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MAXIMAL NESTS OF SUBSPACES, THE MATRIX BRUHAT DECOMPOSITION, AND THE MARRIAGE THEOREM – WITH AN APPLICATION TO GRAPH COLORING*

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Abstract. Using the celebrated Marriage Theorem of P. Hall, we give an elementary combinatorial proof of the theorem that asserts that given two maximal nests $\mathcal{N}_1$ and $\mathcal{N}_2$ in a finite dimensional vector space V, there is an ordered basis of V that generates $\mathcal{N}_1$ and a permutation of that ordered basis that generates $\mathcal{N}_2$. From this theorem one easily obtains the Matrix Bruhat Decomposition. A generalization to matroids is discussed, and an application to graph coloring is given.

Key words. Nests of subspaces, matrix Bruhat decomposition, marriage theorem, graph colorings, matroids.

AMS subject classifications. 15A03, 05C15, 05B35

1. Introduction. Let V be a vector space of finite dimension n over a field F. A family of subspaces of V is a *nest* provided it is totally ordered by set-inclusion. The nest $\mathcal{N} = (V_0, V_1, \ldots, V_n)$ of subspaces of V is a *maximal nest* provided that $\dim V_k = k$ for $k = 0, 1, \ldots, n$. Note that in a maximal nest, $V_0 = \{0\}$ and $V_n = V$. A maximal nest $\mathcal{N}$ can be constructed by choosing an ordered basis $v_1, v_2, \ldots, v_n$ of V and defining V_k to be the subspace of V spanned by $\{v_i : 1 \leq i \leq k\}$. We call $v_1, v_2, \ldots, v_n$ an *ordered basis of the maximal nest* $\mathcal{N}$ and write $\mathcal{N} = [v_1, v_2, \ldots, v_n]$. Every maximal nest is of the form $[v_1, v_2, \ldots, v_n]$ for an appropriate choice of ordered basis.

In [2] Fillmore et al. consider nests over the complex field and, using the nest algebra [1],[1] they prove that for any two maximal nests $\mathcal{N}_1$ and $\mathcal{N}_2$ there is an ordered basis $u_1, u_2, \ldots, u_n$ and a permutation π of $\{1, 2, \ldots, n\}$ such that $\mathcal{N}_1 = [u_1, u_2, \ldots, u_n]$ and $\mathcal{N}_2 = [u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)}]$. This result was obtained much earlier by Steinberg [5] without any restriction on the field.[2] In this note we prove this result about pairs of maximal nests by establishing a connection with the celebrated Marriage Theorem of P. Hall; see [4, pp. 47-51]. We also discuss a possible generalization to matroids and give an application to "doubly-multicolored spanning trees" of connected graphs.

2. Results. The following theorem, which gives necessary and sufficient conditions for two partitions of a set to have a common system of (distinct) representatives, is equivalent to the Marriage Theorem.

THEOREM 2.1. *Let n be a positive integer, and let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be two partitions of a set X. Then there is a permutation π of $\{1, 2, \ldots, n\}$ such that*

$$A_k \cap B_{\pi(k)} \neq \emptyset, \quad (1 \leq k \leq n) \tag{2.1}$$

if and only if for each set $K \subseteq \{1, 2, \ldots, n\}$, $\cup_{i \in K} A_i$ contains at most $|K|$ of the sets $B_1, B_2, \ldots, B_n$.

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[1] The algebra of linear operators that leave every subspace of the nest invariant.

[2] In a private communication, W.E. Longstaff has remarked that the proof given in [2] can be modified to apply to arbitrary fields.

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From Theorem 2.1 we can deduce that given any two maximal nests of V, there is a basis of V that generates each of them [2, 5].

THEOREM 2.2. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two maximal nests of the n-dimensional vector space V. There exists a basis $u_1, u_2, \ldots, u_n$ of V and a unique permutation π of $\{1, 2, \ldots, n\}$, depending on this basis, such that*

$$\mathcal{N}_1 = [u_1, u_2, \ldots, u_n] \quad \text{and} \quad \mathcal{N}_2 = [u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)}]. \tag{2.2}$$

Proof. Let $\mathcal{N}_1 = (V_0, V_1, \ldots, V_n) = [v_1, v_2, \ldots, v_n]$, and $\mathcal{N}_2 = (W_0, W_1, \ldots, W_n) = [w_1, w_2, \ldots, w_n]$. We also let

$$A_k = V_k \setminus V_{k-1} \quad \text{and} \quad B_k = W_k \setminus W_{k-1}, \quad (1 \le k \le n),$$

and

$$A(K) = \bigcup_{i \in K} A_i \quad \text{and} \quad B(K) = \bigcup_{i \in K} B_i, \quad (K \subseteq \{1, 2, \ldots, n\}).$$

