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ITERATIONS OF CONCAVE MAPS, THE PERRON–FROBENIUS THEORY, AND APPLICATIONS TO CIRCLE PACKINGS*

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Abstract. The theory of pseudo circle packings is developed. It generalizes the theory of circle packings. It allows the realization of almost any graph embedding by a geometric structure of circles. The corresponding Thurston's relaxation mapping is defined and is used to prove the existence and the rigidity of the pseudo circle packing. It is shown that iterates of this mapping, starting from an arbitrary point, converge to its unique positive fixed point. The coordinates of this fixed point give the radii of the packing. A key property of the relaxation mapping is its superadditivity. The proof of that is reduced to show that a certain real polynomial in four variables and of degree 20 is always nonnegative. This in turn is proved by using recently developed algorithms from real algebraic geometry. Another important ingredient in the development of the theory is the use of nonnegative matrices and the corresponding Perron–Frobenius theory.

Key words. Circle packings, Discrete conformal geometry, Nonnegative matrices, Perron–Frobenius, Monotone and positive mappings, Min-max principles, Fixed-point theorems, Rigidity of circle packings, Real algebraic geometry, Nonnegative polynomials, Algorithms for sums of squares of polynomials.

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1. Introduction.

A circle packing in the plane is a finite collection of circles whose interiors are disjoint. Thus, for each pair of circles in a circle packing, either the distance between the circles is positive or it is zero, in which case the circles are mutually tangent from the outside. The *nerve* or the *1-skeleton* of a circle packing is the embedded graph whose vertices correspond to the circles of the packing and in which two vertices are connected by an edge iff the corresponding circles are tangent to one another. In the other direction, if G is a graph embedded in the plane, then *a circle-packing realization of* G is a circle packing whose nerve is combinatorially isomorphic to G. All of these notions can be extended to surfaces other than the plane. It is natural to ask which graph embeddings have circle-packing realizations. A beautiful result due to Andreev asserts the following.

Theorem 1.1. *(Andreev.) Any finite triangulation of the 2-sphere has a circle-packing realization. Moreover, any two such realizations can be mapped onto one another by a Möbius transformation.*

This theorem contains two parts: (1) the existence of circle-packing realizations for finite S^2 triangulations and (2) the uniqueness of the packing up to the action of $\mathrm{Aut}(S^2)$. The later is commonly referred to as *rigidity*. This formulation of the theorem of Andreev is due to Thurston [13]; see also [9]. This result was extended by Thurston to any compact and closed surface of a finite genus. Other proofs of Andreev's theorem were given by O. Schramm [14]. A new interest in circle packings

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originated in Thurston's lecture in the conference to celebrate de Branges's solution of the Bieberbach conjecture [2]. There Thurston outlined an algorithmic approach to approximate the Riemann mapping of a simply connected domain onto the open unit disc. He left open a few problems concerning the approximation he suggested to the first-order derivative of the Riemann mapping. These were solved by B. Rodin and D. Sullivan [12], Z.-X. He, O. Schramm, and others.

This paper outlines a new theory of a much wider family of problems. We call these *the realization of a given graph embedding by pseudo circle packings.*

In section 2 we define the notion of *a graph embedding* that fits our geometric context.

In section 3 the notion of *the* $\overline{a}$*-mapping,* $f_{\overline{a}}$*, of a graph embedding* is defined. It is observed that $f_{\overline{a}}:\mathbb{R}^{+|V|}\to\mathbb{R}^{+|V|}$ is an isotone mapping. Also, two normalizations $F_{\overline{a}}$ and $G_{\overline{a}}$ of $f_{\overline{a}}$ are presented. These are tools for the study of $f_{\overline{a}}$. They have the same dynamics as that of $f_{\overline{a}}$ but behave much better under iterations. A main result to be proved in this paper is that the normalizations have a unique fixed point in $\mathbb{R}^{+|V|}$ and that the iterates of the normalizations converge to it independently of the starting point. This is Theorem 3.3, which implies that the dynamics of the iterations of $F_{\overline{a}}$ and of $G_{\overline{a}}$ resembles the dynamics of a contraction $\mathbb{R}^{+|V|}\to\mathbb{R}^{+|V|}$. This statement will be made accurate in section 12. An immediate conclusion of this theorem is Theorem 3.4, which asserts that $f_{\overline{a}}$ has a unique positive *eigenvalue* $\lambda^{(1)}$ and a corresponding projectively unique *eigenpoint* $\overline{r}\in\mathbb{R}^{+|V|}$. These notions are defined via the equality

$$f_{\overline{a}}(\overline{r})=\lambda^{(1)}\cdot\overline{r}.$$

This eigenpoint provides the radii of a pseudo circle-packing realization for the graph embedding G with an angles-parameter vector $\overline{a}$. This is the connection of the $\overline{a}$-mapping with circle packings. In fact this last theorem is a general Andreev-type theorem, applied to the broader context of pseudo circle packings. It has the two features of the existence (of the eigenpoint) and the rigidity (the uniqueness of $\lambda^{(1)}$ and of $\overline{r}$).

In section 4 we explain Andreev's original theorem using our new terminology. In section 5 we prove the rigidity of pseudo circle packings. This is the content of Theorem 5.1. The proof uses the maximum principle for pseudo circle packings [7, 1]. Later on we will prove a much more general rigidity principle (Theorem 12.5). This, however, will invoke the Perron–Frobenius theory for nonnegative matrices. The latter approach gives hope to proving rigidity results for infinite pseudo circle packings as well. Here one may use the infinite theory of Perron–Frobenius. We hope to accomplish this in a subsequent paper.

A key property of the $\overline{a}$-mapping $f_{\overline{a}}$ is its *superadditivity*:

$$\forall\,\overline{r},\overline{s}\in\mathbb{R}^{+|V|},\;\; f_{\overline{a}}(\overline{r})+f_{\overline{a}}(\overline{s})\le f_{\overline{a}}(\overline{r}+\overline{s}).$$

This is part (b) of Theorem 6.1 in section 6. This property lies on a certain geometric inequality, which is presented in Lemma 6.5. Here is one interpretation of this inequality: Suppose that we have three triangles, one with sides of length X, Y, and Z,

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a second with sides of length U, V, and W, and a third with sides of length $(X+U)$, $(Y+V)$, and $(Z+W)$. Let us denote by α the angle in the first triangle between the sides of length X and Y. Let β be the corresponding angle in the second triangle between U and V and let γ be the corresponding angle between $(X+U)$ and $(Y+V)$ in the third triangle. Then

$$\alpha \cdot (X+Y-Z) + \beta \cdot (U+V-W) \leq \gamma \cdot ((X+U)+(Y+V)-(Z+W)).$$

It turns out that proving this inequality is not at all simple. Later on, we dedicate the whole of section 10 to a discussion of concave mappings and geometric inequalities. We introduce there two interesting reductions of our inequality. The original transcendental inequality of Lemma 6.5 is reduced to the nonnegativity of a certain real polynomial in 4 variables and of degree 20. This falls right into methods of real algebraic geometry (Lemma 10.6) [11]. Using a recently developed algorithm, the polynomial is represented as a sum of five squares of other polynomials.

In section 7 we prove elementary properties of the sets

$$A_{i,\theta} = \big\{(r_0,\ldots,r_{|V|-1}) \in \mathbb{R}^{+|V|} \,|\, R_i \geq \theta r_i\ , \\ \text{where } f_{\overline{a}}(r_0,\ldots,r_{|V|-1}) = (R_0,\ldots,R_{|V|-1})\big\}$$

for $0 \leq i < |V|$ and a fixed $\theta > 0$. These sets measure the monotonicity of $f_{\overline{a}}$ in a single coordinate. We show that $A_{i,\theta}$ is a cone (Proposition 7.4), is connected (Proposition 7.5), and has an affine algebraic boundary $\partial A_{i,\theta}$ in $\mathbb{R}^{+|V|}$ (Proposition 7.9).

In section 8 we use Sperner's lemma in order to prove that there exists an $\overline{r} \in \mathbb{R}^{+|V|}$ so that either $f_{\overline{a}}(\overline{r}) \geq \overline{r}$ or $f_{\overline{a}}(\overline{r}) \leq \overline{r}$.

In section 9 we use the Brouwer fixed-point theorem to prove the existence of an eigenvalue and an eigenpoint of $f_{\overline{a}}$ (Theorem 9.2). The key feature in the proof is the superadditivity of $f_{\overline{a}}$, which implies that the sets $A_{i,\theta}$ are in fact convex (Theorem 9.1). This almost completes the proofs of Theorems 3.3 and 3.4. The part that is yet to be proved is that the normalizations $F_{\overline{a}}$ and $G_{\overline{a}}$ behave like contractions.

For that we apply in section 11 the theory of Perron–Frobenius for (finite) nonnegative matrices. This theory enables us to develop a new machinery that proves the general rigidity principle for pseudo circle packings. It also produces a wealth of identities and inequalities for various quantities in our theory. Let us denote by $\lambda^{(n)} = \lambda^{(n)}(G,\overline{a})$ the unique eigenvalue of the mapping $f_{\overline{a}}^{(\circ n)} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$, the nth iterate of $f_{\overline{a}}$. Let us denote by $\lambda^{(n)}(\overline{s}) = \lambda^{(n)}(G,\overline{a})(\overline{s})$ the largest eigenvalue of the symmetric, nonnegative, and irreducible matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$. Then here is a partial list of these identities and inequalities:

(1) $\forall n \in \mathbb{Z}^+$, $\forall \overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$, $f_{\overline{a}}^{(\circ n)}(\overline{r})^T \leq (f_{\overline{a}}^{(\circ n)})'(\overline{s}) \cdot \overline{r}^T$ and $f_{\overline{a}}^{(\circ n)}(\overline{r})^T = (f_{\overline{a}}^{\circ n)})'(\overline{r}) \cdot \overline{r}^T$. (These follow by Theorem 12.2 and Theorem 12.1, respectively.)

(2) If $\overline{r} \in \mathbb{R}^{+|V|}$ is the eigenpoint of $f_{\overline{a}}^{(\circ n)}$, i.e., $f_{\overline{a}}^{(\circ n)}(\overline{r})^T = \lambda^{(n)}\overline{r}^T$, then $\lambda^{(n)} = \rho((f_{\overline{a}}^{(\circ n)})'(\overline{r})) = \lambda^{(n)}(\overline{r})$. (This follows by the proof of Theorem 12.5.)

(3) $\lambda^{(n)}(\overline{s}) \geq \max_{\overline{r}\in\mathbb{R}^{+|V|}, \overline{r}\cdot\overline{r}^T=1} \overline{r} \cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T$ and $\lambda^{n}(\overline{s}) = \max_{\overline{r}\in\mathbb{R}^{+|V|}, \overline{r}\cdot\overline{r}^T=1} \overline{r} \cdot (f_{\overline{a}}^{(\circ n)})'(\overline{s}) \cdot \overline{r}^T$. (These follow by Theorem 12.4 and by its proof, respectively.)

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(4) $\lambda^{(n)} = \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(n)}(\overline{s})$. (This follows by Theorem 12.5.)

(5) $\lambda^{(n)} = \max_{\overline{r}\in\mathbb{R}^{+|V|}, \overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T$. (This follows by Theorem 13.1.)

(6) $\lambda^{(n)} = \max\{\alpha\in\mathbb{R}^+ \,|\, \exists\, \overline{v}\in\mathbb{R}^{+|V|} \text{ such that } f_{\overline{a}}^{(\circ n)}(\overline{v}) \geq \alpha\overline{v}\}$. (This follows by Theorem 13.2.)

The topic in section 12 is the converse of the contraction principle and the Courant–Hilbert min-max theorem applied to rigidity. We prove the strong rigidity theorem (Theorem 12.5). This in turn implies that the iterates of the normalizations converge to their fixed points independently of the starting point (Theorem 12.9). We then use a theorem of C. Bessaga on the converse of the contraction principle, which implies that the normalizations are in fact contractions after an appropriate change of the topology on $\mathbb{R}^{+|V|}$. This concludes the proofs of Theorems 3.3 and 3.4.

Sections 14 and 15 are devoted to the λ-packing property of geometric configurations $(G,\overline{a})$. We give a geometric necessary and sufficient condition on $(G,\overline{a})$ so that $f_{\overline{a}}$ will have λ as its eigenvalue. This is the λ-packing property. The condition is

$$\dim\left(\cap_{i=0}^{|V|-1} A_{i,\lambda}\right) = 1.$$

In section 16 we compute the range of the values of the radii and eigenvalues of any given pseudo circle packing. This is a number theoretical problem (Theorem 16.2, Corollary 16.3).

The following basic lower bound is computed in Proposition 17.2 in section 17:

$$\lambda^{(1)}(G,\overline{a}) \geq \frac{1}{|V|}\left\{\sum_{a_i-\text{closed}} \frac{1}{|\sin(a_i/(2d_i))|} + \sum_{a_i-\text{open}} \frac{1}{|\sin(a_i/(2(d_i-1)))|}\right\} - 1.$$

This estimate is the key in proving Theorem 18.1, *the packing theorem,* in section 18: Given G and $\lambda > 0$, there exists an angles-parameter vector $\overline{a}$ such that the configuration $(G,\overline{a})$ has the λ-packing property. Once more, this is a general Andreev-type theorem.

Finally, in section 19 we compute the expected value of the random variable $\lambda^{(1)}(G,\overline{a})$. This expectation turns out to be $+\infty$. So we find the tight asymptotics of this variable. This is done in Theorem 19.4. We prove the following estimates:

$$\frac{1}{|V|}\sum_{i=0}^{|V|-1} \frac{1}{|\sin(a_i/(2l_i))|} - 1 \leq \lambda^{(1)}(G,\overline{a}) \leq \max_{0\leq i<|V|}\left\{\frac{1}{|\sin(a_i/(2l_i))|} - 1\right\}$$

and, as an immediate conclusion, we get

$$\lambda^{(1)}(G,\overline{a}) = \Omega\left(\frac{1}{\min_{0\leq i<|V|}|a_i|}\right),$$

where we use the Ω notation as in complexity theory in computer science.

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2. The embedding of a graph. We describe here the exact information we need on the embedding of the graph G. V denotes the set of vertices of G. We number the vertices of G by $0 \leq i < |V|$. For each such vertex i we let d_i denote the valence of that vertex in G, i.e., the number of its neighboring vertices. The input given to us has the following structure:

It contains $|V|$ rows. Row number i, $0 \leq i < |V|$, has one of the following two forms. Either it is the finite sequence $j_{i,1}, \ldots, j_{i,d_i}$ or the sequence $j_{i,1}, \ldots, j_{i,d_i}, j_{i,1}$. Here $0 \leq j_{i,k} < |V|$ $(1 \leq k \leq d_i)$ are the d_i neighbors of vertex i ordered (geometrically) counterclockwise.

Thus, for example, an empty line in such a structure stands for an isolated vertex and a line containing a single vertex stands for a leaf.

Remark 2.1. The family of graphs we are thinking of is the family of simple graphs such that the valence d at each and every vertex satisfies $d \geq 3$. So they are connected graphs without any leaves.

