(A)$ such that each column sum of the matrix $D\widetilde{A}$ is nonpositive, and at least one sum is negative. Thus, $\widetilde{A}$ contains a vector y in its rowspace with the property that $-y \gneq 0$. Since $y^T x = 0$ for each x in the null space of $\widetilde{A}$, $\widetilde{A}$ does not have a positive null vector.

(c) implies (d): This follows by noting that some column of DA is nonzero and nonnegative if and only if some column of $(-D)A$ is nonzero and nonpositive.

(d) implies (e): Assume that (d) holds. Suppose that y is a vector and $\widetilde{A}$ is a matrix in $\mathcal{Q}(A)$ such that $y^T\widetilde{A} \geq 0$. Let D be the diagonal matrix whose ith entry is the sign of the ith entry of y. If $y \neq 0$, then, since $y^T\widetilde{A} \geq 0$, each nonzero column of DA has a positive entry. This would contradict (d). Hence, $y = 0$. It now follows from Farkas' Lemma that (e) holds.

(e) implies (f): This is clear.

(f) implies (a): By Farkas' Lemma, $k(\widetilde{A}) = \mathbb{R}^m$ if and only if $k^*(\widetilde{A}) = \{0\}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. Hence (f) implies (a). □

Note that Proposition 2.3 and Thereom 2.4 imply that an L^+-matrix is necessarily an L-matrix. The matrix $\begin{bmatrix} 1 & 1 \end{bmatrix}$ is an example of an L-matrix which is not an L^+-matrix. As an L^+-matrix requires both a positive null vector and linearly independent rows, a nonempty L^+-matrix has more columns than rows.

It is known [5] that the problem of recognizing if an m by n matrix A is not an L-matrix is an NP-complete problem. Using (c) of Theorem 2.4, we see that the problem of recognizing if a given m by n matrix A is not an L^+-matrix is NP. The recognition problem for L-matrices can be polynomially reduced to that for L^+-matrices by noting that the matrix A is an L-matrix if and only if the matrix $\begin{bmatrix} A & -A \end{bmatrix}$ is an L^+-matrix. Hence, the the problem of recognizing if a matrix is not an L^+-matrix is NP-complete.

Now that we have discussed the constrained analog of L-matrices, we turn to the constrained analog of a balanceable matrix. Recall that a balanceable matrix is a sign pattern with the property that it allows a left null vector with no zeros. This is equivalent to the fact that there exists a strict signing D such that each nonzero column of DA has both a positive and a negative entry. A natural generalization to the constrained case, which indeed turns out to be exactly what we need, is to sign patterns whose dual allows a vector having no zero entries. The following lemma gives a signing characterization of such matrices.

LEMMA 2.5. *Let A by an m by n sign pattern. Then A allows a vector in the dual which has no zero entries if and only if there exists a strict signing D such that each nonzero column of DA has a negative entry.*

Proof. First assume that D is a strict signing such that each nonzero column of DA has a negative entry. By emphasizing the negative entries of DA it is possible to create a matrix $\widetilde{A} \in \mathcal{Q}(A)$ such that each column sum of $D\widetilde{A}$ is nonpositive. Hence, each column sum of $-D\widetilde{A}$ is nonnegative, and thus $k^*(\widetilde{A})$ contains a vector with no zero entries.

Conversely, assume that there exist $\widetilde{A} \in \mathcal{Q}(A)$, and $y \in k^*(\widetilde{A})$ such that each entry of y is nonzero. By definition, $y^T\widetilde{A} \geq 0$. Since each entry of y is nonzero, each nonzero column of DA has a positive entry, where $D = diag(y_1, \ldots, y_m)$. Hence each nonzero column of $(-D)A$ has a negative entry. □

Sign patterns that do not satisfy the signing condition in Lemma 2.5 have been previously studied. A matrix B is *central*, if the origin is in the convex hull of its columns. Equivalently, B is central if the constrained system $Bx = 0,\ x \gneq 0$ is consistent. A sign pattern A is *sign-central* provided A requires the property of being central. Thus, A is sign-central if and only if $Ax = 0,\ x \gneq 0$ is a sign-consistent, constrained linear system. Since the coefficient matrix in (4) is not sign-central, we again see that (4) is not sign-consistent.

Sign-central matrices have been studied in [1, 4]. In particular, the following characterization of sign-central matrices is contained in [1]. Using this characterization, it is shown in [1] that the problem of recognizing if a matrix is not sign-central is NP-complete.