We first prove the assertion:

$A(K) \cup \{0\}$ *contains a subspace of dimension $|K|$, namely the subspace U_K spanned by the vectors v_i $(i \in K)$, but no subspace of dimension larger than $|K|$.*

We prove this assertion by induction on $k = |K|$. First suppose that $k = 1$ and $K = \{j\}$. Then every scalar multiple of v_j is in $A_j \cup \{0\} = (V_j \setminus V_{j-1}) \cup \{0\}$ and hence A_j contains the 1-dimensional subspace spanned by v_j. Suppose that $A_j \cup \{0\}$ contains a 2-dimensional subspace U. Since $\dim V_j = j$ and $\dim V_{j-1} = j-1$, $U \cap V_{j-1}$ is a 1-dimensional subspace contradicting $U \subseteq (V_j \setminus V_{j-1}) \cup \{0\}$.

Now suppose that $k > 1$. Let m be the largest integer in K, and let $K' = K \setminus \{m\}$. By induction $A(K') \cup \{0\}$ contains the $(k-1)$-dimensional subspace spanned by the vectors v_i $(i \in K')$. The set A_m contains all vectors of the form $cv_m + u$ where c is a nonzero scalar and u is a vector in V_{m-1}. Since $A(K')$ is contained in V_{m-1}, A_m contains all vectors of the form $cv_m + u$ where c is a nonzero scalar and u is in $A(K')$. Hence $A(K) \cup \{0\}$ contains the k-dimensional subspace spanned by v_i $(i \in K)$.

Suppose that $A(K) \cup \{0\}$ contains a subspace W of dimension $|K| + 1$. Then $W \subseteq V_m$, and since V_{m-1} has codimension 1 in V_m, we have that $W' = W \cap V_{m-1}$ has dimension at least $|K|$. Then $W' \cap A_m = \emptyset$, and thus $W' \subseteq A(K') \cup \{0\}$, contradicting the induction hypothesis.

We now apply Theorem 2.1. Suppose there exists a $K \subseteq \{1, 2, \ldots, n\}$ such that $A(K)$ contains $|K| + 1$ of the sets $B_1, B_2, \ldots, B_n$, say, B_i $(i \in J)$ where $|J| = |K| + 1$. By the assertion applied to $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, $B(J) \cup \{0\}$ contains a $(|K|+1)$-dimensional subspace and $A(K) \cup \{0\}$ does not, and we have a contradiction. By Theorem 2.1 there is a permutation π of $\{1, 2, \ldots, n\}$ and vectors $u_1, u_2, \ldots, u_n$ such that $u_i \in A_i$ and $u_i \in B_{\pi(i)}$, $(1 \le i \le n)$. The vectors $\{u_1, u_2, \ldots, u_n\}$ are a basis of V, and the uniqueness of the permutation π is obvious. The theorem now follows. □

From Theorem 2.2 we can deduce the Matrix Bruhat Decomposition; see, e.g., [6].

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THEOREM 2.3. *Let A be a nonsingular matrix of order n over a field F. Then their exist nonsingular lower triangular matrices L_1 and L_2 of order n and a unique permutation matrix P of order n such that*

$$A = L_2 P L_1.$$

Proof. Consider the two nests $\mathcal{N}_1 = [v_1, v_2, \ldots, v_n]$ and $\mathcal{N}_2 = [w_1, w_2, \ldots, w_n]$ where $v_1, v_2, \ldots, v_n$ are the rows of A and $w_1, w_2, \ldots, w_n$ are the rows of A^2. It follows from Theorem 2.2 that there exists a basis $u_1, u_2, \ldots, u_n$ of F^n, a permutation σ of $\{1, 2, \ldots, n\}$ with corresponding permutation matrix P, and nonsingular lower triangular matrices L_1 and L_2 such that

$$L_1 A = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad L_2^{-1} A^2 = \begin{bmatrix} u_{\sigma(1)} \\ u_{\sigma(2)} \\ \vdots \\ u_{\sigma(n)} \end{bmatrix} = P \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Hence $L_2^{-1} A^2 = P L_1 A$, and since A is nonsingular, $A = L_2 P L_1$. The uniqueness of P follows from the uniqueness of σ as given in Theorem 2.2. □

We can generalize the notion of a nest of subspaces of a vector space to a nest of flats of a matroid. Let $M = (X, \mathcal{I})$ be a matroid [3, 7] on the finite set X, where $\mathcal{I}$ is the collection of its independent sets. Let the rank of M be n. A *maximal nest of the matroid* M is a family $\mathcal{F} = (F_0, F_1, \ldots, F_n)$ where F_k is a flat of M of rank k, $(k = 0, 1, \ldots, n)$. Choosing, for each $k = 1, 2, \ldots, n$, an element x_k in $F_k \setminus F_{k-1}$ we obtain an ordered basis $x_1, x_2, \ldots, x_n$ of M such that $x_1, x_2, \ldots, x_k$ is a basis of F_k. We write $\mathcal{F} = [x_1, x_2, \ldots, x_n]$ and call $x_1, x_2, \ldots, x_n$ an *ordered basis of the maximal nest* $\mathcal{F}$. Note that F_0 is the closure in M of the empty set.