3. The isotone accompanying mappings of an embedding of a graph. Let $\overline{a} = (a_0, \ldots, a_{|V|-1}) \in \mathbb{R}^{|V|}$. We call this vector the *angles-parameter vector.* We define the following $\overline{a}$*-mapping* $f_{\overline{a}}$ *of the embedding of the graph* G as follows:

$$\begin{cases} f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|} \\ f_{\overline{a}}(r_0, \ldots, r_{|V|-1}) = (R_0, \ldots, R_{|V|-1}), \end{cases}$$

where each R_i is a positive real function of $(r_0, \ldots, r_{|V|-1})$ defined by the combinatorial structure of the given embedding of G. Given i $(0 \leq i < |V|)$, we compute R_i implicitly as follows:

$$(r_{j_{i,k}} + r_{j_{i,k+1}})^2 = (R_i + r_{j_{i,k}})^2 + (R_i + r_{j_{i,k+1}})^2 - 2(R_i + r_{j_{i,k}})(R_i + r_{j_{i,k+1}}) \cos \alpha_k$$

for $k = 1, \ldots, l_i - 1$ and

$$\sum_{k=1}^{l_i - 1} \alpha_k = a_i,$$

where l_i is the length of row i (which is either d_i or $d_i + 1$). The geometrical interpretation of this formula is easy. We use the cosine law to express the fact that, if certain values of radii $r_{j_{i,k}}$ are given at the d_i neighboring vertices of i, then R_i is the radius at the vertex i itself that creates a total angle of a_i radians around the vertex i. If row number i has the form $j_{i,1}, \ldots, j_{i,d_i}$, then the angle is *open.* If it has the form $j_{i,1}, \ldots, j_{i,d_i}, j_{i,1}$, then it is *closed* and usually in that case $a_i = 2\pi$. Since the angles α_k, $k = 1, \ldots, l_i - 1$, between successive tangent circles are less than π, in order for that to make sense we need the total angle a_i to be small enough (less than $(d_i - 1)\pi$ or $d_i \pi$, depending on the form of row number i). This is provided that we have a locally flat structure in mind. We call such angles-parameter vectors $\overline{a}$ *admissible.* Intuitively it is clear that there exists exactly one such value R_i and moreover it is clear that $\partial R_i / \partial r_{j_{i,k}} > 0$ for every $0 \leq i < |V|$ and every $1 \leq k \leq l_i$. Thus indeed we have the following proposition.

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PROPOSITION 3.1. *The mapping $f_{\overline{a}}$ is well defined for any admissible value of the vector $\overline{a}$ and it is an isotone mapping $\mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$.*

REMARK 3.2. The definition of $f_{\overline{a}}$ is based on a relation between the total angle at a vertex of G and the corresponding coordinate of $\overline{a}$. The relation is given by $\sum_{k=1}^{l_i-1} \alpha_k = a_i$. The total angle $\sum_{k=1}^{l_i-1} \alpha_k$ can be open or closed. This is a generalization of the notion of *the curvature $K(v)$ at an interior vertex v*; see [9], [7]. The notion of a curvature at a vertex was used for triangulations. It was defined to be the sum of all the angles at v of the 2-simplexes that contain v, minus 2π. Thus, with our notation, $a_i = 2\pi$.

For any vector $\overline{x} = (x_0, \ldots, x_{|V|-1}) \in \mathbb{R}^{+|V|}$ we denote $|\overline{x}| = \sum_{i=0}^{|V|-1} x_i$. Also we let $\pi : \mathbb{R}^{+|V|} \to \mathbb{R}^+$ be the projection mapping onto the first coordinate,

$$\pi(x_0, \ldots, x_{|V|-1}) = x_0.$$

We denote by $F_{\overline{a}} = f_{\overline{a}}/|f_{\overline{a}}|$ the *normalization of $f_{\overline{a}}$ of the first kind.* We also denote by $G_{\overline{a}} = f_{\overline{a}}/(\pi \circ f_{\overline{a}})$ the *normalization of $f_{\overline{a}}$ of the second kind.* We will prove the following theorem.

THEOREM 3.3. *Suppose that we are given an embedding of a graph G with a vertex set V. Suppose that $\overline{a} \in \mathbb{R}^{|V|}$ is an angles-parameter vector. Then*

(i) *$F_{\overline{a}}$ has a unique fixed point $\overline{r}$ in $\mathbb{R}^{+|V|}$. Moreover, if $\overline{x}$ is any point in $\mathbb{R}^{+|V|}$, then*

$$\lim_{n\to\infty} F_{\overline{a}}^{(n)}(\overline{x}) = \overline{r}.$$

(ii) *$G_{\overline{a}}$ has a unique fixed point $\overline{s}$ in $\mathbb{R}^{+|V|}$. Moreover, if $\overline{x}$ is any point in $\mathbb{R}^{+|V|}$, then*

$$\lim_{n\to\infty} G_{\overline{a}}^{(n)}(\overline{x}) = \overline{s}.$$

Here the notation $F_{\overline{a}}^{(n)}$ means the nth iterate of the mapping $F_{\overline{a}}$. A consequence of this theorem is the following.

THEOREM 3.4. *Suppose that we are given an embedding of a graph G with a vertex set V. Then, for any angles-parameter vector $\overline{a} \in \mathbb{R}^{|V|}$, the corresponding $\overline{a}$-mapping $f_{\overline{a}}$ has a unique positive eigenvalue $\lambda = \lambda(\overline{a})$. The fixed points of the normalized mappings $F_{\overline{a}}$ and $G_{\overline{a}}$ are the positive eigenpoints that correspond to this eigenvalue.*

4. Triangulations of the 2-sphere S^2. These will serve to explain the relationship between circle packings that realize graphs and the fixed points of certain $\overline{a}$-mappings $f_{\overline{a}}$ of the embeddings of the graphs for certain special values of $\overline{a}$.

We recall that any (finite) triangulation of S^2 can be thought of as a planar graph. This graph can be embedded in the plane in such a way that three of its vertices, say we number them 0, 1, and 2, form a triangle, and all the other vertices $3, 4, 5, \ldots$ lie in the interior of this triangle. A circle-packing realization of this particular embedding of the triangulation will consist of three congruent circles that are tangent to one

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another from the outside, plus the other circles located in the circular triangle they form. If we connect the centers of tangent circles in this triangulation we will get a graph that is isomorphic to the original triangulation of the sphere minus "the outside triangle." The graph is embedded in the plane in such a manner that its vertices 0, 1, and 2 form an equilateral triangle. All the other vertices are located inside this triangle. Thus, the total angles made at the vertices 0, 1, and 2 are $a_0 = a_1 = a_2 = \pi/3$. These are open angles. The total angles at all the other (inner) vertices are $a_i = 2\pi$, $i = 3, 4, 5, \ldots$. These angles are closed. We have the following theorem.

THEOREM 4.1. *Let T be a planar embedding of an S^2 triangulation. Let us assume that the vertices 0, 1, and 2 of the embedding form a triangle that contains in it all the other vertices of T. Let $\overline{a} = (\pi/3, \pi/3, \pi/3, 2\pi, \ldots, 2\pi) \in \mathbb{R}^{|V|}$. Then any fixed point $\overline{r} = (r_0, r_1, \ldots, r_{|V|-1})$ of $f_{\overline{a}}$ has coordinates that are the radii of a circle-packing realization of T.*

Proof. Given any inner vertex, the circles at its neighbors do not overlap and do not intersect the circle at the vertex. This is because of the choice of $\overline{a}$. Now an inductive argument on the number of "generations" about that circle shows that it is impossible for a circle of a later generation to intersect the original circle. This is because this will imply the existence of a circle that has an intersecting neighbor. ◻

A direct consequence of this simple observation is that the Andreev theorem is equivalent to the following statement: *If $\overline{a}$ and $f_{\overline{a}}$ are as in the theorem above, then $f_{\overline{a}}$ has a fixed point.* Our uniqueness result of the eigenvalue of this $f_{\overline{a}}$ is just the well-known *rigidity result* in the statement of the Andreev theorem. This motivates our study of fixed points of $\overline{a}$-mappings.

5. Rigidity of pseudo circle packings. We first turn our attention to the uniqueness of the eigenvalue of any $\overline{a}$-mapping $f_{\overline{a}}$. We refer to this, geometrically, as *the rigidity of pseudo circle packings.* The reason will become clear after we define this notion. However, in the previous section we managed to identify this uniqueness with the rigidity result of the Andreev theorem for the particular case of triangulations of S^2. We will give an argument to prove that there exists at most one eigenvalue. Also we will demonstrate that the set of all the eigenpoints that correspond to the eigenvalue (if it exists) is one dimensional. By that we mean that the eigenpoints of any such pair are proportional to one another. The proof of the last property makes use of the maximum principle. We will use it here in a similar way to that used in [7] (section 2).

THEOREM 5.1. *Suppose that we are given an embedding of a graph G with a vertex set V. Suppose that $\overline{a} \in \mathbb{R}^{|V|}$ is an angles-parameter vector. Then $f_{\overline{a}}$ has at most one eigenvalue. Moreover, if the eigenvalue exists, then any pair of corresponding eigenpoints are proportional to one another.*

Proof. Let λ and μ be eigenvalues of $f_{\overline{a}}$. By the definition, there exist two points (corresponding eigenpoints) $\overline{r}$ and $\overline{s}$ in $\mathbb{R}^{+|V|}$ such that $f_{\overline{a}}(\overline{r}) = \lambda\overline{r}$ and $f_{\overline{a}}(\overline{s}) = \mu\overline{s}$. Let us denote $\overline{r} = (r_0, \ldots, r_{|V|-1})$ and $\overline{s} = (s_0, \ldots, s_{|V|-1})$. Let us assume that the ratio r_i/s_i, $0 \leq i < |V|$, attains its maximum for $i = 0$. The defining equations of $f_{\overline{a}}$ are homogeneous. Hence we can scale $\overline{s}$ and assume that $\lambda r_0 = \mu s_0$. As agreed

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before, let us denote the neighboring vertices of vertex 0 by $j_{0,1}, \ldots, j_{0,l_0}$. Here they are listed as in row number 0 in the embedding of G. Also $l_0 = d_0$ if the angle about 0 is open and $l_0 = d_0 + 1$ if this angle is closed. If $\lambda \neq \mu$, then we can assume that $\mu < \lambda$. We will now proceed to show that, for $k = 1, \ldots, l_0$, we have the inequalities $r_{j_{0,k}} < s_{j_{0,k}}$. This will give us the desired contradiction and prove that $\mu = \lambda$. Let C_k be the center of the circle with the radius $r_{j_{0,k}}$ (the one that corresponds to the vertex $j_{0,k}$ with respect to $\overline{r}$). Here $k = 1, \ldots, l_0$. Also let us denote by C_0 the center of the circle at vertex 0. This circle has a radius equal to λr_0. Let α_k be the angle formed by the centers $C_k C_0 C_{k+1}$ as in the definition of $f_{\overline{a}}$. Then by the definition $\sum_{k=1}^{l_0-1} \alpha_k = a_0$, a constant. In fact this constant (the total angle about vertex 0) equals the first coordinate of $\overline{a}$. Clearly $|\alpha_k|$ is a nondecreasing function of the radius at $j_{0,k}$ (and that at $j_{0,k+1}$). Hence the sum $|\alpha_{k-1}| + |\alpha_k|$ is strictly increasing in the radius at $j_{0,k}$. So necessarily also $\sum_{k=1}^{l_0-1} |\alpha_k|$ is strictly increasing in the radius at $j_{0,k}$. By our choice of the vertex 0 we have $r_{j_{0,k}}/s_{j_{0,k}} \leq \mu/\lambda < 1$, $k = 1, \ldots, l_0$. Hence we must have $|\alpha_k{}'| > |\alpha_k|$, $k = 1, \ldots, l_0$, where the angle $\alpha_k{}'$ is the parallel angle to α_k but with respect to $\overline{s}$. By the definition of $f_{\overline{a}}$ also $\sum_{k=1}^{l_0-1} \alpha_k{}' = a_0$. This is a contradiction and it proves that $\mu = \lambda$.

We now prove the last part of the theorem. Let λ be the eigenvalue of $f_{\overline{a}}$. Let $\overline{r} = (r_0, \ldots, r_{|V|-1})$ and $\overline{s} = (s_0, \ldots, s_{|V|-1})$ be two eigenpoints that correspond to λ. Thus $f_{\overline{a}}(\overline{r}) = \lambda\overline{r}$ and $f_{\overline{a}}(\overline{s}) = \lambda\overline{s}$. Again let us assume that the ratio r_i/s_i, $0 \leq i < |V|$, attains its maximum for $i = 0$. We rescale $\overline{s}$ and assume that $r_0 = s_0$. We will now proceed to show that, for $k = 1, \ldots, l_0$, we have the equalities $r_{j_{0,k}} = s_{j_{0,k}}$. If we do that it will imply that the ratio function r_i/s_i, $0 \leq i < |V|$, attains its maximum value also on each one of the neighboring vertices of 0, i.e., $k = 1, \ldots, l_0$. Hence this ratio function is in fact a constant function (in fact it equals 1 because of our scaling). Hence $\overline{r} = \overline{s}$, which is exactly what we need to show. We use the same notation C_k as in the first part of the proof, except that here, obviously, C_0 is the center of the unit circle at vertex 0. We have as before

$\sum_{k=1}^{l_0-1} \alpha_k = \sum_{k=1}^{l_0-1} \alpha_k{}' = a_0$. But our choice of the vertex 0 implies this time that $r_{j_{0,k}}/s_{j_{0,k}} \leq 1$, $k = 1, \ldots, l_0$. So the monotonicity of the angles with respect to the radii implies that $|\alpha_k{}'| \geq |\alpha_k|$, $k = 1, \ldots, l_0$, and so, necessarily, $\alpha_k{}' = \alpha_k$ for $k = 1, \ldots, l_0$. This in turn proves that $r_{j_{0,k}} = s_{j_{0,k}}$ for each such k. □

REMARK 5.2. The above proof shows that if $\overline{r}$ and $\overline{s}$ are two eigenpoints of the $\overline{a}$-mapping $f_{\overline{a}}$, then the sequence of ratios r_i/s_i, $0 \leq i < |V|$, is a constant sequence. We can think of a different—more general—definition of the $\overline{a}$-mapping $f_{\overline{a}}$. In this definition we will take $\overline{a} \in \mathbb{R}^D$, where $1 \leq D \leq |V|$. The geometric meaning will be that we preassign values to the total angles only at *a subset of V, the set of all the vertices of G*. These preassigned angles can also be closed or open. The first part of the proof applied to the ratio function only at the vertices that correspond to $\overline{a}$ shows that the corresponding mapping $f_{\overline{a}}$ has at most a single eigenvalue. The second part shows that the maximum principle holds in that case as well. Namely, we define *the boundary vertices of G* to be those vertices to which we did not assign a total angle by the vector $\overline{a}$. Then, if $\overline{r}$ and $\overline{s}$ are two eigenpoints of $f_{\overline{a}}$, the sequence of ratios r_i/s_i, $0 \leq i < |V|$, cannot attain its maximum at a nonboundary vertex unless it is a

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constant function. This is another form of rigidity, more general than the above. We will not make use of it in this paper.

6. Superadditivity of $f_{\overline{a}}$. Our computing experiments indicate clearly that $f_{\overline{a}}$ enjoys the property of being superadditive. This is part (b) of the following.