PROPOSITION 2.6. *Let A be an m by n sign pattern. Then A is a sign-central matrix if and only if for each strict signing D, some column of DA is nonnegative.*

Hence, Lemma 2.5 and Proposition 2.6 show that the sign patterns which have no zero column and allow a vector in the dual with no zero entries are precisely the sign patterns that are not sign-central matrices. If zero columns are allowed, then a matrix can be sign-central and allow a vector in the dual with no zero entries. The sign pattern

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

is one such example. Clearly, Proposition 2.6 implies that a balanceable matrix with no zero column is not sign-central. The matrix $\begin{bmatrix} 1 \end{bmatrix}$ shows that the converse does not hold.

We now summarize the relations between the classes of sign patterns discussed in this section. Two new classes of sign patterns were defined, the L^+-matrices, and the matrices that are not sign-central. These classes generalize L-matrices, and balanceable matrices, respectively. In particular, we have seen that each L^+-matrix is an L-matrix but not conversely, and each matrix which is balanceable and has no column of zeros is not sign-central but not conversely. By Theorem 2.4 every L^+-matrix is a sign-central matrix. The matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

shows that sign-central matrix need not be an L^+-matrix, the sign pattern

$$\begin{bmatrix} 1 \end{bmatrix}$$

shows that an L-matrix need not be sign-central, and the sign pattern

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

shows that a sign-central matrix need not be an L-matrix.

The main result of Section 3 is a characterization of sign-consistent, constrained linear systems like that of Proposition 2.2, but with L^+- rather than L-matrices and with not sign-central rather than balanceable matrices.

3. Sign-consistency. The main result of this section is the development of necessary and sufficient conditions for the constrained linear system $Ax = b$, $x \gneq 0$ to be sign-consistent. We begin with a qualitative version of Farkas' Lemma. Throughout this section we let A denote an m by n sign pattern, and $\mathcal{S}$ denote the set of all signings D such that each nonzero column of DA has a positive entry.

LEMMA 3.1. *Let $Ax = b$, $x \gneq 0$ be a constrained linear system. Assume that $b = [b_1, \ldots, b_m]^T \neq 0$. Then the system is sign-consistent if and only if $d_i b_i \geq 0$ for each signing $D = diag(d_1, \ldots, d_m) \in \mathcal{S}$ and $i = 1, \ldots, m$.*

Proof. We argue both implications by contrapositive. First suppose that $Ax = b$, $x \gneq 0$ is not sign-consistent. Then there exist $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$ such that $\widetilde{A}x = \tilde{b}$, $x \gneq 0$ is not consistent. By Farkas' Lemma, there exists a vector y such that $y^T \widetilde{A} \geq 0$ and $y^T \tilde{b} < 0$. Let $D = \text{diag}(d_1, \ldots, d_m)$ be the signing whose ith entry is the sign of the ith entry of y. Since $y^T \widetilde{A} \geq 0$, $D \in \mathcal{S}$, and since $y^T \tilde{b} < 0$ there exists an i such that $d_i b_i < 0$.

Next suppose that there is a signing $D = \text{diag}(d_1, d_2, \ldots, d_m)$ in $\mathcal{S}$, and an i such that $d_i b_i < 0$. By emphasizing each positive entry of DA one obtains a matrix

ELA

$\widetilde{A} \in \mathcal{Q}(A)$ such that each of the column sums of $D\widetilde{A}$ is nonnegative. By emphasizing the ith entry of b, one obtains a vector $\tilde{b} \in \mathcal{Q}(A)$ such that the column sum of $D\tilde{b}$ is negative. It now follows from Farkas' Lemma that $\widetilde{A}x = \tilde{b},\ x \gneq 0$ is not consistent. Hence $Ax = b,\ x \gneq 0$ is not sign-consistent. $\square$

Using the qualitative version of Farkas' Lemma we immediately obtain the following necessary and sufficient conditions for $Ax = b,\ x \gneq 0$ to be sign-consistent. We define the sets:

$$\begin{aligned}
A_+ &= \{i : \text{the } (i,i)\text{-entry of each } D \in \mathcal{S} \text{ is nonnegative} \\
&\qquad \text{and is positive for at least one such } D\} \\
A_- &= \{i : \text{the } (i,i)\text{-entry of each } D \in \mathcal{S} \text{ is nonpositive} \\
&\qquad \text{and is negative for at least one such } D\} \\
A_0 &= \{i : \text{the } (i,i)\text{-entry of each } D \in \mathcal{S} \text{ is zero}\} \\
A_* &= \{1, 2, \ldots, m\} \setminus (A_+ \cup A_- \cup A_0).
\end{aligned}$$

