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Z-PENCILS*

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Abstract. The matrix pencil $(A, B) = \{tB - A \mid t \in \mathbb{C}\}$ is considered under the assumptions that A is entrywise nonnegative and $B - A$ is a nonsingular M-matrix. As t varies in $[0, 1]$, the Z-matrices $tB - A$ are partitioned into the sets L_s introduced by Fiedler and Markham. As no combinatorial structure of B is assumed here, this partition generalizes some of their work where $B = I$. Based on the union of the directed graphs of A and B, the combinatorial structure of nonnegative eigenvectors associated with the largest eigenvalue of (A, B) in $[0, 1)$ is considered.

Key words. Z-matrix, matrix pencil, M-matrix, eigenspace, reduced graph.

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1. Introduction. The generalized eigenvalue problem $Ax = \lambda Bx$ for $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}^{n,n}$, with inequality conditions motivated by certain economics models, was studied by Bapat et al. [1]. In keeping with this work, we consider the matrix pencil $(A, B) = \{tB - A \mid t \in \mathbb{C}\}$ under the conditions

$$A \text{ is entrywise nonnegative, denoted by } A \geq 0 \tag{1}$$

$$b_{ij} \leq a_{ij} \text{ for all } i \neq j \tag{2}$$

$$\text{there exists a positive vector } u \text{ such that } (B - A)u \text{ is positive.} \tag{3}$$

Note that in [1] A is also assumed to be irreducible, but that is not imposed here. When $Ax = \lambda Bx$ for some nonzero x, the scalar λ is an *eigenvalue* and x is the corresponding *eigenvector* of (A, B). The *eigenspace* of (A, B) associated with an eigenvalue λ is the nullspace of $\lambda B - A$.

A matrix $X \in \mathbb{R}^{n,n}$ is a *Z-matrix* if $X = qI - P$, where $P \geq 0$ and $q \in \mathbb{R}$. If, in addition, $q \geq \rho(P)$, where $\rho(P)$ is the spectral radius of P, then X is an *M-matrix*, and is singular if and only if $q = \rho(P)$. It follows from (1) and (2) that when $t \in [0, 1]$,

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$tB - A$ is a Z-matrix. Henceforth the term *Z-pencil* (A, B) refers to the circumstance that $tB - A$ is a Z-matrix for all $t \in [0, 1]$.

Let $\langle n \rangle = \{1, 2, \ldots, n\}$. If $J \subseteq \langle n \rangle$, then X_J denotes the principal submatrix of X in rows and columns of J. As in [3], given a nonnegative $P \in \mathbb{R}^{n,n}$ and an $s \in \langle n \rangle$, define

$$\rho_s(P) = \max_{|J|=s}\{\rho(P_J)\}$$

and set $\rho_{n+1}(P) = \infty$. Let L_s denote the set of Z-matrices in $\mathbb{R}^{n,n}$ of the form $qI - P$, where $\rho_s(P) \le q < \rho_{s+1}(P)$ for $s \in \langle n \rangle$, and $-\infty < q < \rho_1(P)$ when $s = 0$. This gives a partition of all Z-matrices of order n. Note that $qI - P \in L_0$ if and only if $q < p_{ii}$ for some i. Also, $\rho_n(P) = \rho(P)$, and L_n is the set of all (singular and nonsingular) M-matrices.

We consider the Z-pencil (A, B) subject to conditions (1)–(3) and partition its matrices into the sets L_s. Viewed as a partition of the Z-matrices $tB - A$ for $t \in [0, 1]$, our result provides a generalization of some of the work in [3] (where $B = I$). Indeed, since no combinatorial structure of B is assumed, our Z-pencil partition is a consequence of a more complicated connection between the Perron-Frobenius theory for A and the spectra of $tB - A$ and its submatrices.