THEOREM 6.1. *Suppose that we are given an embedding of a graph G with a vertex set V. Suppose that $\overline{a} \in \mathbb{R}^{|V|}$ is an angles-parameter vector. Then*

(i) *for any $\overline{r} \in \mathbb{R}^{+|V|}$ and for any $t > 0$ we have*

$$f_{\overline{a}}(t\overline{r}) = t f_{\overline{a}}(\overline{r}) \quad \textit{and}$$

(ii) *for any $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ we have*

$$f_{\overline{a}}(\overline{r}) + f_{\overline{a}}(\overline{s}) \leq f_{\overline{a}}(\overline{r} + \overline{s}).$$

REMARK 6.2. Part (i) follows immediately from the definition of $f_{\overline{a}}$. This is so because the defining equations of $f_{\overline{a}}$ are homogeneous. Hence only part (ii) needs to be proved.

REMARK 6.3. The definition of $f_{\overline{a}}$ implies that it will suffice to prove the following. Suppose that the equations below hold:

$$\sum_{k=1}^{l-1} \cos^{-1}\left(1 - \frac{2r_k r_{k+1}}{(R + r_k)(R + r_{k+1})}\right) = \alpha,$$

$$\sum_{k=1}^{l-1} \cos^{-1}\left(1 - \frac{2s_k s_{k+1}}{(S + s_k)(S + s_{k+1})}\right) = \alpha,$$

$$\sum_{k=1}^{l-1} \cos^{-1}\left(1 - \frac{2(r_k + s_k)(r_{k+1} + s_{k+1})}{(T + r_k + s_k)(T + r_{k+1} + s_{k+1})}\right) = \alpha,$$

where $l \geq 2$, $\alpha > 0$, $r_k, s_k, R, S, T > 0$, and each $\cos^{-1}\beta$ lies in $(0, \pi)$. Then $R + S \leq T$.

REMARK 6.4. If we use the following identity:

$$\frac{r_k r_{k+1}}{(R + r_k)(R + r_{k+1})} = \frac{1}{2}(1 - \cos\alpha_k) = \sin^2\left(\frac{\alpha_k}{2}\right),$$

for some $0 < \alpha_k < \pi$, and similar identities for the other two equations, then we see that the above three equations are equivalent to the following three equations:

$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{r_k r_{k+1}}{(R + r_k)(R + r_{k+1})}}\right\} = \frac{\alpha}{2},$$

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$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{s_k s_{k+1}}{(S+s_k)(S+s_{k+1})}}\right\} = \frac{\alpha}{2},$$

$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{(r_k+s_k)(r_{k+1}+s_{k+1})}{(T+r_k+s_k)(T+r_{k+1}+s_{k+1})}}\right\} = \frac{\alpha}{2}.$$

Our branch of the function $\sin^{-1} x$ is an increasing function for $0 \le x \le 1$ and so, given the first two equations, we need to prove that

$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{(r_k+s_k)(r_{k+1}+s_{k+1})}{(R+S+r_k+s_k)(R+S+r_{k+1}+s_{k+1})}}\right\} \ge \frac{\alpha}{2}$$

because this is equivalent to proving that $R + S \le T$.

We now proceed to give a complete proof of the theorem. It will be based on the following lemma.

Lemma 6.5. *If $a, b, c, d, R, S > 0$, then*

$$R\sin^{-1}\left\{\sqrt{\frac{ab}{(R+a)(R+b)}}\right\} + S\sin^{-1}\left\{\sqrt{\frac{cd}{(S+c)(S+d)}}\right\}$$
$$\le (R+S)\sin^{-1}\left\{\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right\}.$$

Remark 6.6. The lemma has two simple geometric interpretations:

(1) Three circles of radii R, a, and b that are mutually tangent to one another from the outside form a Euclidean triangle. The vertices of the triangle are the centers of the circles. The sides of the triangle have the following lengths: $R + a$, $a + b$, and $R + b$. Similarly, three circles of radii S, c, and d form a triangle of sides $S + c$, $c + d$, and $S+d$. Finally, a third such triangle is formed by three circles of radii $R+S+a+c$, $a + c + b + d$, and $R + S + b + d$. We note that the sides of the third triangle have lengths that are the sums of the respective sides of the first two triangles. On the other hand, the three sets of triples of circles also form three circular triangles. The vertices of these triangles are the tangency points of pairs of circles in each triple. The lemma implies that the circular sides of the third (largest) triangle are greater than or equal to the sums of the respective circular sides of the first two circular triangles.

(2) Let us consider three Euclidean triangles, one with sides of length X, Y, and Z, and an angle α between X and Y; a second with sides of length U, V, and W, and an angle β between U and V; and a third with sides of length $(X + U)$, $(Y + V)$, and $(Z + W)$, and an angle γ between $(X + U)$ and $(Y + V)$. Then

$$\alpha \cdot (X + Y - Z) + \beta \cdot (U + V - W) \le \gamma \cdot ((X + U) + (Y + V) - (Z + W)).$$

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Proof (of theorem on superadditivity of $f_{\overline{a}}$). Let us assume that

$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{r_k r_{k+1}}{(R+r_k)(R+r_{k+1})}}\right\} = \frac{\alpha}{2}$$

and

$$\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{s_k s_{k+1}}{(S+s_k)(S+s_{k+1})}}\right\} = \frac{\alpha}{2}.$$

Then, using the lemma, we obtain the following:

$$\begin{aligned}
\frac{\alpha}{2} &= \left(\frac{R}{R+S}\right)\frac{\alpha}{2} + \left(\frac{S}{R+S}\right)\frac{\alpha}{2} \\
&= \left(\frac{R}{R+S}\right)\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{r_k r_{k+1}}{(R+r_k)(R+r_{k+1})}}\right\} \\
&\quad + \left(\frac{S}{R+S}\right)\sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{s_k s_{k+1}}{(S+s_k)(S+s_{k+1})}}\right\} \\
&= \sum_{k=1}^{l-1}\left\{\left(\frac{R}{R+S}\right)\sin^{-1}\left\{\sqrt{\frac{r_k r_{k+1}}{(R+r_k)(R+r_{k+1})}}\right\}\right. \\
&\quad \left. + \left(\frac{S}{R+S}\right)\sin^{-1}\left\{\sqrt{\frac{s_k s_{k+1}}{(S+s_k)(S+s_{k+1})}}\right\}\right\} \\
&\le \sum_{k=1}^{l-1} \sin^{-1}\left\{\sqrt{\frac{(r_k+s_k)(r_{k+1}+s_{k+1})}{(R+S+r_k+r_{k+1})(R+S+r_{k+1}+s_{k+1})}}\right\}.
\end{aligned}$$

As noted before, this is equivalent to

$$f_{\overline{a}}(\overline{r}) + f_{\overline{a}}(\overline{s}) \le f_{\overline{a}}(\overline{r}+\overline{s}). \qquad \square$$

We will return in section 10 to talk about the lemma above. We will also introduce some other interesting inequalities and reduction techniques. We now proceed to investigate the existence of eigenvalues of the $\overline{a}$-mapping $f_{\overline{a}}$. An important application of the superadditivity of $f_{\overline{a}}$ will be to prove the convexity of the sets $A_{i,1}$ on which $f_{\overline{a}}$ is increasing in its ith coordinate. These sets will be defined in the next section. Using this convexity result, we will be able to use the Brouwer fixed-point theorem in order to prove the existence of an eigenvalue of $f_{\overline{a}}$.

7. Monotonicity in a single coordinate, the sets $A_{i,\theta}$. We assume that we are given an embedding of a graph G with a set of vertices V. We also fix $\overline{a} \in \mathbb{R}^{|V|}$ and consider the $\overline{a}$-mapping $f_{\overline{a}}$.

DEFINITION 7.1.

$$\begin{aligned}
A_{i,\theta} = \big\{&(r_0,\dots,r_{|V|-1}) \in \mathbb{R}^{+|V|} \mid R_i \ge \theta r_i, \\
&\text{where } f_{\overline{a}}(r_0,\dots,r_{|V|-1}) = (R_0,\dots,R_{|V|-1})\big\}
\end{aligned}$$

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for $0 \leq i < |V|$ and a fixed $\theta > 0$.

In this section we will prove a few elementary and useful properties of the sets $A_{i,\theta}$. First we note the following.

REMARK 7.2. The boundary $\partial A_{i,\theta}$ contains the intersection of the ith hyperplane of $\mathbb{R}^{|V|}$, $\{X_i = 0\}$, with $\partial\mathbb{R}^{+|V|}$. The reason for this is the following: If we consider a point $(r_0, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_{|V|-1})$ on this hyperplane, where

$$r_0, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{|V|-1}$$

are all fixed-positive numbers, then by the definition above we have

$$(r_0, \ldots, r_{i-1}, \epsilon, r_{i+1}, \ldots, r_{|V|-1}) \in A_{i,\theta}$$

for $\epsilon > 0$ small enough.

DEFINITION 7.3. Let $\overline{r} \in \mathbb{R}^{+|V|}$ and $0 \leq i < |V|$. $\overline{r}_i$ will denote the vector in $\mathbb{R}^{|V|}$ that has identical coordinates to those of $\overline{r}$ at the d_i locations of the neighbors of i and also at i itself. It has zeros at the other coordinates.

We will denote by $\mathbb{R}_i$ the set of all the vectors in $\mathbb{R}^{|V|}$ that have zeros at the d_i locations of the neighbors of i and also at i itself and have positive coordinates elsewhere.

PROPOSITION 7.4. *If* $\overline{r} \in A_{i,\theta}$, $0 \leq i < |V|$, $\theta > 0$, *and* $\lambda > 0$, *then* $\lambda\overline{r}_i + \mathbb{R}_i \subseteq A_{i,\theta}$.

Proof. By the definitions it follows that

$$\partial A_{i,\theta} = \left\{ (r_0, \ldots, r_{|V|-1}) \in \mathbb{R}^{+|V|} \;\middle|\; \sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(\theta r_i + r_{j_{i,k}})(\theta r_i + r_{j_{i,k+1}})}\right) = a_i \right\}.$$

Here $\partial A_{i,\theta}$ is the boundary of $A_{i,\theta}$ relative to $\mathbb{R}^{+|V|}$. This shows that the defining equation for $\partial A_{i,\theta}$ is homogeneous in r_i and in its d_i neighbors, $r_{j_{i,1}}, \ldots, r_{j_{i,d_i}}$, but is independent of the other $|V| - d_i - 1$ coordinates. ∎

PROPOSITION 7.5. $A_{i,\theta}$ *is a connected set,* $0 \leq i < |V|$, $\theta > 0$.

Proof. It is enough to show that $\partial A_{i,\theta}$ is a connected set. This is equivalent to showing that $\partial A_{i,\theta}$ is an arcwise connected set. To see this let $\overline{r}, \overline{s} \in \partial A_{i,\theta}$. Then

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(\theta r_i + r_{j_{i,k}})(\theta r_i + r_{j_{i,k+1}})}\right) = a_i$$

and

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2s_{j_{i,k}} s_{j_{i,k+1}}}{(\theta s_i + s_{j_{i,k}})(\theta s_i + s_{j_{i,k+1}})}\right) = a_i.$$

For any admissible $(t_1, \ldots, t_{l_i}) \in \mathbb{R}^{+l_i}$ there exists a unique $t > 0$ such that

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2t_k t_{k+1}}{(\theta t + t_k)(\theta t + t_{k+1})}\right) = a_i.$$

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This follows by Proposition 3.1. Moreover, t depends continuously on $(t_1, \ldots, t_{l_i})$. Thus we can deform $(r_{j_{i,1}}, \ldots, r_{j_{i,d_i}})$ to $(s_{j_{i,1}}, r_{j_{i,2}}, \ldots, r_{j_{i,d_i}})$ by $(\alpha r_{j_{i,1}}, \ldots, r_{j_{i,d_i}})$, where α lies between 1 and $s_{j_{i,1}}/r_{j_{i,1}}$. For each α there corresponds (continuously) an $r_i(\alpha)$. We do that for each of the d_i coordinates and this defines a path from $\overline{r}$ to $\overline{s}$ that lies in $\partial A_{i,\theta}$. □

PROPOSITION 7.6. *If $i \neq k$, then $int(A_{i,\theta}) \cap int(A_{k,\theta}) \neq \emptyset$ and $int(A_{i,\theta}) \cap A^c_{k,\theta} \neq \emptyset$.*

Proof. If $\{j_{i,1}, \ldots, j_{i,d_i}\} \cap \{j_{k,1}, \ldots, j_{k,d_k}\} = \emptyset$ the claim is clear. If not we separate into two cases.

Case 1: $k \notin \{j_{i,1}, \ldots, j_{i,d_i}\}$. We can take $r_{j_{i,l}} = 1$ for $1 \leq l \leq d_i$ and $r_{j_{k,l}} = 1$ for $1 \leq l \leq d_k$. We also take r_i and r_k small enough. If

$$f_{\overline{a}}(\ldots, r_i, \ldots, r_k, \ldots) = (\ldots, R_i, \ldots, R_k, \ldots),$$

then we have $\theta r_i < R_i$ and $\theta r_k < R_k$.

Case 2: $k \in \{j_{i,1}, \ldots, j_{i,d_i}\}$. In this case i is a neighbor of k and vice versa. Since G is a simple graph we either have $d_k > 3$ or $d_l > 3$. Let us assume that $d_k > 3$. We take $r_i = 1$ and $r_k = 1 + \epsilon$, where $\epsilon > 0$. We take for all the other neighbors of i and k huge radii. This completes the second case and proves that $int(A_{i,\theta}) \cap int(A_{k,\theta}) \neq \emptyset$ whenever $i \neq k$.

A similar proof works for $int(A_{i,\theta}) \cap A^c_{k,\theta} \neq \emptyset$. □

Next we make a simple observation regarding the extension to the boundary of $f_{\overline{a}}$.

PROPOSITION 7.7. *The mapping $f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$ can continuously be extended to $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$.*

Proof. This is a straightforward verification through the defining equations

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(r_i + r_{j_{i,k}})(r_i + r_{j_{i,k+1}})}\right) = a_i, \ \ 0 \leq i < |V|. \qquad \square$$

REMARK 7.8. With the aid of the last proposition we can view (by extension) the sets $A_{i,\theta}$ as subsets of $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$. We end this section with one more proposition that extends the contents of Propositions 7.4 and 7.5.

PROPOSITION 7.9. *$\partial A_{i,\theta}$ is the connected intersection of $\mathbb{R}^{+|V|}$ and of a $(|V|-1)$-dimensional* affine algebraic variety.

Proof. We will point out how to give the algebraic equation of the affine variety. For that we consider the defining equation of $\partial A_{i,\theta}$:

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(\theta r_i + r_{j_{i,k}})(\theta r_i + r_{j_{i,k+1}})}\right) = a_i.$$

We observe that this is, in fact, algebraic in its variables $r_i, r_{j_{i,1}}, \ldots, r_{j_{i,d_i}}$. To see this we just recall from elementary trigonometry that, if

$$\cos^{-1} X + \cos^{-1} Y = \cos^{-1} Z,$$

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then (up to a sign)

$$Z = XY - \sqrt{1-X^2}\sqrt{1-Y^2}.$$

Applying this $l_i - 2$ times to the defining equation gives us

$$\text{ALGEBRAIC FORM IN}(r_i, r_{j_{i,1}}, \ldots, r_{j_{i,d_i}}) = \cos a_i. \qquad \square$$

8. Eigenpoints of $f_{\overline{a}}$.

Remark 8.1. The definitions imply that the set of fixed points of $f_{\overline{a}}$ in $\mathbb{R}^{+|V|}$ is exactly the following:

$$\mathbb{R}^{+|V|} \cap \left(\cap_{i=0}^{|V|-1} \partial A_{i,1} \right).$$

In other words this is the set of all the eigenpoints of $f_{\overline{a}}$ that correspond to the eigenvalue $\lambda = 1$.