Corollary 3.2. *Let $Ax = b,\ x \gneq 0$ be a constrained linear system with $b \neq 0$. Then the system is sign-consistent if and only if $j \in A_+ \cup A_0$ whenever $b_j > 0$, and $j \in A_- \cup A_0$ whenever $b_j < 0$.*

Note in particular that if $j \in A_*$, and $Ax = b,\ x \gneq 0$ is sign-consistent then $b_j = 0$. For the constrained system $Ax = b,\ x \gneq 0$ arising in Problem 2, we have

$$S = \{\text{diag}(-1, -1)\},\ A_+ = \emptyset,\ A_- = \{1, 2\},\ A_0 = \emptyset, \ \text{ and } A_* = \emptyset.$$

Hence, by Corollary 3.2, the system arising in Problem 2 is sign-consistent.

The sets A_+, A_-, A_0, and A_* can be described in terms of certain submatrices of A. First we give a structure theorem for sign patterns that is analogous to Proposition 2.1. In Lemma 3.3 (and later in Theorem 3.4) we must regard an empty matrix as an L^+-matrix if and only if it has no rows, and an empty matrix as not sign-central if and only if it has no columns.

Lemma 3.3. *Let A be an m by n sign pattern. Then the rows and columns of A can be permuted to obtain a matrix of the form*

$$\begin{bmatrix} A_N & O \\ * & A_{L^+} \end{bmatrix},$$

where A_{L^+} is an L^+-matrix, and A_N is not sign-central. Moreover, up to permutation of rows and columns, A_{L^+} and A_N are unique.

Proof. First we establish the existence. If A is an L^+-matrix, then we may take A_{L^+} to be A, and A_N to be the empty 0 by 0 matrix. Otherwise, by Theorem 2.4, there is a signing D such that each nonzero column of DA has a positive entry. Among all such signings choose D such that it has the largest number of nonzero entries. Without loss of generality we may assume that $D = D' \oplus O_{m-k}$, where D' is a strict signing of order k. If $k = m$, we take A_{L^+} to be an empty matrix, and A_N to be the matrix consisting of the nonzero columns of A. Otherwise, by permuting the columns of A we may assume that

$$A = \begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix},$$

where A_1 has k rows, each of its columns is nonzero, and A_2 has at least one column. By the choice of D, for each signing E of order $m-k$ some column of EA_2 is nonzero and nonpositive. Hence, by Theorem 2.4, A_2 is an L^+-matrix. By Proposition 2.6, A_1 is not sign-central. The existence part of the proof is now completed by letting $A_N = A_1$, and $A_{L^+} = A_2$.

We now argue the uniqueness. Since A_1 is not sign-central, there exists an $\widetilde{A}_1 \in \mathcal{Q}(A_1)$ whose null space does not contain a nonzero nonnegative vector. Thus, every nonnegative vector in the null space of a matrix of the form

$$\begin{bmatrix} \widetilde{A}_1 & O \\ * & A_2 \end{bmatrix}$$

has its first k nonzero entries equal to 0. Since A_2 is an L^+-matrix, Theorem 2.4 implies that each $\widetilde{A}$ in $\mathcal{Q}(A)$ has a nonnegative, null vector each of whose last $n-k$ coordinates is nonzero. Thus, it follows that the columns of A_N are precisely those i for which some $\widetilde{A} \in \mathcal{Q}(A)$ has no nonnegative null vector whose ith coordinate is nonzero. This determines the columns of A_N, and hence of A_{L^+}. The rows of A_{L^+} are determined, since they are precisely the nonzero rows in the submatrix whose columns intersect A_{L^+}. ◻

We now identify the sets A_+, A_-, A_0 and A_*, in terms of A_{L^+} and A_N. We let e_i denote the vector with a 1 in position i and 0's elsewhere.

THEOREM 3.4. *Let A be an m by n sign pattern, and assume that A has the form*

$$\begin{bmatrix} A_N & O \\ * & A_{L^+} \end{bmatrix},$$

where A_{L^+} is an L^+-matrix with k rows, and A_N is not sign-central. Then the following hold.

(a) $A_0 = \{k+1, \ldots, m\}$,

(b) $A_+ = \{i :$ the matrix $[A_N \ -e_i]$ is sign-central$\}$,

(c) $A_- = \{i :$ the matrix $[A_N \ e_i]$ is sign-central$\}$.

Proof. Let D be signing in $\mathcal{S}$. Since A_{L^+} is an L^+-matrix, each of the last $(m-k)$ diagonal entries of D are 0. Hence $\{k+1, \ldots, m\} \subseteq A_0$. Since A_N is not sign-central, there exists a strict signing E of order k such that $E \oplus O_{m-k} \in \mathcal{S}$. Hence $A_0 \subseteq \{k+1, \ldots, m\}$. Therefore, (a) holds.