Conditions (2) and (3) imply that $B - A$ is a nonsingular M-matrix and thus its inverse is entrywise nonnegative; see [2, N_{38}, p. 137]. This, together with (1), gives $(B - A)^{-1}A \ge 0$. Perron-Frobenius theory is used in [1] to identify an eigenvalue $\rho(A, B)$ of the pencil (A, B), defined as

$$\rho(A, B) = \frac{\rho\left((B - A)^{-1}A\right)}{1 + \rho\left((B - A)^{-1}A\right)}.$$

Our partition involves $\rho(A, B)$ and the eigenvalues of the subpencils (A_J, B_J). Our Z-pencil partition result, Theorem 2.4, is followed by examples where as t varies in $[0, 1]$, $tB - A$ ranges through some or all of the sets L_s for $0 \le s \le n$. In Section 3 we turn to a consideration of the combinatorial structure of nonnegative eigenvectors associated with $\rho(A, B)$. This involves some digraph terminology, which we introduce at the beginning of that section.

In [3], [5] and [7], interesting results on the spectra of matrices in L_s, and a classification in terms of the inverse of a Z-matrix, are established. These results are of course applicable to the matrices of a Z-pencil; however, as they do not directly depend on the form $tB - A$ of the Z-matrix, we do not consider them here.

2. Partition of Z-pencils.

We begin with two observations and a lemma used to prove our result on the Z-pencil partition.

Observation 2.1. *Let (A, B) be a pencil with $B - A$ nonsingular. Given a real $\mu \ne -1$, let $\lambda = \frac{\mu}{1+\mu}$. Then the following hold.*
(i) *$\lambda \ne 1$ is an eigenvalue of (A, B) if and only if $\mu \ne -1$ is an eigenvalue of $(B - A)^{-1}A$.*
(ii) *λ is a strictly increasing function of $\mu \ne -1$.*
(iii) *$\lambda \in [0, 1)$ if and only if $\mu \ge 0$.*

Proof. If μ is an eigenvalue of $(B-A)^{-1}A$, then there exists nonzero $x \in \mathbb{R}^n$ such that $(B-A)^{-1}Ax = \mu x$. It follows that $Ax = \mu(B-A)x$ and if $\mu \neq -1$, then $Ax = \frac{\mu}{1+\mu}Bx = \lambda Bx$. Notice that λ cannot be 1 for any choice of μ. The reverse argument shows that the converse is also true. The last statement of (i) is obvious. Statements (ii) and (iii) follow easily from the definition of λ. □

Note that $\lambda = 1$ is an eigenvalue of (A, B) if and only if $B - A$ is singular.

OBSERVATION 2.2. *Let (A, B) be a pencil satisfying* (2), (3). *Then the following hold.*
(i) *For any nonempty $J \subseteq \langle n\rangle$, $B_J - A_J$ is a nonsingular M-matrix.*
(ii) *If in addition* (1) *holds, then the largest real eigenvalue of (A, B) in $[0, 1)$ is $\rho(A, B)$.*

Proof. (i) This follows since (2) and (3) imply that $B - A$ is a nonsingular M-matrix (see [2, I_{27}, p. 136]) and since every principal submatrix of a nonsingular M-matrix is also a nonsingular M-matrix; see [2, p. 138].

(ii) This follows from Observation 2.1, since $\mu = \rho((B-A)^{-1}A)$ is the maximal positive eigenvalue of $(B-A)^{-1}A$. □

LEMMA 2.3. *Let (A, B) be a pencil satisfying* (1)-(3). *Let $\mu = \rho\left((B-A)^{-1}A\right)$ and $\rho(A, B) = \frac{\mu}{1+\mu}$. Then the following hold.*
(i) *For all $t \in (\rho(A, B), 1]$, $tB - A$ is a nonsingular M-matrix.*
(ii) *The matrix $\rho(A, B)B - A$ is a singular M-matrix.*
(iii) *For all $t \in (0, \rho(A, B))$, $tB - A$ is not an M-matrix.*
(iv) *For $t = 0$, either $tB - A$ is a singular M-matrix or is not an M-matrix.*

Proof. Recall that (1) and (2) imply that $tB-A$ is a Z-matrix for all $0 < t \leq 1$. As noted in Observation 2.2 (i), $B-A$ is a nonsingular M-matrix and thus its eigenvalues have positive real parts [2, G_{20}, p. 135], and the eigenvalue with minimal real part is real [2, Exercise 5.4, p. 159]. Since the eigenvalues are continuous functions of the entries of a matrix, as t decreases from $t = 1$, $tB - A$ is a nonsingular M-matrix for all t until a value of t is encountered for which $tB - A$ is singular. Results (i) and (ii) now follow by Observation 2.2 (ii).