In general the set of all the eigenpoints of $f_{\overline{a}}$ that correspond to an eigenvalue $\lambda > 0$ is

$$\mathbb{R}^{+|V|} \cap \left(\cap_{i=0}^{|V|-1} \partial A_{i,\lambda} \right).$$

If $\lambda > 1$ then it is a subset of the set

$$\mathbb{R}^{+|V|} \cap \left(\cap_{i=0}^{|V|-1} int(A_{i,1}) \right).$$

If $\lambda < 1$ then it is a subset of the set

$$\mathbb{R}^{+|V|} \cap \left(\cap_{i=0}^{|V|-1} A_{i,1}^c \right).$$

Definition 8.2. An $\overline{a}$ *pseudo circle packing* consists of three objects:
(a) an embedding of a graph G with the vertex sequence V;
(b) a vector $\overline{a} = (a_0, \ldots, a_{|V|-1}) \in \mathbb{R}^{|V|}$;
(c) a sequence of $|V|$ circles with the radii $\overline{r} = (r_0, \ldots, r_{|V|-1}) \in \mathbb{R}^{+|V|}$.

There is a bijection between the vertices in V and the sequence of circles. Two circles are called *neighbors* if the corresponding vertices are joined by an edge. The radii have such values that $\overline{r}$ is an eigenpoint of the $\overline{a}$-mapping $f_{\overline{a}}$. Sometimes we will say that this is *an $\overline{a}$ pseudo circle-packing realization of the embedding of the graph.*

Theorem 8.3. *$f_{\overline{a}}$ has $|V|$ defining equations. Any $\overline{a}$ pseudo circle-packing realization of the embedding of the graph is a positive simultaneous solution of an algebraic system of $|V|$ equations in $|V| + 1$ unknowns. The unknowns are the $|V|$ coordinates of the eigenpoint of $f_{\overline{a}}$ and the corresponding eigenvalue $\lambda(\overline{a})$. The equations are homogeneous in the first $|V|$ unknowns but not in $\lambda(\overline{a})$. The algebraic system depends only on the combinatorics of the embedding of the graph.*

Proof. Recall that our graphs are simple and that the valence at every vertex is at least 3. So the number of the defining equations for $f_{\overline{a}}$ is exactly $|V|$ by its definition.

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By the definition of an $\overline{a}$ pseudo circle-packing realization of the embedding of the graph, it follows that the equations that characterize the pseudo circle packing are given by

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(\lambda r_i + r_{j_{i,k}})(\lambda r_i + r_{j_{i,k+1}})}\right) = a_i, \;\; 0 \le i < |V|.$$

These are homogeneous in the r_i, $0 \le i < |V|$, but not in λ. Finally, that the system is an algebraic system follows as in the proof of Proposition 7.9. □

Remark 8.4. It is well known [6] (chapter I, section 7, pages 47–55) that over $\mathbb{C}$ any homogeneous system of n equations in $m > n$ unknowns has a nontrivial solution. This kind of result does not suffice in our case even for $\lambda = 1$, for we need positive solutions. On the top of this, in our case $m = n$ and hence the general theory on intersections of projective spaces over $\mathbb{C}$ does not even apply. Thus, in order to prove Theorem 3.3, we will have to take advantage of the specific structure of the systems that correspond to pseudo circle packings. We will outline an iterative process that converges to the desired fixed point. A priori it seems that this process might lead to zero or infinite radii. However, we will show that this does not happen.

The following proposition is a variant of the *Kantorovich lemma* [10] and the homogeneous nature of $f_{\overline{a}}$.

Proposition 8.5. *If there exist two points $\overline{s}_1, \overline{s}_2 \in \mathbb{R}^{+|V|}$ such that $f_{\overline{a}}(\overline{s}_1) \le \overline{s}_1$ and $f_{\overline{a}}(\overline{s}_2) \ge \overline{s}_2$, then there are points $\overline{r} \in \mathbb{R}^{+|V|}$ such that $f_{\overline{a}}(\overline{r}) = \overline{r}$.*

Proof. Let $t > 0$ be such that $\overline{s}_2 \le t\overline{s}_1$. Since $f_{\overline{a}}$ is isotone (by Proposition 3.1), it follows that $f_{\overline{a}}(\overline{s}_2) \le f_{\overline{a}}(t\overline{s}_1)$. By the hypothesis, this implies that

$$\overline{s}_2 \le f_{\overline{a}}(\overline{s}_2) \le f_{\overline{a}}(t\overline{s}_1) \le t\overline{s}_1.$$

Hence the sequence of iterates $f_{\overline{a}}^{(m)}(\overline{s}_2)$ is monotone increasing and bounded from above and so $\overline{r} = \lim_{m\to\infty} f_{\overline{a}}^{(m)}(\overline{s}_2)$ is the desired fixed point. □

We will not be able to prove that two points as in Proposition 8.5 exist because this is not true. In fact, our computer experience shows that if we iterate $f_{\overline{a}}$ starting from an arbitrary point, then usually the sequence of iterates will have entries that grow to infinity or that approach zero (unless we start at a fixed point of $f_{\overline{a}}$). So our strategy will be to show (using Sperner's lemma) that there always exists a point of one of the two types. Thus, the iterates of $f_{\overline{a}}$ starting at this point will be monotone and hence will converge, maybe to infinity or to zero. Then we will pass to a normalized $\overline{a}$-mapping, i.e., $F_{\overline{a}}$ or $G_{\overline{a}}$, and use its iterates starting from that point in order to show that $f_{\overline{a}}$ has an eigenpoint. This stage of the proof, using the normalized $\overline{a}$-mapping, will use Brouwer's fixed-point theorem and the fact that the $\partial A_{i,1}$ are parts of an affine variety.

Proposition 8.6. *Consider the $\overline{a}$-mapping $f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$. Then there exist points $\overline{r} \in \mathbb{R}^{+|V|}$ such that either $f_{\overline{a}}(\overline{r}) \ge \overline{r}$ or $f_{\overline{a}}(\overline{r}) \le \overline{r}$.*

Proof. Let us consider $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$ and the extended (to $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$) sets $A_{i,1}$.

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The defining equation for $\partial A_{i,1}$ is

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_{i,k}} r_{j_{i,k+1}}}{(r_i + r_{j_{i,k}})(r_i + r_{j_{i,k+1}})}\right) = a_i,$$

where the valence $d_i \geq 3$. It follows that $A_{i,1}$ (the extended set) contains the full intersection of the $(|V|-1)$-dimensional hyperplane $H_i = \{X_i = 0\}$ with $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$. This is so because, when we fix positive values of the coordinates of all the radii $(r_0, \ldots, r_{|V|-1})$ except for r_i and take $r_i > 0$ very small, then, by the defining equation, we get a point that lies in $A_{i,1} \cap \mathbb{R}^{+|V|}$ (see Remark 7.2). Now there are two cases to consider.

Case 1:

$$\cup_{i=0}^{|V|-1} A_{i,1} \neq \overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}.$$

Then there must be a point $\overline{r} \in \mathbb{R}^{+|V|}$ (not just in $\overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}$) such that $\overline{r} \in \cap_{i=0}^{|V|-1} A_{i,1}^c$. This means that $f_{\overline{a}}(\overline{r}) \leq \overline{r}$.

Case 2:

$$\cup_{i=0}^{|V|-1} A_{i,1} = \overline{\mathbb{R}^{+|V|}} - \{\overline{0}\}.$$

Let us consider a $(|V|-1)$-dimensional hyperplane H that intersects each of the $|V|$ positive axes of coordinates. The set $S = H \cap \overline{\mathbb{R}^{+|V|}}$ is a $(|V|-1)$-dimensional simplex. The $|V|$ sets $A_{i,1} \cap S$ for $0 \leq i < |V|$ form a closed covering of S. By Sperner's lemma we obtain

$$S \cap \left(\cap_{i=0}^{|V|-1} A_{i,1}\right) \neq \emptyset. \tag{8.1}$$

We note that the set in (8.1) can't have an intersection with $\partial \mathbb{R}^{+|V|}$. Hence also

$$S \cap \left(\cap_{i=0}^{|V|-1} A_{i,1}\right) \subseteq \mathbb{R}^{+|V|}. \tag{8.2}$$

By (8.1) and (8.2) there exist points $\overline{r} \in \mathbb{R}^{+|V|}$ such that $f_{\overline{a}}(\overline{r}) \geq \overline{r}$. ∎

Remark 8.7. The last proposition implies that there are always points $\overline{r} \in \mathbb{R}^{+|V|}$ for which $\lim_{m\to\infty} f_{\overline{a}}^{(m)}(\overline{r})$ exists in the broad sense. However, the limit might "blow" to infinity or "shrink" to zero. Indeed it is this phenomenon that we experience while executing computer experiments. Thus the dynamics of the $\overline{a}$-mapping $f_{\overline{a}}$ almost captures solutions of our algebraic system. But it is not good enough.

This leads to the idea of *normalization*. We want a mapping with dynamics similar to that of $f_{\overline{a}}$ but for which the iterates are confined to always stay bounded away from zero and from ∞. Two such normalizations work and we now proceed to describe them both. For the reader's convenience we repeat formally the definitions that were given before the statement of Theorem 3.3 in section 3.

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DEFINITION 8.8. Let $\overline{x} = (x_0, \ldots, x_{|V|-1}) \in \mathbb{R}^{+|V|}$. We denote the l_1-norm of $\overline{x}$ by $|\overline{x}| = \sum_{k=0}^{|V|-1} x_i$. Let $\overline{a} \in \mathbb{R}^{|V|}$. We denote the normalization of the first kind of $f_{\overline{a}}$ by $F_{\overline{a}} = f_{\overline{a}}/|f_{\overline{a}}|$. We will also use the notation

$$C = \left\{ \overline{x} \in \mathbb{R}^{+|V|} \mid |\overline{x}| = 1 \right\}.$$

Then $F_{\overline{a}} : \mathbb{R}^{+|V|} \to C$.

REMARK 8.9. It follows by the homogeneous property of $f_{\overline{a}}$ that we have $F_{\overline{a}}(C) = F_{\overline{a}}(\mathbb{R}^{+|V|})$ and that $F_{\overline{a}}$ can be extended continuously to $\overline{C}$.

DEFINITION 8.10. Let $\overline{x} = (x_0, \ldots, x_{|V|-1}) \in \mathbb{R}^{+|V|}$. We denote the projection function on the first coordinate by $\pi : \mathbb{R}^{+|V|} \to \mathbb{R}$. Thus $\pi(\overline{x}) = x_0$. Let $\overline{a} \in \mathbb{R}^{|V|}$. We denote the normalization of the second kind of $f_{\overline{a}}$ by $G_{\overline{a}} = f_{\overline{a}}/(\pi \circ f_{\overline{a}})$. We will also use the notation

$$D = \left\{ \overline{x} \in \mathbb{R}^{+|V|} \mid \pi(\overline{x}) = 1 \right\}.$$

Then $G_{\overline{a}} : \mathbb{R}^{+|V|} \to D$.

REMARK 8.11. It follows by the homogeneous property of $f_{\overline{a}}$ that we have $G_{\overline{a}}(D) = G_{\overline{a}}(\mathbb{R}^{+|V|})$ and that $G_{\overline{a}}$ can be extended continuously to $\overline{D}$.

Our software implementation used the normalization of the second kind. The next section will show how the superadditivity of $f_{\overline{a}}$ and the Brouwer fixed-point theorem are used to prove the existence of an eigenvalue of $f_{\overline{a}}$. We also recall that our rigidity result on $f_{\overline{a}}$, in Theorem 5.1, implies that the eigenvalue is unique.

9. The existence of an eigenvalue of $f_{\overline{a}}$. Here is an important application of the superadditivity of $f_{\overline{a}}$.

THEOREM 9.1. *For each $k = 0, \ldots, |V| - 1$ and any $\theta > 0$, the set $A_{k,\theta}$ is a convex subset of $\mathbb{R}^{+|V|}$.*

Proof. Let $\overline{r} = (r_0, \ldots, r_{|V|-1}), \overline{s} = (s_0, \ldots, s_{|V|-1}) \in A_{k,\theta}$. Let $0 \le t \le 1$. Assume that

$$f_{\overline{a}}(\overline{r}) = (R_0, \ldots, R_{|V|-1}), \quad f_{\overline{a}}(\overline{s}) = (S_0, \ldots, S_{|V|-1}).$$

Then, by the definition of $A_{k,\theta}$, we have the inequalities

$$\theta r_k \le R_k, \quad \theta s_k \le S_k.$$

By the superadditivity of $f_{\overline{a}}$, Theorem 6.1, we have

$$\begin{aligned} f_{\overline{a}}(t\overline{r} + (1-t)\overline{s}) &\ge f_{\overline{a}}(t\overline{r}) + f_{\overline{a}}((1-t)\overline{s}) \\ &= t f_{\overline{a}}(\overline{r}) + (1-t) f_{\overline{a}}(\overline{s}). \end{aligned}$$

If we denote

$f_{\overline{a}}(t\overline{r} + (1-t)\overline{s}) = (T_0, \ldots, T_{|V|-1})$, then this inequality applied to the kth coordinate shows that

$$T_k \ge tR_k + (1-t)S_k.$$

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But $tR_k + (1-t)S_k \geq \theta(tr_k + (1-t)s_k)$ and hence

$$T_k \geq \theta(tr_k + (1-t)s_k).$$

This proves that $t\overline{r} + (1-t)\overline{s} \in A_{k,\theta}$ and so $A_{k,\theta}$ is a convex set. ☐

We will use the Brouwer fixed-point theorem as it appears in [10].

THEOREM 9.2. *(The Brouwer fixed-point theorem.) Let $G : \overline{C} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous on the compact, convex set $\overline{C}$, and suppose that $G(\overline{C}) \subseteq \overline{C}$. Then G has a fixed point in $\overline{C}$.*

Here is the existence theorem of the eigenvalue of $f_{\overline{a}}$.