Suppose that $i \in A_+$. If E is a strict signing of order k such that $E \oplus O \in \mathcal{S}$, then the last column of $E[A_N \ -e_i]$ is nonpositive. Otherwise, some column of EA_N is nonpositive. Hence, by Proposition 2.2, $[A_N \ -e_i]$ is sign-central. Similarly, if $i \in A_-$, then $[A_N \ e_i]$ is sign-central.

Now suppose that $[A_N \ -e_i]$ is sign-central. Since A_N is not sign-central, there exists a strict signing E such that each column of EA_N has a positive entry. Since

$[A_N\ -e_i]$ is sign-central, the (i,i)-entry of E is positive. Hence, $E \oplus O_{m-k}$ is a signing in $\mathcal{S}$ whose (i,i)-entry is positive.

Let D be a signing in $\mathcal{S}$. As previously noted, $D = E \oplus O_{m-k}$ for some signing E of order k. Without loss of generality we may assume that $E = F \oplus O_\ell$, where F is a strict signing, and that

$$A_N = \begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix}$$

where A_2 has ℓ rows and A_1 has no zero columns. Since A_N is not sign-central, A_2 is not sign central. Hence, by Proposition 2.6, there is a strict signing G such that each nonzero column of GA_2 has a positive entry. Since A_N has no zero column, each column of $(F \oplus G)A_N$ has a positive entry. Since $[A_N\ -e_i]$ is sign-central, Proposition 2.6 implies that the (i,i)-entry of $F \oplus G$ is positive. Hence the (i,i)-entry of D is nonnegative. Therefore, $i \in A_+$. A similar argument shows that if $[A_N\ e_i]$ is sign-central, then $i \in A_-$. □

As the recognition problems for L^+-matrices and sign-central matrices are NP-complete, Theorem 3.4 shows that the problem of determining if the system $Ax = b,\ x \gneq 0$ is not sign-consistent is NP-complete.

Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

It is easy to verify that $A_N = A[\{1,2,3,4\},\{1,2,3,4\}]$, and that $A_{L^+} = A[\{5,6\},\{5,6,7,8\}]$. The only i such that $[A_N\ -e_i]$ is sign-central is $i = 2$, and there is no j such that $[A_N\ e_j]$ is sign-central. Hence it follows from Theorem 3.4, that $Ax = b,\ x \gneq 0$ is sign-consistent if and only if b is a nonzero sign pattern with $b_1 = b_3 = b_4 = 0$, and $b_2 \geq 0$.

We conclude this section by studying systems $Ax = b,\ x \gneq 0$ where A is a square matrix of order n that allows invertibility and $b \neq 0$. König's Theorem (see Theorem 1.2.1 in [2]) implies that an n by n matrix A allows invertibility if and only if A does not contain an r by s zero submatrix for any positive integers with $r + s > n$.

Suppose that $Ax = b,\ x \gneq 0$ is sign-consistent. Since a nonempty L^+-matrix has more columns than rows, the fact that A allows invertibility and Lemma 3.3, imply that A_{L^+} is empty, and hence $A = A_N$ is not sign-central. By Proposition 2.1, we may assume that A has the form

$$\begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix},$$

where A_1 is a balanceable matrix with no zero column, and A_2 is an L-matrix. We write b as

$$\begin{bmatrix} b^{(1)} \\ b^{(2)} \end{bmatrix}$$

to conform to this partition of A. Since L-matrices have at least as many columns as rows, the fact that A allows invertibility implies that both A_1 and A_2 are square matrices with no zero columns. Let k be the order of A_1. Since A_1 is balanceable and has no zero column, there is a strict signing E such that each column of A_1 contains both a positive and a negative entry. Hence both $E \oplus O_{n-k}$ and its negative are in $\mathcal{S}$. Thus by Corollary 3.2, $b^{(1)} = 0$.