To prove (iii), consider $t \in (0, \rho(A, B))$. Since $(B-A)^{-1}A \geq 0$, there exists an eigenvector $x \geq 0$ such that $(B-A)^{-1}Ax = \mu x$. Then $Ax = \rho(A, B)Bx$ and $(tB - A)x = (t - \rho(A, B))\, Bx \leq 0$ since $Bx = \frac{1}{\rho(A,B)}Ax \geq 0$. By [2, A_5, p. 134], $tB - A$ is not a nonsingular M-matrix. To complete the proof (by contradiction), suppose $\alpha B - A$ is a singular M-matrix for some $\alpha \in (0, \rho(A, B))$. Since there are finitely many values of t for which $tB - A$ is singular, we can choose $\beta \in (\alpha, \rho(A, B))$ such that $\beta B - A$ is nonsingular. Let $\epsilon = \frac{\beta - \alpha}{\alpha}$. Then $(1 + \epsilon)(\alpha B - A)$ is a singular M-matrix and

$$(1+\epsilon)(\alpha B - A) + \gamma I = \beta B - A - \epsilon A + \gamma I \leq \beta B - A + \gamma I$$

since $A \geq 0$ by (1). By [2, C_9, p. 150], $\beta B - A - \epsilon A + \gamma I$ is a nonsingular M-matrix for all $\gamma > 0$, and hence $\beta B - A + \gamma I$ is a nonsingular M-matrix for all $\gamma > 0$ by [4, 2.5.4, p. 117]. This implies that $\beta B - A$ is also a (nonsingular) M-matrix ([2, C_9, p. 150]), contradicting the above. Thus we can also conclude that $\alpha B - A$ cannot be a singular M-matrix for any choice of $\alpha \in (0, \rho(A, B))$, establishing (iii). For (iv),

$-A$ is a singular M-matrix if and only if it is, up to a permutation similarity, strictly triangular. Otherwise, $-A$ is not an M-matrix. □

THEOREM 2.4. *Let* (A, B) *be a pencil satisfying* (1)-(3). *For* $s = 1, 2, \ldots, n$ *let*

$$\sigma_s = \max_{|J|=s}\{\rho\left((B_J - A_J)^{-1}A_J\right)\}, \quad \tau_s = \frac{\sigma_s}{1+\sigma_s},$$

and $\tau_0 = 0$. *Then for* $s = 0, 1, \ldots, n-1$ *and* $\tau_s \leq t < \tau_{s+1}$, *the matrix* $tB - A \in L_s$. *For* $s = n$ *and* $\tau_n \leq t \leq 1$, *the matrix* $tB - A \in L_n$.