THEOREM 9.3. *Let V be the set of vertices of a graph G. Let $\overline{a} \in \mathbb{R}^{|V|}$ be an angles-parameter vector. Then the normalization of the first kind $F_{\overline{a}}$ of $f_{\overline{a}}$ has a unique fixed point in $\mathbb{R}^{+|V|}$ (and hence $f_{\overline{a}}$ has a unique positive eigenvalue in $\mathbb{R}^{+|V|}$).*

Proof. Let $\theta > 0$ be such that $\emptyset \neq \cap_{k=0}^{|v|-1} A_{k,\theta} \subseteq \mathbb{R}^{+|V|}$. Let us denote, as usual, $C = \{\overline{x} \in \mathbb{R}^{+|V|} \mid |\overline{x}| = 1\}$. Then, by the definition of $F_{\overline{a}}$, we have

$$F_{\overline{a}} = \frac{f_{\overline{a}}}{|f_{\overline{a}}|} : C \to C.$$

Each $A_{k,\theta}$, $k = 0, \ldots, |V|-1$, is a convex cone in $\mathbb{R}^{+|V|}$, which is a closed set (for convexity we used Theorem 9.1). Hence $\cap_{k=0}^{|V|-1} A_{k,\theta}$ is a nonempty closed subset of $\mathbb{R}^{+|V|}$ and is convex (being the intersection of convex sets). Since $f_{\overline{a}}(\cap_{k=0}^{|V|-1} A_{k,\theta}) \subseteq \cap_{k=0}^{|V|-1} A_{k,\theta}$ ($f_{\overline{a}}(\overline{r}) \geq \theta\overline{r} \Rightarrow f_{\overline{a}}(f_{\overline{a}}(\overline{r})) \geq f_{\overline{a}}(\theta\overline{r}) = \theta f_{\overline{a}}(\overline{r})$), it follows that $F_{\overline{a}}(C \cap (\cap_{k=0}^{|V|-1} A_{k,\theta})) \subseteq C \cap (\cap_{k=0}^{|V|-1} A_{k,\theta})$. The set $C \cap (\cap_{k=0}^{|V|-1} A_{k,\theta})$ is compact and convex (for C is also convex). The Brouwer fixed-point theorem implies that $F_{\overline{a}}$ has a fixed point in $C \cap (\cap_{k=0}^{|V|-1} A_{k,\theta})$. If $\overline{r_0}$ is such a fixed point, then $F_{\overline{a}}(\overline{r_0}) = \overline{r_0}$. By the definition of $F_{\overline{a}}$, it follows that

$$f_{\overline{a}}(\overline{r_0}) = |f_{\overline{a}}(\overline{r_0})|\, \overline{r_0}.$$

Hence $\lambda = |f_{\overline{a}}(\overline{r_0})|$ is an eigenvalue of $f_{\overline{a}}$. The converse also holds, i.e., if λ is a positive eigenvalue of $f_{\overline{a}}$ with an eigenpoint $\overline{r_0}$ such that $|\overline{r_0}| = 1$, then $f_{\overline{a}}(\overline{r_0}) = \lambda\overline{r_0}$. Hence we have $|f_{\overline{a}}(\overline{r_0})| = \lambda|\overline{r_0}| = \lambda$ and so $F_{\overline{a}}(\overline{r_0}) = \overline{r_0}$. Thus $\overline{r_0}$ is a fixed point of $F_{\overline{a}}$. The uniqueness of the fixed point of $F_{\overline{a}}$ follows by the uniqueness of the eigenvalue of $f_{\overline{a}}$ (the rigidity theorem). ☐

We now return to the proof of the superadditivity of $f_{\overline{a}}$. This involves an interesting set of inequalities and concavity of certain functions.

10. Concave functions and certain inequalities. The inequality of Lemma 6.5 is

$$R\sin^{-1}\left\{\sqrt{\frac{ab}{(R+a)(R+b)}}\right\} + S\sin^{-1}\left\{\sqrt{\frac{cd}{(S+c)(S+d)}}\right\}$$
$$\leq (R+S)\sin^{-1}\left\{\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right\}$$

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for any $a, b, c, d, R, S > 0$. Using the double-angle formula this inequality is equivalent to

$$R\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}+S\cos^{-1}\left\{1-\frac{2cd}{(S+c)(S+d)}\right\}$$
$$\leq (R+S)\cos^{-1}\left\{1-\frac{2(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}\right\}.$$

Here is a suggestion of Paul Federbush, from the University of Michigan, to try to prove the inequality. Let's define the function

$$g(t)=(tR+(1-t)S)$$
$$\times\sin^{-1}\left\{\sqrt{\frac{(ta+(1-t)c)(tb+(1-t)d)}{(t(R+a)+(1-t)(S+c))(t(R+b)+(1-t)(S+d))}}\right\}$$

for $0 \leq t \leq 1$. If $g(t)$ is concave in t, then

$$\frac{1}{2}\left(g(0)+g(1)\right)\leq g\left(\frac{1}{2}\right).$$

This is the inequality of Lemma 6.5. To check concavity it suffices to verify that $g''(t) \leq 0$ for $0 \leq t \leq 1$. Here is the advantage of checking that over working with the original inequality.

REMARK 10.1. $g''(t)$ is an algebraic expression, i.e., $\sin^{-1}$ does not appear in it any more.

Similarly we could define a function $h(t)$ that involves $\cos^{-1}$ but not the square root function. When $h''(t)$ is written as a quotient, the denominator involves the square root but is easily seen to be positive. The numerator is a polynomial in (t, R, a, b, S, c, d). It determines the sign of $h''(t)$. This polynomial is of degree 6 in t, R, and S (separately) and of degree 5 in a, b, c, and d (separately). The total degree of the polynomial is 16 and the task is to prove that it is nonpositive for all $0 \leq t \leq 1$ and all $R, a, b, S, c, d > 0$.

We mention here an integral representation formula for our function. It is of interest by itself.

THEOREM 10.2. *If $R, a, b > 0$, then*

$$R\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}=R^{3/2}b^{1/2}\int_0^a\frac{dx}{(R+x)\sqrt{x(R+x+b)}}$$
$$=R^{3/2}a^{1/2}\int_0^b\frac{dx}{(R+x)\sqrt{x(R+x+a)}}.$$

Proof. We consider three circles of radii a, b, and R that are mutually tangent to one another from the outside. The triangle with their centers as its vertices has edges of lengths $R+a$, $a+b$, and $R+b$. By Lemma 3.2 in [7], it follows that

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$$\frac{\partial}{\partial(\log a)}\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}=\frac{h}{R+a}, \tag{10.1}$$

where h is the distance from O, the intersection point of the three common tangents of the circles, to the edge of length $R+a$. In our particular disc pattern h equals the radius r of the incircle of our triangle. To see this, let A, B, C be the three tangency points of the circles. Then $\overline{OA}$ is orthogonal to the edge of length $R+a$ and $\overline{OB}$ is orthogonal to that of length $a+b$. Hence the angle OBA equals the angle OAB and so $h=|\overline{OA}|=|\overline{OB}|$. Similarly $h=|\overline{OC}|$. Let $s=R+a+b$ be the semiperimeter of the triangle and let F be its area. Then, by elementary geometry,

$$h=\frac{F}{s}.$$

By the Heron formula, $F=\sqrt{sRab}$, and so

$$h=\frac{\sqrt{sRab}}{s}=\sqrt{\frac{Rab}{s}}=\sqrt{\frac{Rab}{R+a+b}}.$$

Going back to (10.1),

$$\frac{\partial}{\partial(\log a)}\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}=\left(\frac{1}{R+a}\right)\sqrt{\frac{Rab}{R+a+b}},$$

so that

$$a\frac{\partial}{\partial a}\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}=\left(\frac{1}{R+a}\right)\sqrt{\frac{Rab}{R+a+b}}$$

and hence

$$\begin{aligned}R\cos^{-1}\left\{1-\frac{2ab}{(R+a)(R+b)}\right\}&=\int_0^a\frac{R}{x(R+x)}\sqrt{\frac{Rxb}{R+x+b}}dx\\&=R^{3/2}b^{1/2}\int_0^a\frac{dx}{(R+x)\sqrt{x(R+x+b)}}.\end{aligned}$$

We can change the roles of a and b by working similarly with the edge of the triangle of length $R+b$. $\square$

We now proceed to give another technique of proving the inequality of Lemma 6.5. There are two ideas involved in it. The first idea is summarized in the following.

THEOREM 10.3. *Suppose that there exists a twice-differentiable, surjective, and strictly increasing function $f : I\rightarrow[0,1]$ that satisfies the following two conditions:*

(1) $f''(1-f^2)+f\cdot(f')^2\le 0$ *on I.*

(2)

$$\begin{aligned}\left(\frac{R}{R+S}\right)f^{-1}\left(\sqrt{\frac{ab}{(R+a)(R+b)}}\right)+\left(\frac{S}{R+S}\right)f^{-1}\left(\sqrt{\frac{cd}{(S+c)(S+d)}}\right)&\\ \le f^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right)&\end{aligned}$$

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for all $a, b, c, d, R, S > 0$. Then the inequality of Lemma 6.5 holds true.

Proof. Consider the function $y = \sin^{-1} f(x)$ for $x \in I$. Then

$$\frac{dy}{dx} = \frac{f'}{\sqrt{1-f^2}},$$

$$\frac{d^2y}{dx^2} = \frac{f''(1-f^2) + f \cdot (f')^2}{(1-f^2)^{3/2}}.$$

By condition (1) we get $d^2y/dx^2 \leq 0$ on I and hence y is concave in I. So, for any $x, z \in I$ and for any $0 \leq t \leq 1$, we have

$$t \sin^{-1} f(x) + (1-t) \sin^{-1} f(z) \leq \sin^{-1} f(tx + (1-t)z). \tag{10.2}$$

We make the following choice:

$$x = f^{-1}\left(\sqrt{\frac{ab}{(R+a)(R+b)}}\right), \; z = f^{-1}\left(\sqrt{\frac{cd}{(S+c)(S+d)}}\right), \; t = \left(\frac{R}{R+S}\right).$$

Then, by (10.2), we get

$$\begin{aligned} &\left(\frac{R}{R+S}\right) \sin^{-1}\left(\sqrt{\frac{ab}{(R+a)(R+b)}}\right) \\ &+ \left(\frac{S}{R+S}\right) \sin^{-1}\left(\sqrt{\frac{cd}{(S+c)(S+d)}}\right) \leq \sin^{-1} f(tx + (1-t)z). \end{aligned} \tag{10.3}$$

By condition (2) we have

$$tx + (1-t)z \leq f^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right)$$

and, since f is increasing and also $\sin^{-1}$ is increasing, we get

$$\sin^{-1} f(tx + (1-t)y) \leq \sin^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right). \tag{10.4}$$

Lemma 6.5 follows from (10.3) and (10.4). ☐

Special cases.

(I) $f(x) = \sin x, \; I = [0, \pi/2]$. Then, in this case, we have

$$f''(1-f^2) + f \cdot (f')^2 = -\sin x \cos^2 x + \sin x \cos^2 x \equiv 0$$

and condition (1) of the theorem is satisfied. Condition (2) is the inequality of Lemma 6.5 and so the theorem is correct trivially in this case.

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(II) $f(x) = 1 - 1/x,\ I = [1, \infty]$. Then, in this case, we have

$$f''(1-f^2) + f \cdot (f')^2 = -\frac{2}{x^3}\left(1 - \left(1 - \frac{1}{x}\right)^2\right) + \left(1 - \frac{1}{x}\right) \cdot \frac{1}{x^4}$$
$$= \frac{1}{x^5} - \frac{3}{x^4} < 0$$

for $x \geq 1$. So condition (1) is satisfied. Condition (2) and the conclusion of the theorem prove the following.

Lemma 10.4. *If for every $a, b, c, d, R, S > 0$ the following inequality is true:*

$$\left(\frac{R}{R+S}\right)\left(\frac{1}{1 - \sqrt{(ab)/[(R+a)(R+b)]}}\right) + \left(\frac{S}{R+S}\right)\left(\frac{1}{1 - \sqrt{(cd)/[(S+c)(S+d)]}}\right) \leq \frac{1}{1 - \sqrt{[(a+c)(b+d)]/[(R+S+a+c)(R+S+b+d)]}},$$

then the inequality of Lemma 6.5 *is true.*

It might have been easier to continue from here if f^{-1} of Theorem 10.3 were concave. Unfortunately this is *never* so, which might be a reason for the difficulty in proving the inequality of the last lemma. This is explained in the following.

Remark 10.5. The function $f(x)$ of Theorem 10.3 is concave on I. For f, $(1 - f^2) \geq 0$, so by condition (1) we get, on I,

$$f'' \leq \frac{-f \cdot (f')^2}{1 - f^2} \leq 0.$$

Hence f^{-1} is convex on $[0, 1]$.

The second idea in this approach (after that of Theorem 10.3) is an elementary trick to get rid of the square root functions in Lemma 10.4. Let us denote

$$\alpha = \sqrt{\frac{a}{R+a}},\ \beta = \sqrt{\frac{b}{R+b}},\ \gamma = \sqrt{\frac{c}{S+c}},\ \delta = \sqrt{\frac{d}{S+d}}.$$

Then $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. Also α, β are independent except for $\alpha = 1$ iff $\beta = 1$. That happens only if $R = 0$. γ, δ are independent except for $\gamma = 1$ iff $\delta = 1$. That happens only if $S = 0$. For the inverse transformations we have

$$a = \left(\frac{\alpha^2}{1-\alpha^2}\right) R,\ b = \left(\frac{\beta^2}{1-\beta^2}\right) R,\ c = \left(\frac{\gamma^2}{1-\gamma^2}\right) S,\ d = \left(\frac{\delta^2}{1-\delta^2}\right) S.$$

With this notation, the left-hand side of the inequality in Lemma 10.4 is

$$\left(\frac{R}{R+S}\right)\left(\frac{1}{1-\alpha\beta}\right) + \left(\frac{S}{R+S}\right)\left(\frac{1}{1-\gamma\delta}\right) = \frac{R(1-\gamma\delta) + S(1-\alpha\beta)}{(R+S)(1-\alpha\beta)(1-\gamma\delta)}.$$

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As for the right-hand side, we have

$$I_1 = \sqrt{\frac{a+c}{R+S+a+c}} = \sqrt{\frac{\alpha^2(1-\gamma^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma^2)R+(1-\alpha^2)S}},$$

$$I_2 = \sqrt{\frac{b+d}{R+S+b+d}} = \sqrt{\frac{\beta^2(1-\delta^2)R+\delta^2(1-\beta^2)S}{(1-\delta^2)R+(1-\beta^2)S}}.$$

Plugging these into the inequality of Lemma 10.4 we get

$$\frac{R(1-\gamma\delta)+S(1-\alpha\beta)}{(R+S)(1-\alpha\beta)(1-\gamma\delta)} \le \frac{1}{1-I_1I_2}.$$

Hence,

$$I_1I_2 \ge \frac{R\alpha\beta(1-\gamma\delta)+S\gamma\delta(1-\alpha\beta)}{R(1-\gamma\delta)+S(1-\alpha\beta)}.$$

On squaring both sides we conclude that, in order to prove Lemma 6.5, it suffices to prove the following.