Since A is an SNS-matrix, each matrix in $\mathcal{Q}(A)$ is invertible. Hence each system $\widetilde{A}x = \widetilde{b}$, $(\widetilde{A} \in \mathcal{Q}(A)$, $\widetilde{b} \in \mathcal{Q}(b))$ has exactly one solution, namely $x = (\widetilde{A_2})^{-1}\widetilde{b^{(2)}}$. It follows from Cramer's rule that for $j = 1, 2, \ldots, n+1$, the matrix C_j obtained from $[A_2\ b^{(2)}]$ by deleting column j has the property that either $\det \widetilde{C_j} \geq 0$ for all $\widetilde{C_j} \in \mathcal{Q}(C_j)$ or $\det \widetilde{C_j} \leq 0$ for all $\widetilde{C_j} \in \mathcal{Q}(C_j)$. Thus, for each j each nonzero term in the standard determinant expansion of C_j has the same sign. Hence, by Theorem 1.2.5 in [3], the determinants of the matrices in $\mathcal{Q}(C_j)$ all have the same sign. It now follows by Cramer's rule, that each vector $(\widetilde{A_2})^{-1}\widetilde{b^{(2)}}$, where $\widetilde{A_2} \in \mathcal{Q}(A_2)$, and $\widetilde{b^{(2)}} \in \mathcal{Q}(b^{(2)})$, has the same sign pattern. Hence, the unconstrained system $A_2x = b^{(2)}$ is sign-solvable, and $(A_2)^{-1}b^{(2)} \geq 0$.

Therefore, we have shown one direction of the following result. The converse follows immediately from the definitions.

COROLLARY 3.5. *Let*

$$A = \begin{bmatrix} A_1 & O \\ * & A_2 \end{bmatrix} \tag{6}$$

be a sign pattern where A_1 is a square, balanceable matrix with no zero columns, and A_2 is an SNS-matrix. Let

$$b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \end{bmatrix}$$

be a nonzero vector partitioned to agree with that of A. Then $Ax = b$, $x \gneq 0$ is sign-consistent if and only if $b^{(1)} = 0$, the linear system $A_2x = b^{(2)}$ is sign-solvable, and $(A_2)^{-1}b^{(2)} \geq 0$.

We note that Corollary 3.5 and the polynomial-time recognition algorithm for SNS-matrices given in [7] imply that there is a polynomial-time algorithm for recognizing if the constrained linear system $Ax = b, x \gneq 0$ is sign-consistent, in the case that A is square and allows invertibility. To see this, assume that A is square and allows invertibility. Using the polynomial-time recognition algorithm for SNS-matrices in [7], one can put A into the form in (6), where A_2 is an SNS-matrix, and A_1 is balanceable, in polynomial-time. Using Cramer's rule, one sees that $A_2x = b^{(2)}$ is sign-solvable if and only if each of the matrices C_j defined above is an SNS-matrix. Thus, using the polynomial-time algorithm for recognizing SNS-matrices, we can determine if $A_2x = b^{(2)}$ is sign-solvable in polynomial-time.

4. S-matrices, and matrices that do not allow centrality. In this section we introduce and study two classes of sign patterns that arise naturally when we consider sign-solvable, constrained linear systems.

We begin by describing the characterization of sign-solvable linear systems in [5]. This characterization is in terms of L-matrices and S^*-matrices. An m by $(m+1)$ sign pattern A is an S^*-matrix provided each of its m by m submatrices is an SNS-matrix. Equivalently, (see Corollary 1.2.10 in [3]), the m by $m+1$ sign-pattern A is an S^*-matrix provided there exists a sign pattern vector s such that A requires the property that the null space is spanned by a vector with sign pattern s.

The following paraphrased result of Klee, Ladner and Manber (see also Theorem 1.2.12 in [3]) shows that recognizing sign-solvable linear systems can be reduced to recognizing L-matrices and S^*-matrices.

Proposition 4.1. *Let $A = [a_{ij}]$ be an m by n matrix, and let b be a nonzero m by 1 vector. Assume that $z = (z_1, z_2, \ldots, z_n)^T$ is a solution to the (unconstrained) linear system $Ax = b$. Let*

$$\beta = \{j : z_j \neq 0\} \text{ and } \alpha = \{i : a_{ij} \neq 0 \text{ for some } j \in \beta\}.$$

Then the (unconstrained) system $Ax = b$ is sign-solvable if and only if the matrix

$$[A[\alpha, \beta] \; -b[\alpha]]$$

is an S^*-matrix and the matrix $A(\alpha, \beta)^T$ is an L-matrix.

In Section 5, we generalize this result to constrained linear systems. The classes of matrices involved are S-matrices, and matrices that do not allow centrality.

An *S-matrix* is an S^*-matrix whose null space is spanned by a positive vector. By Theorem 2.4 an S-matrix is an L^+-matrix and hence a sign-central matrix. S-matrices have been extensively studied (see Chapter 4 of [3]), and as shown in [5] can be recognized in polynomial-time.