Let $\mathcal{G} = (G_0, G_1, \ldots, G_n)$ be another maximal nest of M, and define $A_k = F_k \setminus F_{k-1}$ and $B_k = G_k \setminus G_{k-1}$, $(k = 1, 2, \ldots, n)$. Using Theorem 2.1 we can assert that there exists a basis $u_1, u_2, \ldots, u_n$ and a permutation π of $\{1, 2, \ldots, n\}$ such that

$$\mathcal{F} = [u_1, u_2, \ldots, u_n] \quad \text{and} \quad \mathcal{G} = [u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)}]$$

if and only if $\cup_{i \in J} A_i$ contains at most $|J|$ of the sets $B_1, B_2, \ldots, B_n$ for each $J \subseteq \{1, 2, \ldots, n\}$. Unlike for vector spaces, this last condition need not hold for arbitrary matroids. For example, in a matroid of rank n on a set X of $n+1$ elements every proper subset of which is independent (that is, X is a circuit), this condition does not hold.

Let K_{n+1} be the complete graph with $n+1$ vertices $1, 2, \ldots, n+1$ and edge set $E = \{ij : 1 \le i < j \le n+1\}$, and let M_{n+1} be the cycle matroid of K_{n+1} on its set of edges.[3] A flat F of M_{n+1} is obtained by choosing a subset U of vertices and a partition of U into sets $U_1, U_2, \ldots, U_s$; the flat F consists of the union of

[3]A subset of edges is *independent* in M_{n+1} if and only if it does not contain a cycle; the rank of M_{n+1} is n.

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the edges of the complete graphs induced on the U_i and has rank equal to $|U| - s$. By taking $s = 1$ and $|U| = t + 1$, we obtain a *special flat* of M_{n+1} of rank t, the set of edges of the complete graph induced on a subset of $t + 1$ vertices. A *special maximal nest* of $M(K_{n+1})$ corresponds to a maximal chain $X_1 \subset X_2 \subset \cdots \subset X_{n+1}$ of subsets of the vertex set $\{1, 2, \ldots, n+1\}$ with $|X_k| = k$ for $k = 1, 2, \ldots, n+1$. Let $X_k = \{i_1, i_2, \ldots, i_k\}$, $(k = 1, 2, \ldots, n+1)$ The set A_k above consists of all the edges joining vertex i_{k+1} to vertices $i_1, i_2, \ldots, i_k$.[4] The sets B_k have a similar description corresponding to a different permutation of $1, 2, \ldots, n+1$. It is easy to check that in this setting, the set $A(J)$, respectively $B(J)$, contains a complete graph on $|J|$ vertices (namely the vertices j with $j \in J$) but does not contain a complete graph of $|J| + 1$ vertices. It follows that $A(J)$ can contain at most $|J|$ of the sets B_i. We thus have the following conclusion.

COROLLARY 2.4. *Let K_{n+1} be the complete graph with vertices $1, 2, \ldots, n+1$. Let $i_1, i_2, \ldots, i_{n+1}$ be a permutation of $1, 2, \ldots, n+1$. Suppose we color the edges joining vertex $k+1$ to vertices $\{1, 2, \ldots, k\}$ with color k, and independently color the edges joining vertex i_{k+1} to vertices $\{i_1, i_2, \ldots, i_k\}$ with color k', $(k = 1, 2, \ldots, n)$. Then K_{n+1} has a spanning tree T such that no two edges of T have the same color in the first coloring and no two edges of T have the same color in the second coloring.*

The corollary asserts the existence of a spanning tree of a complete graph which is multicolored (no two edges of the same color) in both the colorings, a *doubly-multicolored spanning tree*. The corollary does not hold in the context of arbitrary maximal nests of $M(K_{n+1})$. For example, when $n = 4$, $\emptyset \subseteq \{12\} \subseteq \{12, 34\} \subseteq E$ and $\emptyset \subseteq \{13\} \subseteq \{13, 24\} \subseteq E$ are two maximal nests for which there does not exist a doubly multicolored spanning tree.

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[4] Thus A_k is the set of edges of a star and $A_0 = \emptyset$.