Lemma 10.6. *If $R, S > 0$ and $0 < \alpha, \beta, \gamma, \delta < 1$, then*

$$\left(\frac{\alpha^2(1-\gamma^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma^2)R+(1-\alpha^2)S}\right)\left(\frac{\beta^2(1-\delta^2)R+\delta^2(1-\beta^2)S}{(1-\delta^2)R+(1-\beta^2)S}\right) \ge \left(\frac{R\alpha\beta(1-\gamma\delta)+S\gamma\delta(1-\alpha\beta)}{R(1-\gamma\delta)+S(1-\alpha\beta)}\right)^2.$$

Proof (of Lemma 10.6). Let us define

$$E = \left(\frac{\alpha^2(1-\gamma^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma^2)R+(1-\alpha^2)S}\right)\left(\frac{\beta^2(1-\delta^2)R+\gamma^2(1-\alpha^2)S}{(1-\delta^2)R+(1-\alpha^2)S}\right) - \left(\frac{R\alpha\beta(1-\gamma\delta)+S\gamma\delta(1-\alpha\beta)}{R(1-\gamma\delta)+S(1-\alpha\beta)}\right)^2.$$

Then

$$E = \frac{RS(R+S)[(1-\gamma\delta)L\cdot R+(1-\alpha\beta)M\cdot S]}{[(1-\gamma^2)R+(1-\alpha^2)S][(1-\delta^2)R+(1-\alpha^2)S][(1-\gamma\delta)R+(1-\alpha\beta)S]^2},$$

where we have

$$\begin{aligned} L = {} & \alpha^2\beta^2(\alpha-\beta)^2+(\alpha-\beta)^2\gamma^3\delta^3 \\ & +\{(\alpha\delta)^2(1-\alpha\beta)(1+\alpha\beta-2\beta^2)-(\alpha\delta)(\beta\gamma)(2-4\alpha\beta+\beta\alpha^3+\alpha\beta^3) \\ & +(\beta\gamma)^2(1-\alpha\beta)(1+\alpha\beta-2\alpha^2)\} \\ & +\{\gamma^2\beta(1-\alpha\beta)(2\alpha-\beta-\alpha\beta^2)-\gamma\delta(\alpha^2+\beta^2+2\alpha^3\beta^3-4\alpha^2\beta^2) \\ & +\delta^2\alpha(1-\alpha\beta)(2\beta-\alpha-\alpha^2\beta)\}\gamma\delta, \end{aligned}$$

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or, as a polynomial in γ and δ,

$$\begin{aligned}L = {}& \alpha^2\beta^2(\alpha-\beta)^2 + \beta^2(1-\alpha\beta)(1+\alpha\beta-2\alpha^2)\gamma^2\\ &+ \alpha^2(1-\alpha\beta)(1+\alpha\beta-2\beta^2)\delta^2 - \alpha\beta(2+\alpha\beta^3-4\alpha\beta+\beta\alpha^3)\gamma\delta\\ &+ \beta(1-\alpha\beta)(2\alpha-\beta-\alpha\beta^2)\gamma^3\delta - (\alpha^2+\beta^2+2\alpha^3\beta^3-4\alpha^2\beta^2)\gamma^2\delta^2\\ &+ \alpha(1-\alpha\beta)(2\beta-\alpha-\alpha^2\beta)\gamma\delta^3 + (\alpha-\beta)^2\gamma^3\delta^3,\end{aligned}$$

and where $M = M(\alpha,\beta,\gamma,\delta) = L(\gamma,\delta,\alpha,\beta)$. Thus it suffices to prove that, for any $0 < \alpha,\beta,\gamma,\delta < 1$, we have $L(\alpha,\beta,\gamma,\delta) \ge 0$. For this will also imply that $M(\alpha,\beta,\gamma,\delta) \ge 0$ for any such choice. This, in turn, will show that $E \ge 0$ for every choice of $R, S > 0$ and $0 < \alpha,\beta,\gamma,\delta < 1$ and hence will prove Lemma 10.6. To check the nonnegativity of $L(\alpha,\beta,\gamma,\delta)$ we make the substitutions

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{x^2}{1+x^2}, \frac{y^2}{1+y^2}, \frac{z^2}{1+z^2}, \frac{w^2}{1+w^2}\right)$$

and clear the denominators. This gives us a nonnegative polynomial in $\mathbb{R}[x,y,z,w]$. To check its nonnegativity one may use techniques of real algebraic geometry [11]. This polynomial has a representation as a sum of five squares of other polynomials. $\square$

Here is another interesting inequality regarding the denominators in Lemma 10.6.

Proposition 10.7. *If $R, S > 0$ and $0 < \alpha,\beta,\gamma,\delta < 1$, then*

$$((1-\gamma^2)R + (1-\alpha^2)S)((1-\delta^2)R + (1-\beta^2)S) \le ((1-\gamma\delta)R + (1-\alpha\beta)S)^2.$$

In fact we have the following identity:

$$\begin{aligned}&((1-\gamma\delta)R + (1-\alpha\beta)S)^2 - ((1-\gamma^2)R + (1-\alpha^2)S)((1-\delta^2)R + (1-\beta^2)S)\\ &\quad = (\gamma-\delta)^2R^2 - ((\alpha\delta-\beta\gamma)^2 - (\alpha-\beta)^2 - (\gamma-\delta)^2)RS + (\alpha-\beta)^2S^2.\end{aligned}$$

The discriminant of this quadratic form in R and S is

$$\begin{aligned}\Delta &= ((\alpha\delta-\beta\gamma)^2 - (\alpha-\beta)^2 - (\gamma-\delta)^2)^2 - 4(\alpha-\beta)^2(\gamma-\delta)^2\\ &= (\alpha\delta-\beta\gamma+\alpha-\beta+\gamma-\delta)(\alpha\delta-\beta\gamma-\alpha+\beta-\gamma+\delta)\\ &\quad\times(\alpha\delta-\beta\gamma+\alpha-\beta-\gamma+\delta)(\alpha\delta-\beta\gamma-\alpha+\beta+\gamma-\delta).\end{aligned}$$

We note that the inequality of Proposition 10.7 relates the denominators of the two sides of the inequality of Lemma 10.6. The numerators, however, are not comparable. However, we have the following identity here:

$$\begin{aligned}&(\alpha^2(1-\gamma^2)R + \gamma^2(1-\alpha^2)S)(\beta^2(1-\delta^2)R + \delta^2(1-\beta^2)S) - (\alpha\beta(1-\gamma\delta)R\\ &\qquad +\gamma\delta(1-\alpha\beta)S)^2\\ &\quad = -\alpha^2\beta^2(\gamma-\delta)^2R^2 + (\beta^2\gamma^2 - \beta^2\gamma^2\delta^2 - \alpha^2\beta^2\gamma^2 + \alpha^2\delta^2 - \alpha^2\beta^2\delta^2 - \alpha^2\gamma^2\delta^2\\ &\qquad -2\alpha\beta\gamma\delta + 2\alpha\beta\gamma^2\delta^2 + 2\alpha^2\beta^2\gamma\delta)RS - \gamma^2\delta^2(\alpha-\beta)^2S^2.\end{aligned}$$

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The discriminant of this quadratic form in R and S is

$$\Delta_1 = (\alpha\beta\gamma + \alpha\gamma\delta - \alpha\delta - \alpha\beta\delta + \beta\gamma - \beta\gamma\delta)(\alpha\beta\gamma - \alpha\gamma\delta + \alpha\delta - \alpha\beta\delta - \beta\gamma + \beta\gamma\delta)$$
$$\times (\alpha\beta\gamma - \alpha\gamma\delta - \alpha\delta - \alpha\beta\delta + \beta\gamma + \beta\gamma\delta)(\alpha\beta\gamma + \alpha\gamma\delta + \alpha\delta - \alpha\beta\delta - \beta\gamma - \beta\gamma\delta).$$

We note that we are interested here in quadratic forms that are nonnegative for nonnegative values of their variables. Probably the next result is known. We provide the proof for the convenience of the reader.

THEOREM 10.8. *$AR^2 + 2BRS + CS^2 \geq 0$ for all $R, S \geq 0$ iff* (i) $A, B, C \geq 0$ or (ii) $A \geq 0$*,* $B < 0$*,* $B^2 \leq AC$*.*

Proof. In one direction: First assume that $AR^2 + 2BRS + CS^2 \geq 0$ for all $R, S \geq 0$. Plugging in $S = 0$ we get $AR^2 \geq 0$ for all $R \geq 0$ and hence necessarily $A \geq 0$. Similarly $C \geq 0$. If also $B \geq 0$ then we have case (i). Otherwise $B < 0$. We claim that in this case $A > 0$. Otherwise $A = 0$ and the form is $S(2BR + CS)$. The sign of the form is the same as that of $2BR + CS$. If we let $S \to 0^+$, then $2BR + CS < 0$ for $B < 0$, which is a contradiction. Hence if $B < 0$, then $A > 0$. Similarly $C > 0$. Now we have

$$AR^2 + 2BRS + CS^2 = A\left(R + \frac{BS}{A}\right)^2 + \left(\frac{AC - B^2}{A}\right)S^2.$$

Since $B < 0$, we can choose $R, S > 0$ so that $R + BS/A = 0$. Hence $AC - B^2 \geq 0$ is a necessity and we have case (ii).

In the opposite direction: We assume that (i) or (ii) holds. In case (i) we have $A, B, C \geq 0$ and this clearly implies that $AR^2 + 2BRS + CS^2 \geq 0$ for all $R, S \geq 0$. In case (ii), by $B < 0$ we get $B^2 > 0$ and by $AC \geq B^2$ it follows that $AC > 0$. Hence, again we have the identity

$$AR^2 + 2BRS + CS^2 = A\left(R + \frac{BS}{A}\right)^2 + \left(\frac{AC - B^2}{A}\right)S^2.$$

So by (ii) we obtain

$$AR^2 + 2BRS + CS^2 \geq \left(\frac{AC - B^2}{A}\right)S^2 \geq 0$$

for all R, S (not just $R, S \geq 0$). $\square$

By this theorem it follows that, in order to prove Proposition 10.7, i.e., to prove that

$$(\gamma - \delta)^2R^2 - ((\alpha\delta - \beta\gamma)^2 - (\alpha - \beta)^2 - (\gamma - \delta)^2)RS + (\alpha - \beta)^2S^2 \geq 0$$

for all $R, S > 0$, $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ we need to show that case (ii) holds. This means we need to show that the assumptions

(1) $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ and

(2) $(\alpha\delta - \beta\gamma)^2 > (\alpha - \beta)^2 + (\gamma - \delta)^2$

imply that

$$\Delta = ((\alpha\delta - \beta\gamma)^2 - (\alpha - \beta)^2 - (\gamma - \delta)^2)^2 - 4(\alpha - \beta)^2(\gamma - \delta)^2 \leq 0.$$

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Proof (of Proposition 10.7). We make the assumptions (1) and (2) above. Also, without any restrictions, we assume that $\alpha \geq \beta$, $\gamma \geq \delta$. Then $1-\alpha \leq 1-\beta$, $1-\gamma \leq 1-\delta$, and all these numbers are nonnegative. To show that $\Delta \leq 0$ is equivalent to showing that

$$((\alpha\delta-\beta\gamma)^2-(\alpha-\beta)^2-(\gamma-\delta)^2)^2 \leq 4(\alpha-\beta)^2(\gamma-\delta)^2.$$

By assumption (2), the number under the square on the left side is nonnegative as are those on the right side. So we can extract square roots to get

$$(\alpha\delta-\beta\gamma)^2-(\alpha-\beta)^2-(\gamma-\delta)^2 \leq 2(\alpha-\beta)(\gamma-\delta)$$

or

$$(\alpha\delta-\beta\gamma)^2 \leq (\alpha-\beta+\gamma-\delta)^2$$

or

$$0 \leq (\alpha-\beta+\gamma-\delta-\alpha\delta+\beta\gamma)(\alpha-\beta+\gamma-\delta+\alpha\delta-\beta\gamma).$$

We will now show that both factors are nonnegative. This will establish the proof. The first factor is $\alpha-\beta+\gamma-\delta-\alpha\delta+\beta\gamma = \alpha(1-\delta)-\beta(1-\gamma)+\gamma-\delta$. Here we have $\alpha \geq \beta$, $1-\delta \geq 1-\gamma$ and so $\alpha(1-\delta) \geq \beta(1-\gamma)$. Also $\gamma-\delta \geq 0$ and hence indeed

$$\alpha-\beta+\gamma-\delta-\alpha\delta+\beta\gamma \geq 0.$$

The second factor is $\alpha-\beta+\gamma-\delta+\alpha\delta-\beta\gamma = \gamma(1-\beta)-\delta(1-\alpha)+\alpha-\beta$, which is nonnegative as well, for similar reasons. □

We give one more reduction of Lemma 6.5, which is based on Theorem 10.3.

LEMMA 10.9. (i) *If for every $a,b,c,d,R,S>0$ there exists an $n>0$ such that the following inequality is true:*

$$\left(\frac{R}{R+S}\right)\left(\frac{1}{1-\sqrt{ab/((R+a)(R+b))}}\right)^{1/n}+\left(\frac{S}{R+S}\right)\left(\frac{1}{1-\sqrt{cd/((S+c)(S+d))}}\right)^{1/n}$$
$$\leq \left(\frac{1}{1-\sqrt{(a+c)(b+d)/((R+S+a+c)(R+S+b+d))}}\right)^{1/n},$$

then the inequality of Lemma 6.5 *is true.*

(ii) *If for every $a,b,c,d,R,S>0$ the following inequality is true:*

$$\left\{1-\sqrt{\frac{ab}{(R+a)(R+b)}}\right\}^{R/(R+S)}\left\{1-\sqrt{\frac{cd}{(S+c)(S+d)}}\right\}^{S/(R+S)}$$
$$\geq 1-\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}},$$

then the inequality of Lemma 6.5 *is true.*

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Proof. We first note that the proof of Theorem 10.3 gives us, in fact, a stronger theorem.

THEOREM 10.3′. *Let $a, b, c, d, R, S > 0$ be fixed. Suppose that there exists a twice-differentiable, surjective, and strictly increasing function $f : I \to [0, 1]$ that satisfies the following two conditions:*
(1) $f''(1 - f^2) + f \cdot (f')^2 \le 0$ *on* I,
(2)

$$\left(\frac{R}{R+S}\right) f^{-1}\left(\sqrt{\frac{ab}{(R+a)(R+b)}}\right) + \left(\frac{S}{R+S}\right) f^{-1}\left(\sqrt{\frac{cd}{(S+c)(S+d)}}\right) \le f^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right),$$

then

$$\left(\frac{R}{R+S}\right) \sin^{-1}\left(\sqrt{\frac{ab}{(R+a)(R+b)}}\right) + \left(\frac{S}{R+S}\right) \sin^{-1}\left(\sqrt{\frac{cd}{(S+c)(S+d)}}\right) \le \sin^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right).$$

A proof of (i). Let us fix $a, b, c, d, R, S > 0$. By the assumption there exists a corresponding $n > 0$. We define

$$f(x) = 1 - \frac{1}{x^n}, \quad I = [1, \infty].$$

Then $f'(x) = n/x^{n+1}$, $f''(x) = -n(n+1)/x^{n+2}$, and so

$$\begin{aligned} f''(1-f^2) + f \cdot (f')^2 &= \frac{-n(n+1)}{x^{n+2}}\left(1 - \left(1 - \frac{1}{x^n}\right)^2\right) + \left(1 - \frac{1}{x^n}\right) \cdot \left(\frac{n}{x^{n+1}}\right)^2 \\ &= \frac{-n(n+1)}{x^{n+2}} \cdot \frac{1}{x^n} \cdot \left(2 - \frac{1}{x^n}\right) + \left(1 - \frac{1}{x^n}\right) \cdot \frac{n^2}{x^{2n+2}} \\ &= \frac{-2n(n+1)}{x^{2n+2}} + \frac{n(n+1)}{x^{3n+2}} + \frac{n^2}{x^{2n+2}} - \frac{n^2}{x^{3n+2}} \\ &= \frac{n}{x^{3n+2}} - \frac{n(n+2)}{x^{2n+2}} = \frac{n}{x^{3n+2}}\left(1 - (n+2)x^n\right) < 0 \end{aligned}$$

for $x \in I$ since $n > 0$. This proves that $f(x)$ satisfies condition (1) of Theorem 10.3′. Condition (2) of Theorem 10.3′ is satisfied by our assumption on n. Hence, by Theorem 10.3′, the inequality of Lemma 6.5 holds for our fixed $a, b, c, d, R, S > 0$. But this is true for any choice of $a, b, c, d, R, S > 0$ and hence the conclusion of part (i).