Since a matrix has linearly independent columns if and only if it has no nonzero null vector, the condition that $A(\alpha, \beta)^T$ be an L-matrix is equivalent to the condition that no matrix in $\mathcal{Q}(A(\alpha, \beta)^T)$ has a nonzero null vector. This leads us to consider matrices M such that no matrix in $\mathcal{Q}(M)$ contains a nonzero, nonnegative null vector. These are precisely the sign patterns that do not allow centrality. We now give a useful characterization of matrices which do not allow centrality.

Theorem 4.2. *Let A be an m by n matrix. Then the following are equivalent.*
(a) There exists a positive integer k and permutation matrices P and Q such that the first k rows of PAQ have the form

$$\begin{bmatrix} x_1 & O & \cdots & O \\ * & x_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & x_k \end{bmatrix},$$

where x_j is either a positive or a negative row vector for $j = 1, 2, \ldots, k$.
(b) The row space of each matrix in the qualitative class of A contains a positive vector.
(c) A does not allow centrality.
(d) Each submatrix $A[\{1, 2, \ldots, m\}, \beta]$ of A contains either a nonzero, nonnegative row or a nonzero, nonpositive row.

Proof. First we assume (a), and prove that for each matrix in the qualitative class of PAQ there exists a positive vector in the span of its first k rows. We prove this by induction on k. If $k = 1$, then the first row of each matrix in the qualitative class of A is either positive or negative, and hence (b) holds. Assume $k \geq 2$ and proceed by induction. Let $P\widetilde{A}Q$ be a matrix in $\mathcal{Q}(PAQ)$, let ℓ be the number of columns of x_1. By the inductive hypothesis, there is a linear combination of rows 2, 3,..., k of $P\widetilde{A}Q$ which has positive entries in all but the first ℓ entries. For λ sufficiently large, this linear combination along with either λx_1 or $-\lambda x_1$ will be a positive vector. Hence, (b) holds.

Next we assume (b) and prove that (c) holds. Let $\widetilde{A} \in \mathcal{Q}(A)$. Since the row space of $\widetilde{A}$ contains a positive vector, $\widetilde{A}$ is not central. Hence, (c) holds.

To prove that (c) implies (d) we prove the contrapositive. Thus, assume that there exists a β such that each nonzero row of $A[\{1, 2, \ldots, m\}, \beta]$ contains both a positive and a negative entry. Then there exists a matrix $\widetilde{A} \in \mathcal{Q}(A)$ such that each vector in the row space of $\widetilde{A}$ has the sum of its coordinates indexed by β equal to 0. Hence, A allows centrality.

Finally, we prove that (d) implies (a). Assume that (d) holds. The proof is by induction on m, the number of rows of A. If $m = 1$, then the condition implies that A contains either a positive row or a negative row, and hence (a) holds. Assume that $m \geq 2$, and proceed by induction. The assumptions imply that A contains a nonzero, nonpositive row or a nonzero, nonnegative row. Without loss of generality we may assume that the first $\ell > 0$ entries are the nonzero entries of the first row of A, and each of these are positive. If $\ell = n$, then we may take $k = 1$, and x_1 to be the first row of A. Otherwise, let $B = A[\{2, \ldots, m\}, \{\ell + 1, \ldots, n\}]$. The assumptions on A imply that each submatrix $B[\{1, \ldots, m-1\}, \beta]$ of B contains either a nonzero nonpositive row or a nonzero, nonnegative row. Hence, by induction, up to permutation of rows and columns the first $k-1$ rows of B have the form

$$\begin{bmatrix} x_2 & O & \cdots & O \\ * & x_3 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & x_k \end{bmatrix},$$

where each x_j $(j = 2, 3, \ldots, k)$ is positive or negative. It now follows, using the first k rows of A, that (a) holds. Hence, by induction, (d) implies (a).

The proof is now complete. □

We note that the proof of (d) implies (a) provides a polynomial-time algorithm for determining whether a matrix, A, does not allow centrality. Namely,

1. If A has exactly one row, then A does not allow centrality if and only if A is

positive or A is negative. Otherwise,
2. If A does not have a nonzero, nonpositive row or a nonzero, nonnegative row, then A allows centrality. Otherwise,
3. Let j be a nonzero, nonpositive row or a nonzero, nonnegative row. Let B be the submatrix of A obtained by deleting row j and each column which contains a nonzero entry in the jth row. Replace A by B and go to step 1.