Proof. Fiedler and Markham [3, Theorem 1.3] show that for $1 \leq s \leq n-1$, $X \in L_s$ if and only if all principal submatrices of X of order s are M-matrices, and there exists a principal submatrix of order $s+1$ that is not an M-matrix. Consider any nonempty $J \subseteq \langle n\rangle$ and $t \in [0, 1]$. Conditions (1) and (2) imply that $tB_J - A_J$ is a Z-matrix. By Observation 2.2 (i), $B_J - A_J$ is a nonsingular M-matrix. Let $\mu_J = \rho\left((B_J - A_J)^{-1}A_J\right)$. Then by Observation 2.2 (ii), $\tau_J = \frac{\mu_J}{1+\mu_J}$ is the largest eigenvalue in $[0, 1)$ of the pencil (A_J, B_J). Combining this with Observation 2.2 (i) and Lemma 2.3, the matrix $tB_J - A_J$ is an M-matrix for all $\tau_J \leq t \leq 1$, and $tB_J - A_J$ is not an M-matrix for all $0 < t < \tau_J$. If $1 \leq s \leq n-1$ and $|J| = s$, then $tB_J - A_J$ is an M-matrix for all $\tau_s \leq t \leq 1$. Suppose $\tau_s < \tau_{s+1}$. Then there exists $K \subseteq \langle n\rangle$ such that $|K| = s+1$ and $tB_K - A_K$ is not an M-matrix for $0 < t < \tau_{s+1}$. Thus by [3, Theorem 1.3] $tB - A \in L_s$ for all $\tau_s \leq t < \tau_{s+1}$. When $s = n$, since $B - A$ is a nonsingular M-matrix, $tB - A \in L_n$ for all t such that $\rho(A, B) = \tau_n \leq t \leq 1$ by Lemma 2.3 (i). For the case $s = 0$, if $0 < t < \tau_1$, then $tB - A$ has a negative diagonal entry and thus $tB - A \in L_0$. For $t = 0$, $tB - A = -A$. If $a_{ii} \neq 0$ for some $i \in \langle n\rangle$, then $-A \in L_0$; if $a_{ii} = 0$ for all $i \in \langle n\rangle$, then $\tau_1 = \tau_0 = 0$, namely, $-A \in L_s$ for some $s \geq 1$. □

We continue with illustrative examples.

EXAMPLE 2.5. Consider

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix},$$

for which $\tau_2 = 2/3$ and $\tau_1 = 1/2$. It follows that

$$tB - A \in \begin{cases} L_0 & \text{if } 0 \leq t < 1/2 \\ L_1 & \text{if } 1/2 \leq t < 2/3 \\ L_2 & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

That is, as t increases from 0 to 1, $tB - A$ belongs to all the possible Z-matrix classes L_s.

EXAMPLE 2.6. Consider the matrices in [1, Example 5.3], that is,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 3 & 0 & -1 \\ -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{bmatrix}.$$

Referring to Theorem 2.4, $\tau_4 = \rho(A,B) = \frac{4+\sqrt{6}}{10} = \tau_3 = \tau_2$ and $\tau_1 = 1/3$. It follows that

$$tB - A \in \begin{cases} L_0 & \text{if } 0 \le t < 1/3 \\ L_1 & \text{if } 1/3 \le t < \frac{4+\sqrt{6}}{10} \\ L_4 & \text{if } \frac{4+\sqrt{6}}{10} \le t \le 1. \end{cases}$$

Notice that for $t \in [0,1]$, $tB - A$ ranges through only L_0, L_1 and L_4.

EXAMPLE 2.7. Now let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In contrast to the above two examples, $tB - A \in L_2$ for all $t \in [0,1]$. Note that, in general, $tB - A \in L_n$ for all $t \in [0,1]$ if and only if $\rho(A,B) = 0$.

3. Combinatorial Structure of the Eigenspace Associated with $\rho(A,B)$. Let $, = (V,E)$ be a *digraph*, where V is a finite vertex set and $E \subseteq V \times V$ is the edge set. If $,' = (V,E')$, then $, \cup ,' = (V, E \cup E')$. Also write $,' \subseteq ,$ when $E' \subseteq E$. For $j \ne k$, a *path of length* $m \ge 1$ from j to k in , is a sequence of vertices $j = r_1, r_2, \ldots, r_{m+1} = k$ such that $(r_s, r_{s+1}) \in E$ for $s = 1, \ldots, m$. As in [2, Ch. 2], if $j = k$ or if there is a path from vertex j to vertex k in , , then j has *access to* k (or k is *accessed from* j). If j has access to k and k has access to j, then j and k *communicate*. The communication relation is an equivalence relation, hence V can be partitioned into equivalence classes, which are referred to as the *classes* of , .