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A proof of (ii). Let $a, b, c, d, R, S > 0$. Let us denote

$$t_1 = \sqrt{\frac{ab}{(R+a)(R+b)}},\ \ t_2 = \sqrt{\frac{cd}{(S+c)(S+d)}},$$
$$t_3 = \sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}},\ \ \alpha = \frac{R}{R+S}.$$

Thus,

$$1-\alpha = \frac{S}{R+S}.$$

Let us assume that

$$(1-t_1)^{\alpha}(1-t_2)^{1-\alpha} > 1-t_3$$

so that

$$-\alpha\log(1-t_1) - (1-\alpha)\log(1-t_2) < -\log(1-t_3).$$

We claim that in this case there exists an $n > 0$ for which the inequality of part (i) is satisfied and is sharp, i.e., with no equality. To see this we note that, for any $n > 0$, $0 < t < 1$, the binomial expansion gives us

$$\left(\frac{1}{1-t}\right)^{1/n} = (1-t)^{-1/n} = 1 + \frac{1}{n}\cdot t + \frac{1}{2}\cdot\frac{1}{n}\cdot\left(\frac{1}{n}+1\right)\cdot t^2 + \cdots.$$

Hence,

$$\begin{aligned}
&\alpha\left(\frac{1}{1-t_1}\right)^{1/n} + (1-\alpha)\left(\frac{1}{1-t_2}\right)^{1/n}\\
&\quad = \alpha\left\{1 + \frac{1}{n}\cdot t_1 + \frac{1}{2}\cdot\frac{1}{n}\cdot\left(\frac{1}{n}+1\right)\cdot t_1^2 + \cdots\right\}\\
&\qquad + (1-\alpha)\left\{1 + \frac{1}{n}\cdot t_2 + \frac{1}{2}\cdot\frac{1}{n}\cdot\left(\frac{1}{n}+1\right)\cdot t_2^2 + \cdots\right\}\\
&\quad = 1 + \frac{1}{n}\left\{\alpha t_1 + (1-\alpha)t_2\right\} + \frac{1}{2}\cdot\frac{1}{n}\cdot\left(\frac{1}{n}+1\right)\left\{\alpha t_1^2 + (1-\alpha)t_2^2\right\} + \cdots\\
&\quad = 1 + \frac{1}{n}\left\{\alpha\left(t_1 + \frac{t_1^2}{2} + \cdots\right) + (1-\alpha)\left(t_2 + \frac{t_2^2}{2} + \cdots\right)\right\} + O\left(\frac{1}{n^2}\right)\\
&\quad = 1 + \frac{1}{n}\left\{\alpha\left(-\log(1-t_1)\right) + (1-\alpha)\left(-\log(1-t_2)\right)\right\} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Similarly we have the expansion

$$\left(\frac{1}{1-t_3}\right)^{1/n} = 1 + \frac{1}{n}\left(-\log(1-t_3)\right) + O\left(\frac{1}{n^2}\right).$$

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By our assumption we conclude that for all $n > 0$ large enough we have

$$\alpha\left(\frac{1}{1-t_1}\right)^{1/n} + (1-\alpha)\left(\frac{1}{1-t_2}\right)^{1/n} < \left(\frac{1}{1-t_3}\right)^{1/n},$$

which proves the assertion. If $a, b, c, d, R, S > 0$ are such that we have equality,

$$(1-t_1)^{\alpha}(1-t_2)^{1-\alpha} = 1-t_3,$$

then we make small perturbations

$$a' = a+\epsilon_1, b' = b+\epsilon_2, c' = c+\epsilon_3, d' = d+\epsilon_4, R' = R+\epsilon_5, S' = S+\epsilon_6$$

such that with the perturbed parameters we get strict inequality:

$$(1-t_1')^{\alpha'}(1-t_2')^{1-\alpha'} > 1-t_3'.$$

By the first part of the proof, the inequality of Lemma 6.5 holds with a', b', c', d', R', S'. We now let $\epsilon_j' \to 0$, $1 \le j \le 6$, which is clearly possible to do. $\square$

Remark 10.10. The fact that we can have $\epsilon_j' \to 0$ while keeping the strict inequality

$$(1-t_1')^{\alpha'}(1-t_2')^{1-\alpha'} > 1-t_3'$$

follows by the permanence principle for holomorphic functions. For, if we had the opposite situation, then by the permanence principle we should have had the identity

$$\begin{aligned}\{1-t_1(a',b',c',d',R',S')\}^{R'/(R'+S')}&\{1-t_2(a',b',c',d',R',S')\}^{S'/(R'+S')}\\ &\equiv 1-t_3(a',b',c',d',R',S')\end{aligned}$$

in some open subset of $\mathbb{R}^6$, $|a'-a|, |b'-b|, |c'-c|, |d'-d|, |R'-R|, |S'-S| < \epsilon$. Since the functions t_1, t_2, and t_3 are real holomorphic in the real variables (a', b', c', d', R', S') we conclude that the above identity should have been true in all of $(\mathbb{R}^+)^6$. This is false! We can take, for example,

$$a = 1, b = 2, c = 3, d = 4, R = 5, S = 6.$$

Remark 10.11. It is interesting to note that, if we consider in part (ii) of the previous lemma only the linear term in the binomial expansion (i.e., the coefficients of t_1, t_2, and t_3), we obtain the following inequality, which seems to be correct for every $a, b, c, d, R, S > 0$:

$$\begin{aligned}\left(\frac{R}{R+S}\right)\sqrt{\frac{ab}{(R+a)(R+b)}} + \left(\frac{S}{R+S}\right)&\sqrt{\frac{cd}{(S+c)(S+d)}}\\ &\le \sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}.\end{aligned}$$

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We mimic our reduction of Lemma 6.5 to Lemma 10.6 and obtain that in order to prove Lemma 6.5 it suffices to prove the following.

LEMMA 10.12. *If $0 < \alpha, \beta, \gamma, \delta, t < 1$, then the following inequality is true:*

$$\left(\frac{t\alpha^2(1-\gamma^2)+(1-t)\gamma^2(1-\alpha^2)}{t(1-\gamma^2)+(1-t)(1-\alpha^2)}\right)\left(\frac{t\beta^2(1-\delta^2)+(1-t)\delta^2(1-\beta^2)}{t(1-\delta^2)+(1-t)(1-\beta^2)}\right) \geq \left(1-(1-\alpha\beta)^t(1-\gamma\delta)^{1-t}\right)^2.$$

Proof.

$$\alpha^2 = \frac{a}{R+a} \Longrightarrow a = \frac{\alpha^2 R}{1-\alpha^2}, \qquad \beta^2 = \frac{b}{R+b} \Longrightarrow b = \frac{\beta^2 R}{1-\beta^2},$$
$$\gamma^2 = \frac{c}{S+c} \Longrightarrow c = \frac{\gamma^2 S}{1-\gamma^2}, \qquad \delta^2 = \frac{d}{S+d} \Longrightarrow d = \frac{\delta^2 S}{1-\delta^2}.$$

We substitute

$$\frac{a+c}{R+S+a+c} = \frac{\alpha^2 R(1-\gamma^2)+\gamma^2 S(1-\alpha^2)}{R(1-\gamma^2)+S(1-\alpha^2)},$$

$$\frac{b+d}{R+S+b+d} = \frac{\beta^2 R(1-\delta^2)+\delta^2 S(1-\beta^2)}{R(1-\delta^2)+S(1-\beta^2)},$$

so the inequality of Lemma 10.9(ii) becomes

$$(1-\alpha\beta)^t(1-\gamma\delta)^{1-t} \geq 1-\sqrt{\left(\frac{t\alpha^2(1-\gamma^2)+(1-t)\gamma^2(1-\alpha^2)}{t(1-\gamma^2)+(1-t)(1-\alpha^2)}\right)\left(\frac{t\beta^2(1-\delta^2)+(1-t)\delta^2(1-\beta^2)}{t(1-\delta^2)+(1-t)(1-\beta^2)}\right)}.$$

This is exactly the inequality of Lemma 10.12, where we define

$$t = \frac{R}{R+S}. \qquad \square$$

11. Monotonicity and convexity of $f_{\overline{a}}$ and the Perron–Frobenius theory. Let $\overline{a} \in \mathbb{R}^{|V|}$ be an angles-parameter vector. Then the mapping

$$\begin{cases} f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|} \\ f_{\overline{a}}(r_0, \ldots, r_{|V|-1}) = (R_0, \ldots, R_{|V|-1}), \end{cases}$$

which is implicitly defined by the equations

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_i,k}r_{j_i,k+1}}{(R_i+r_{j_i,k})(R_i+r_{j_i,k+1})}\right) = a_i, \quad 0 \leq i < |V|$$

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(see section 3), satisfies the following three properties:

(1) $f_{\overline{a}}$ is an isotone mapping (Proposition 3.1). This means that

$$\overline{r}, \overline{s} \in \mathbb{R}^{+|V|},\ \overline{r} \leq \overline{s} \Longrightarrow f_{\overline{a}}(\overline{r}) \leq f_{\overline{a}}(\overline{s}).$$

(2) $f_{\overline{a}}$ is a homogeneous mapping (Theorem 6.1(i)). This means that

$$\overline{r} \in \mathbb{R}^{+|V|},\ t > 0 \Longrightarrow f_{\overline{a}}(t\overline{r}) = t f_{\overline{a}}(\overline{r}).$$

(3) $f_{\overline{a}}$ is a superadditive mapping (Theorem 6.1(ii)). This means that

$$\overline{r}, \overline{s} \in \mathbb{R}^{+|V|} \Longrightarrow f_{\overline{a}}(\overline{r}) + f_{\overline{a}}(\overline{s}) \leq f_{\overline{a}}(\overline{r} + \overline{s}).$$

Remark 11.1. Property (3) implies property (1).

We recall definition 13.3.1 on page 448 of [10].

Definition 11.2. A mapping $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is convex on a convex subset $D_0 \subseteq D$ if $F(t\overline{r} + (1-t)\overline{s}) \leq tF(\overline{r}) + (1-t)F(\overline{s})$ whenever $\overline{r}, \overline{s} \in D_0$ and $0 < t < 1$.

Theorem 11.3. *$-f_{\overline{a}}$ is convex on $\mathbb{R}^{+|V|}$.*

Proof. Let $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ and let $0 < t < 1$. Then we have

$$\begin{aligned} -f_{\overline{a}}(t\overline{r} + (1-t)\overline{s}) &\leq -f_{\overline{a}}(t\overline{r}) - f_{\overline{a}}((1-t)\overline{s}) \quad \text{(property (3))} \\ &= t\left(-f_{\overline{a}}(\overline{r})\right) + (1-t)\left(-f_{\overline{a}}(\overline{s})\right) \quad \text{(property (2))}. \end{aligned}$$ ◻

As an immediate consequence of the above we obtain the following.

Theorem 11.4.

(i) *For every $\overline{r} \in \mathbb{R}^{+|V|}$ we have $f_{\overline{a}}'(\overline{r}) \geq 0$.*

(ii) *For every $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ we have*

$$\left(f_{\overline{a}}(\overline{s}) - f_{\overline{a}}(\overline{r})\right)^T \leq f_{\overline{a}}'(\overline{r}) \cdot (\overline{s} - \overline{r})^T.$$

(iii) *For every $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ we have*

$$\left(f_{\overline{a}}'(\overline{s}) - f_{\overline{a}}'(\overline{r})\right) \cdot (\overline{s} - \overline{r})^T \leq \overline{0}.$$

(iv) *For every $\overline{r} \in \mathbb{R}^{+|V|}$ and every $\overline{h} \in \mathbb{R}^{|V|}$ we have*

$$f_{\overline{a}}''(\overline{r})\overline{h}\overline{h} \leq 0.$$

Proof. (i) This follows by property (1). For every i, $0 \leq i < |V|$, the component function $R_i(r_0, r_1, \ldots, r_{|V|-1})$ is nondecreasing in r_j, $0 \leq j < |V|$. Hence

$$\frac{\partial R_i}{\partial r_j} \geq 0, \quad 0 \leq i, j < |V|.$$

(ii) This follows by Theorem 11.3 and Theorem 13.3.2 on page 448 of [10], equation (3).

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(iii) and (iv) follow by Theorem 11.3 and Theorem 13.3.2 on page 448 of [10], equations (4) and (5), respectively. □

THEOREM 11.5. *For every* $\overline{r} \in \mathbb{R}^{+|V|}$ *we have*

$$f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^T = f_{\overline{a}}(\overline{r})^T.$$

Proof. This is essentially Euler's theorem on homogeneous functions of degree 1. Namely, if $f_{\overline{a}}(\overline{r}) = (R_0, \ldots, R_{|V|-1})$, then by property (2) we have for $0 \le i < |V|$

$$R_i(tr_0, \ldots, tr_{|V|-1}) = tR_i(r_0, \ldots, r_{|V|-1}).$$

Here $\overline{r} = (r_0, \ldots, r_{|V|-1})$. So by Euler's theorem, mentioned above, we have

$$\sum_{j=0}^{|V|-1} r_j \frac{\partial R_i}{\partial r_j} = R_i, \quad 0 \le i < |V|.$$

This is just the matrix identity

$$f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^T = f_{\overline{a}}(\overline{r})^T. \qquad \square$$

REMARK 11.6. In this paper the vectors in $\mathbb{R}^n$ are row vectors. This is the reason that we need the transposed operation in our expressions above.

THEOREM 11.7. *For every* $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ *we have*

$$f_{\overline{a}}(\overline{r})^T \le f_{\overline{a}}'(\overline{s}) \cdot \overline{r}^T.$$

Proof. By Theorem 11.4(ii) we have

$$f_{\overline{a}}(\overline{r})^T - f_{\overline{a}}(\overline{s})^T \le f_{\overline{a}}'(\overline{s}) \cdot (\overline{r}^T - \overline{s}^T).$$

So

$$f_{\overline{a}}(\overline{r})^T - f_{\overline{a}}(\overline{s})^T \le f_{\overline{a}}'(\overline{s}) \cdot \overline{r}^T - f_{\overline{a}}'(\overline{s}) \cdot \overline{s}^T.$$

By Theorem 11.5 we have the identity

$$f_{\overline{a}}'(\overline{s}) \cdot \overline{s}^T = f_{\overline{a}}(\overline{s})^T,$$

so we can cancel this quantity on both sides of the matrix inequality to obtain

$$f_{\overline{a}}(\overline{r})^T \le f_{\overline{a}}'(\overline{s}) \cdot \overline{r}^T. \qquad \square$$

COROLLARY 11.8. *For every* $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ *and for every* $k \in \mathbb{Z}^+ \cup \{0\}$ *we have*

$$f_{\overline{a}}^{(\circ k)}(\overline{r})^T \le (f_{\overline{a}}'(\overline{s}))^k \cdot \overline{r}^T,$$

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where the notation $f_{\overline{a}}^{(\circ k)}(\overline{r})$ *stands for* k *applications of the mapping* $f_{\overline{a}}$ *on the vector* $\overline{r}$.