We now summarize this section. For unconstrained linear systems, understanding sign-solvability reduces to understanding S^*-matrices and L-matrices. We've indicated that each S^*-matrix is an L^+-matrix, and hence a sign-central matrix. The analog for constrained linear systems, are S-matrices. Clearly, each S-matrix is an S^*-matrix, but not conversely. The correct generalization of L-matrices to the constrained, sign-solvable setting is based on the characterization that A^T is an L-matrix if and only if the null space of A^T does not allow a nonzero vector. This leads to the class of sign patterns that do not allow centrality. Clearly, if A^T is an L-matrix, then A does not allow centrality. The matrix $[1,1]^T$ shows that the converse does not hold.

5. Sign-solvability. In this section we give necessary and sufficient conditions for a constrained linear system to be sign-solvable, and present a polynomial-time algorithm for recognizing such linear systems. The following result characterizes sign-solvable constrained homogeneous linear systems. In Theorem 5.1 and Corollary 5.2 we say that an empty matrix does not allow centrality if and only if it has no columns.

THEOREM 5.1. *Let $A=[a_{ij}]$ be an m by n matrix. Assume that $z=(z_1,z_2,\ldots,z_n)^T$ is a solution of the constrained, linear system $Ax=0,\ x \gneq 0$. Let*

$$\beta=\{j : z_j \neq 0\} \text{ and } \alpha=\{i : a_{ij}\neq 0 \text{ for some } j\in\beta\}.$$

Then $Ax=0, x\gneq 0$ is sign-solvable if and only if the matrix

$$A[\alpha,\beta]$$

is an S-matrix and the matrix $A(\alpha,\beta)$ does not allow centrality.

Proof. Without loss of generality we assume that $\alpha=\{1,2,\ldots,k\}$ and that $\beta=\{1,2,\ldots,\ell\}$ for some nonnegative integers k and ℓ. It follows from the definitions of α and β that

$$A=\begin{bmatrix} A_1 & A_3 \\ O & A_2 \end{bmatrix}$$

where A_1 is a k by ℓ matrix with no row of zeros. The linear system $Ax=0$ can be rewritten as

$$A_1x^{(1)}+A_3x^{(2)}=0$$

$$A_2x^{(2)}=0.$$

Assume that the constrained system $Ax = 0,\ x \gneq 0$ is sign-solvable. Then every nonzero, nonnegative vector

$$\begin{bmatrix} \tilde{x}^{(1)} \\ \tilde{x}^{(2)} \end{bmatrix}$$

which solves a system $\widetilde{A}x = 0$ for some $\widetilde{A} \in \mathcal{Q}(A)$ satisfies $\tilde{x}^{(2)} = 0$. Hence the constrained system $A_1x^{(1)} = 0,\ x^{(1)} \gneq 0$ is sign-solvable and every solution has no zero entries. Since A_1 has no zero row, Theorem 2.4 implies that A_1 is an L^+-matrix. Since a nonempty L^+-matrix has more columns than rows, $\ell \geq k+1$. If $\ell > k+1$, then the null space of A_1 has dimension at least 2, and hence there is a nonzero, nonnegative solution to $A_1x^{(1)} = 0$ which is not positive. Hence $\ell = k+1$. Since A_1 is an L^+-matrix, A_1 is an L-matrix. Hence each matrix in $\mathcal{Q}(A_1)$ has null space of dimension 1, and it follows that A_1 is an S-matrix.

We next show that A_2 does not allow centrality. Suppose to the contrary that A_2 allows centrality. First consider the case that A_2 is empty. Then A_2 is 0 by $n-k$, and $n-k > 0$. Let $x^{(2)}$ be any nonnegative, nonzero vector. Since A_1 is an L^+-matrix, by Theorem 2.4, there is a vector $x^{(1)}$ such that $A_1x^{(1)} = -A_3x^{(2)}$, and $x^{(1)} \gneq 0$. It follows that

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}$$

is a solution to $Ax = 0$ whose zero pattern is different from that of z. This contradicts the sign-solvability of $Ax = 0, x \gneq 0$.

Next consider the case that A_2 is not empty. Then there is a central matrix $\widetilde{A}_2 \in \mathcal{Q}(A_2)$. Let $x^{(2)}$ be a nonnegative, nonzero vector in the null space of $\widetilde{A}_2$. Since A_1 is an L^+-matrix, by Theorem 2.4, there is a vector $x^{(1)}$ such that $A_1x^{(1)} = -A_3x^{(2)}$, and $x^{(1)} \gneq 0$. It follows that

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}$$

is a solution to

$$\begin{bmatrix} A_1 & A_3 \\ O & \widetilde{A}_2 \end{bmatrix} x = 0$$

whose zero pattern is different than that of z. This contradicts the sign-solvability of $Ax = 0,\ x \gneq 0$. Therefore, A_2 does not allow centrality.