The *digraph* of $X = [x_{ij}] \in \mathbb{R}^{n,n}$, denoted by $\mathcal{G}(X) = (V,E)$, consists of the vertex set $V = \langle n \rangle$ and the set of directed edges $E = \{(j,k) \mid x_{jk} \ne 0\}$. If j has access to k for all distinct $j,k \in V$, then X is *irreducible* (otherwise, *reducible*). It is well known that the rows and columns of X can be simultaneously reordered so that X is in block lower triangular *Frobenius normal form*, with each diagonal block irreducible. The irreducible blocks in the Frobenius normal form of X correspond to the classes of $\mathcal{G}(X)$.

In terminology similar to that of [6], given a digraph , , the *reduced graph* of , , $\mathcal{R}(,) = (V', E')$, is the digraph derived from , by taking

$$V' = \{J \mid J \text{ is a class of } ,\}$$

and

$$E' = \{(J,K) \mid \text{there exist } j \in J \text{ and } k \in K \text{ such that } j \text{ has access to } k \text{ in } ,\}.$$

When $, = \mathcal{G}(X)$ for some $X \in \mathbb{R}^{n,n}$, we denote $\mathcal{R}(,)$ by $\mathcal{R}(X)$.

Suppose now that $X = qI - P$ is a singular M-matrix, where $P \ge 0$ and $q = \rho(P)$. If an irreducible block X_J in the Frobenius normal form of X is singular, then $\rho(P_J) = q$ and we refer to the corresponding class J as a *singular class* (otherwise, a *nonsingular class*). A singular class J of $\mathcal{G}(X)$ is called *distinguished* if when J is accessed from a class $K \neq J$ in $\mathcal{R}(X)$, then $\rho(P_K) < \rho(P_J)$. That is, a singular class J of $\mathcal{G}(X)$ is distinguished if and only if J is accessed only from itself and nonsingular classes in $\mathcal{R}(X)$.

We paraphrase now Theorem 3.1 of [6] as follows.

THEOREM 3.1. *Let $X \in \mathbb{R}^{n,n}$ be an M-matrix and let $J_1, \ldots, J_p$ denote the distinguished singular classes of $\mathcal{G}(X)$. Then there exist unique (up to scalar multiples) nonnegative vectors $x^1, \ldots, x^p$ in the nullspace of X such that*

$$x_j^i \begin{cases} = 0 \text{ if } j \text{ does not have access to a vertex in } J_i \text{ in } \mathcal{G}(X) \\ > 0 \text{ if } j \text{ has access to a vertex in } J_i \text{ in } \mathcal{G}(X) \end{cases}$$

for all $i = 1, 2, \ldots p$ and $j = 1, 2, \ldots, n$. Moreover, every nonnegative vector in the nullspace of X is a linear combination with nonnegative coefficients of $x^1, \ldots, x^p$.

We apply the above theorem to a Z-pencil, using the following lemma.

LEMMA 3.2. *Let (A, B) be a pencil satisfying* (1) *and* (2)*. Then the classes of $\mathcal{G}(tB - A)$ coincide with the classes of $\mathcal{G}(A) \cup \mathcal{G}(B)$ for all $t \in (0, 1)$.*

Proof. Clearly $\mathcal{G}(tB - A) \subseteq \mathcal{G}(A) \cup \mathcal{G}(B)$ for all scalars t. For any $i \neq j$, if either $b_{ij} \neq 0$ or $a_{ij} \neq 0$, and if $t \in (0, 1)$, conditions (1) and (2) imply that $tb_{ij} < a_{ij}$ and hence $tb_{ij} - a_{ij} \neq 0$. This means that apart from vertex loops, the edge sets of $\mathcal{G}(tB - A)$ and $\mathcal{G}(A) \cup \mathcal{G}(B)$ coincide for all $t \in (0, 1)$. □

THEOREM 3.3. *Let (A, B) be a pencil satisfying* (1)–(3) *and let*

$$, = \begin{cases} \mathcal{G}(A) \cup \mathcal{G}(B) & \text{if } \rho(A, B) \neq 0 \\ \\ \mathcal{G}(A) & \text{if } \rho(A, B) = 0. \end{cases}$$