Conversely, assume that

$$A[\alpha, \beta]$$

is an S-matrix and the matrix $A(\alpha,\beta)$ does not allow centrality. Let $\widetilde{A}$ be a matrix in $\mathcal{Q}(A)$. Then

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_1 & \widetilde{A}_3 \\ O & \widetilde{A}_2 \end{bmatrix}$$

where $\widetilde{A}_i$ belongs to $\mathcal{Q}(A_i), (i=1,2,3)$. The linear system $\widetilde{A}x = 0$ is of the form

$$\widetilde{A}_1 x^{(1)} + \widetilde{A}_3 x^{(2)} = 0$$

$$\widetilde{A}_2 x^{(2)} = 0.$$

Since $\widetilde{A}_2$ is not central, and $x^{(2)}$ is constrained to be nonnegative, $x^{(2)}$ must equal 0. Since A_1 is an S-matrix, the above equations have a unique nonzero, nonnegative solution

$$\tilde{z} = \begin{bmatrix} \tilde{z}^{(1)} \\ \tilde{z}^{(2)} \end{bmatrix},$$

where $\tilde{z}^{(2)} = 0$ and $\tilde{z}^{(1)}$ is positive. Therefore $Ax = 0$ is a sign-solvable constrained linear system. □

We now generalize Proposition 4.1 to nonhomogeneous sign-solvable constrained linear systems.

Corollary 5.2. *Let $A = [a_{ij}]$ be an m by n matrix, and let b be a nonzero m by 1 vector. Assume that $z = (z_1, z_2, \ldots, z_n)^T$ is a solution of the constrained linear system $Ax = b$, $x \gneq 0$. Let*

$$\beta = \{j : z_j \neq 0\} \text{ and } \alpha = \{i : a_{ij} \neq 0 \text{ for some } j \in \beta\}.$$

Then the constrained linear system $Ax = b$, $x \gneq 0$ is sign-solvable if and only if $b[\overline{\alpha}] = 0$, the matrix

$$[A[\alpha,\beta] \ \ -b[\alpha]]$$

is an S-matrix and the matrix $A(\alpha,\beta)$ does not allow centrality.

Proof. The corollary follows from Theorem 5.1 by noting that $Ax = b$, $x \gneq 0$ is sign-solvable if and only if $[A \ -b]y = 0$, $y \gneq 0$ is sign-solvable, and the last coordinate of each solution is a nonzero. □

As already noted, a polynomial-time algorithm for recognizing whether or not a matrix is an S-matrix is given in [5]. In Section 4, a polynomial-time algorithm for recognizing whether a matrix does not allow centrality is given. Thus, Theorem 5.1 and Corollary 5.2 now imply the following polynomial-time algorithm for determining if the constrained linear system $Ax = b$, $x \gneq 0$ is sign-solvable.

1. If $Ax = b$, $x \gneq 0$ has no solution, then the constrained system is not sign-solvable. Otherwise,

2. Let β be the support of a solution to $Ax = b,\ x \gneq 0$, and let α be the indices of the nonzero rows of $A[\{1,2,\ldots,m\},\beta]$. If the matrix $A(\alpha,\beta)$ allows centrality or if $b[\overline{\alpha}] \neq 0$, then the constrained system is not sign-solvable. Otherwise,
3. If $b \neq 0$ and the matrix $[A[\alpha,\beta]\ \ -b[\alpha]]$ is not an S-matrix, or if $b = 0$ and the matrix $A[\alpha,\beta]$ is not an S-matrix, then the constrained system is not sign-solvable. Otherwise, the constrained system is sign-solvable.

We note that (see [3] or [5]) the problem of recognizing (unconstrained) sign-solvable linear systems is NP-complete. This is because the problem of recognizing L-matrices (as opposed to recognizing matrices which do not allow centrality) is NP-complete.

We conclude this section with some examples. Let

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $x = [1,1,0,0,0]^T$ is a solution to $Ax = b, x \gneq 0$. Since the matrix $\left[\ A[\{1,2\},\{1,2\}\ \ \ -b[\{1,2\}]\ \right]$ is an S-matrix and the matrix $A[\{3,4\},\{3,4,5\}]$ does not allow centrality, it follows from Corollary 5.2 that the constrained system $Ax = b, x \gneq 0$ is sign-solvable.

For the constrained linear system $Ax = b$ arising in Problem 2, we see that we may take $\beta = \{2\}$ and $\alpha = \{1,2\}$. Since, $A(\alpha,\beta)$ is a 0 by 2 matrix, it allows centrality. Hence by Corollary 5.2, the constrained linear system $Ax = b,\ x \gneq 0$ is not sign-solvable.

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