Let $J_1, \ldots, J_p$ denote the classes of , such that for each $i = 1, 2, \ldots, p$,
(i) *$(\rho(A, B)B - A)_{J_i}$ is singular, and*
(ii) *if J_i is accessed from a class $K \neq J_i$ in $\mathcal{R}(,)$, then $(\rho(A, B)B - A)_K$ is nonsingular.*
Then there exist unique (up to scalar multiples) nonnegative vectors $x^1, \ldots, x^p$ in the eigenspace associated with the eigenvalue $\rho(A, B)$ of (A, B) such that

$$x_j^i \begin{cases} = 0 \text{ if } j \text{ does not have access to a vertex in } J_i \text{ in } , \\ > 0 \text{ if } j \text{ has access to a vertex in } J_i \text{ in } , \end{cases}$$

for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n$. Moreover, every nonnegative vector in the eigenspace associated with the eigenvalue $\rho(A, B)$ is a linear combination with nonnegative coefficients of $x^1, \ldots, x^p$.

Proof. By Lemma 2.3 (ii), $\rho(A, B)B - A$ is a singular M-matrix. Thus

$$\rho(A, B)B - A = qI - P = X,$$

where $P \geq 0$ and $q = \rho(P)$. When $\rho(A, B) = 0$, the result follows from Theorem 3.1 applied to $X = -A$. When $\rho(A, B) > 0$, by Lemma 3.2, , $= \mathcal{G}(X)$. Class J of , is singular if and only if $\rho(P_J) = q$, which is equivalent to $(\rho(A, B)B - A)_J$

being singular. Also a singular class J is distinguished if and only if for all classes $K \neq J$ that access J in $\mathcal{R}(X)$, $\rho(P_K) < \rho(P_J)$, or equivalently $(\rho(A,B)B - A)_K$ is nonsingular. Applying Theorem 3.1 gives the result. □

We conclude with a generalization of Theorem 1.7 of [3] to Z-pencils. Note that the class J in the following result is a singular class of $\mathcal{G}(A) \cup \mathcal{G}(B)$.

THEOREM 3.4. *Let (A,B) be a pencil satisfying (1)–(3) and let $t \in (0, \rho(A,B))$. Suppose that J is a class of $\mathcal{G}(tB - A)$ such that $\rho(A,B) = \frac{\mu}{1+\mu}$, where $\mu = \rho((B_J - A_J)^{-1}A_J)$. Let $m = |J|$. Then $tB - A \in L_s$ with*

$$s \begin{cases} \leq n-1 & \text{if } m = n \\ < m & \text{if } m < n. \end{cases}$$

Proof. That $tB - A \in L_s$ for some $s \in \{0, 1, \ldots, n\}$ follows from Theorem 2.4. By Lemma 2.3 (iii), if $t \in (0, \rho(A,B))$, then $tB - A \notin L_n$ since $\rho(A,B) = \tau_n$. Thus $s \leq n-1$. When $m < n$, under the assumptions of the theorem, we have $\tau_n = \rho(A,B) = \frac{\mu}{1+\mu} \leq \tau_m$ and hence $\tau_m = \tau_{m+1} = \ldots = \tau_n$. It follows that $s < m$. □

We now apply the results of this section to Example 2.6, which has two classes. Class $J = \{2,4\}$ is the only class of $\mathcal{G}(A) \cup \mathcal{G}(B)$ such that $(\rho(A,B)B - A)_J$ is singular, and J is accessed by no other class. By Theorem 3.3, there exists an eigenvector x of (A,B) associated with $\rho(A,B)$ with $x_1 = x_3 = 0$, $x_2 > 0$ and $x_4 > 0$. Since $|J| = 2$, by Theorem 3.4, $tB - A \in L_0 \cup L_1$ for all $t \in (0, \rho(A,B))$, agreeing with the exact partition given in Example 2.6.

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