Proof. For $k = 0$ we have $f_{\overline{a}}^{(\circ 0)}(\overline{r})^T = \overline{r}^T$ and $(f_{\overline{a}}{}'(\overline{s}))^0 = I_{|V|}$, the identity $|V| \times |V|$ matrix. Hence

$$\overline{r}^T = f_{\overline{a}}^{(\circ 0)}(\overline{r})^T = (f_{\overline{a}}{}'(\overline{s}))^0 \cdot \overline{r}^T = I_{|V|} \cdot \overline{r}^T.$$

For $k = 1$ the inequality follows by Theorem 11.7. So let us assume inductively that

$$f_{\overline{a}}^{(\circ k)}(\overline{r})^T \leq (f_{\overline{a}}{}'(\overline{s}))^k \cdot \overline{r}^T.$$

Then for $k + 1$ we have

$$\begin{aligned} f_{\overline{a}}^{(\circ (k+1))}(\overline{r})^T &= f_{\overline{a}}(f_{\overline{a}}^{(\circ k)}(\overline{r}))^T \quad \text{(Theorem 11.7)} \\ &\leq (f_{\overline{a}}{}'(\overline{s})) \cdot f_{\overline{a}}^{(\circ k)}(\overline{r})^T \quad \text{(induction assumption)} \\ &\leq (f_{\overline{a}}{}'(\overline{s})) \cdot (f_{\overline{a}}{}'(\overline{s}))^k \cdot \overline{r}^T = (f_{\overline{a}}{}'(\overline{s}))^{k+1} \cdot \overline{r}^T. \qquad \square \end{aligned}$$

THEOREM 11.9. *For every* $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$, *if* $\rho(f_{\overline{a}}{}'(\overline{s})) < 1$, *then* $(I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1}$ *exists and is nonnegative and*

$$\sum_{k=0}^{\infty} f_{\overline{a}}^{(\circ k)}(\overline{r})^T \leq (I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1} \cdot \overline{r}^T.$$

In particular under the assumptions of the theorem we have for every $\overline{r} \in \mathbb{R}^{+|V|}$

$$\lim_{k \to \infty} f_{\overline{a}}^{(\circ k)}(\overline{r})^T = \overline{0}.$$

Here we use the standard notation $\rho(A)$ *for the spectral radius of the matrix* A.

Proof. By Corollary 11.8 we have for every $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ and for every $k \in \mathbb{Z}^+ \cup \{0\}$

$$f_{\overline{a}}^{(\circ k)}(\overline{r})^T \leq (f_{\overline{a}}{}'(\overline{s}))^k \cdot \overline{r}^T.$$

Hence for every $N \in \mathbb{Z}^+ \cup \{0\}$

$$\sum_{k=0}^{N} f_{\overline{a}}^{(\circ k)}(\overline{r})^T \leq \left\{ \sum_{k=0}^{N} (f_{\overline{a}}{}'(\overline{s}))^k \right\} \cdot \overline{r}^T.$$

If $\rho(f_{\overline{a}}{}'(\overline{s})) < 1$ then most of the claims follow by Neumann's lemma (page 45 in [10]). In fact by Theorem 11.4(i) we have $f_{\overline{a}}{}'(\overline{s}) \geq 0$ so Theorem 2.4.5 on page 53 in [10] implies that $(I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1}$ exists and is nonnegative iff $\rho(f_{\overline{a}}{}'(\overline{s})) < 1$. By Neumann's lemma we have the identity

$$(I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1} = \sum_{k=0}^{\infty} (f_{\overline{a}}{}'(\overline{s}))^k$$

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and the proof is completed. ∎

Here is an interesting consequence.

THEOREM 11.10.

(i) *Let* $\lambda(\overline{a})$ *be the unique eigenvalue of* $f_{\overline{a}}$. *If there exists an* $\overline{s} \in \mathbb{R}^{+|V|}$ *such that* $\rho(f_{\overline{a}}{}'(\overline{s})) < 1$, *then* $0 < \lambda(\overline{a}) < 1$.

(ii) *Conversely, if* $0 < \lambda(\overline{a}) < 1$ *and if* $\overline{r} \in \mathbb{R}^{+|V|}$ *is an eigenvector of* $f_{\overline{a}}$, *then* $\rho(f_{\overline{a}}{}'(\overline{r})) = \lambda(\overline{a}) < 1$.

Proof. (i) Suppose that $\rho(f_{\overline{a}}{}'(\overline{s})) < 1$ for some $\overline{s} \in \mathbb{R}^{+|V|}$. Suppose also that $\overline{r} \in \mathbb{R}^{+|V|}$ is an eigenvector of $f_{\overline{a}}$. Then by Theorem 11.9 we have the inequality

$$\sum_{k=0}^{\infty} f_{\overline{a}}^{(\circ k)}(\overline{r})^T \leq (I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1} \cdot \overline{r}^T.$$

In this case we also have the identity $f_{\overline{a}}(\overline{r})^T = \lambda \overline{r}^T$ (where we shorten $\lambda = \lambda(\overline{a})$) and by induction we have for every $k \in \mathbb{Z}^+ \cup \{0\}$ the identity $f_{\overline{a}}^{(\circ k)}(\overline{r})^T = \lambda^k \overline{r}^T$. Indeed for $k = 0$ this is just

$$\overline{r}^T = f_{\overline{a}}^{(\circ 0)}(\overline{r})^T = \lambda^0 \overline{r}^T$$

and for $k = 1$ this is $f_{\overline{a}}(\overline{r})^T = \lambda \overline{r}^T$). If we assume that

$$f_{\overline{a}}^{(\circ k)}(\overline{r})^T = \lambda^k \overline{r}^T,$$

then for $k + 1$ we have

$$\begin{aligned} f_{\overline{a}}^{(\circ (k+1))}(\overline{r})^T &= f_{\overline{a}}\left(f_{\overline{a}}^{(\circ (k))}(\overline{r})\right)^T \quad \text{(induction hypothesis)} \\ &= f_{\overline{a}}(\lambda^k \overline{r})^T \quad \text{(property (2))} \\ &= \lambda^k f_{\overline{a}}(\overline{r})^T \quad \text{(the case } k = 1) \\ &= \lambda^k (\lambda \overline{r})^T = \lambda^{k+1} \overline{r}^T. \end{aligned}$$

Hence by substituting this into the inequality we get

$$\left(\sum_{k=0}^{\infty} \lambda^k\right) \overline{r}^T \leq (I_{|V|} - f_{\overline{a}}{}'(\overline{s}))^{-1} \cdot \overline{r}^T.$$

Hence $0 < \sum_{k=0}^{\infty} \lambda^k < \infty$, which is equivalent to $0 < \lambda < 1$.

(ii) Let us suppose that $0 < \lambda < 1$ and that $\overline{r} \in \mathbb{R}^{+|V|}$ satisfies the eigenvector equation for $f_{\overline{a}}$

$$f_{\overline{a}}(\overline{r}) = \lambda \overline{r}.$$

By Theorem 11.5 we have $f_{\overline{a}}(\overline{r})^T = f_{\overline{a}}{}'(\overline{r}) \cdot \overline{r}^T$ and so

$$f_{\overline{a}}{}'(\overline{r}) \cdot \overline{r}^T = \lambda \overline{r}^T$$

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or

$$(\lambda I_{|V|} - f_{\overline{a}}'(\overline{r})) \cdot \overline{r}^T = \overline{0}^T.$$

This shows that λ is also an eigenvalue of the matrix $f_{\overline{a}}'(\overline{r})$ and that $\overline{r}^T$ is a corresponding eigenvector. Since $\overline{r} \in \mathbb{R}^{+|V|}$ we have $\overline{r}^T \geq \overline{0}$. In fact each component of $\overline{r}^T$ is a positive number. By Theorem 11.4(i) $f_{\overline{a}}'(\overline{r})$ is a nonnegative matrix. We also claim that $f_{\overline{a}}'(\overline{r})$ is irreducible. To see that we use Theorem 2.3.5 on page 47 of [10]. We need to show that for any two indices $0 \leq i, j < |V|$ there is a sequence of nonzero elements of the matrix $f_{\overline{a}}'(\overline{r})$ located in entries of the form

$$(i, i_1),\ (i_1, i_2), \ldots, (i_m, j).$$

This follows by the following geometrical properties:

(a) G is a connected graph.

(b) The valence of every vertex of G is at least 3.

(c) If $f_{\overline{a}}(x_0, \ldots, x_{|V|-1}) = (R_0, \ldots, R_{|V|-1})$ and l is a vertex of G adjacent to vertex k, then

$$\frac{\partial R_k}{\partial x_l} > 0.$$

We proved that $\overline{r}^T$ is a positive eigenvector of the nonnegative and irreducible matrix $f_{\overline{a}}'(\overline{r})$. By the *Perron–Frobenius theory* of nonnegative matrices, it follows that $\overline{r}^T$ is an eigenvector of $f_{\overline{a}}'(\overline{r})$ that corresponds to $\rho(f_{\overline{a}}'(\overline{r}))$; see Theorem 1.4 on page 27 of [4] or example 10 on page 58 of [8]. Hence we have the equality $\rho(f_{\overline{a}}'(\overline{r})) = \lambda(\overline{a})$, which concludes the proof of part (ii) of our theorem. ◻

It is useful to note that in the course of our proof of Theorem 11.10(ii) we also proved incidentally the following theorem.

THEOREM 11.11. *If $\lambda(\overline{a})$ is the unique eigenvalue of $f_{\overline{a}}$ and if $\overline{r} \in \mathbb{R}^{+|V|}$ is a positive corresponding eigenvector, then*

$$\lambda(\overline{a}) = \rho(f_{\overline{a}}'(\overline{r})),$$

the spectral radius of the nonnegative and irreducible matrix $f_{\overline{a}}'(\overline{r})$, and $\overline{r}^T$ is a positive eigenvector of this matrix.

12. A converse of the contraction principle, the min-max theorem, and rigidity. Suppose that G is a graph embedding, $\overline{a} \in \mathbb{R}^{|V|}$ is an angles-parameter vector, and $f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$ is the corresponding $\overline{a}$-mapping. Let $G_{\overline{a}}$ be the normalization of $f_{\overline{a}}$ of the second kind. The main purpose of this section and of the next section is to prove that $G_{\overline{a}}$ is a contraction on $(\mathbb{R}^{+|V|}, d)$ with an appropriate metric d on $\mathbb{R}^{+|V|}$ that is continuous relative to the standard Euclidean metric. In particular this will imply that $G_{\overline{a}}$ has a unique fixed point in $\mathbb{R}^{+|V|}$ and that the iterates $G_{\overline{a}}^{(\circ n)}(\overline{r})$ converge to this fixed point for any $\overline{r} \in \mathbb{R}^{+|V|}$. We recall the equation of Theorem 11.5:

$$f_{\overline{a}}(\overline{r})^T = f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^T,\ \ \overline{r} \in \mathbb{R}^{+|V|}.$$

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This equation generalizes to iterates of $f_{\overline{a}}$ as follows.

THEOREM 12.1.

$$\forall \overline{r} \in \mathbb{R}^{+|V|},\, \forall n \in \mathbb{Z}^{+} \cup \{0\},\;\; f_{\overline{a}}^{(\circ n)}(\overline{r})^{T} = \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{r}) \cdot \overline{r}^{T}.$$

Proof.

Method 1. The equation $f_{\overline{a}}(\overline{r})^{T} = f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^{T}$ is a consequence of Euler's theorem on homogeneous functions of degree 1. Hence it suffices to prove that $f_{\overline{a}}^{(\circ n)}$ is homogeneous of degree 1. This follows easily by induction on n:

$$\begin{aligned} f_{\overline{a}}^{(\circ(n+1))}(t\overline{r}) &= f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(t\overline{r})\right) = f_{\overline{a}}^{(\circ n)}\left(t f_{\overline{a}}(\overline{r})\right) \\ &= t f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{r})\right) = t f_{\overline{a}}^{(\circ(n+1))}(\overline{r}). \end{aligned}$$

Method 2. We use induction on n starting from the equation
$f_{\overline{a}}(\overline{r})^{T} = f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^{T}$ and the chain rule:

$$\begin{aligned} f_{\overline{a}}^{(\circ(n+1))}(\overline{r})^{T} &= f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{r})\right)^{T} = \left(f_{\overline{a}}^{(\circ n)}\right)'\left(f_{\overline{a}}(\overline{r})\right) \cdot f_{\overline{a}}(\overline{r})^{T} \\ &= \left(f_{\overline{a}}^{(\circ n)}\right)'\left(f_{\overline{a}}(\overline{r})\right) f_{\overline{a}}'(\overline{r}) \cdot \overline{r}^{T} = \left(f_{\overline{a}}^{(\circ(n+1))}\right)'(\overline{r}) \cdot \overline{r}^{T}. \qquad \square \end{aligned}$$

Here is another useful fact on iterations of $f_{\overline{a}}$. It generalizes Theorem 11.7.

THEOREM 12.2.

$$\forall \overline{r}, \overline{s} \in \mathbb{R}^{+|V|},\, \forall n \in \mathbb{Z}^{+} \cup \{0\},\;\; f_{\overline{a}}^{(\circ n)}(\overline{r})^{T} \leq \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{r}^{T}.$$

Proof. The case $n = 0$ is clear and the case $n = 1$ was established in Theorem 11.7. We first prove that $f_{\overline{a}}^{(\circ n)}$ is a superadditive mapping. We use induction on n. By the superadditivity of $f_{\overline{a}}$ and the fact that $f_{\overline{a}}^{(\circ n)}$ is isotone it follows that

$$\begin{aligned} f_{\overline{a}}^{(\circ(n+1))}(\overline{r} + \overline{s}) &= f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{r} + \overline{s})\right) \\ &\geq f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{r}) + f_{\overline{a}}(\overline{s})\right). \end{aligned}$$

Hence the induction hypothesis implies that

$$\begin{aligned} f_{\overline{a}}^{(\circ(n+1))}(\overline{r} + \overline{s}) &\geq f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{r})\right) + f_{\overline{a}}^{(\circ n)}\left(f_{\overline{a}}(\overline{s})\right) \\ &= f_{\overline{a}}^{(\circ(n+1))}(\overline{r}) + f_{\overline{a}}^{(\circ(n+1))}(\overline{s}). \end{aligned}$$

As in Theorem 11.3 this implies that $-f_{\overline{a}}^{(\circ n)}$ is convex on $\mathbb{R}^{+|V|}$. As in Theorem 11.4(ii) this implies the inequality

$$\left(f_{\overline{a}}^{(\circ n)}(\overline{s}) - f_{\overline{a}}^{(\circ n)}(\overline{r})\right)^{T} \leq \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{r}) \cdot (\overline{s} - \overline{r})^{T}.$$

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So

$$f_{\overline{a}}^{(\circ n)}(\overline{r})^T - f_{\overline{a}}^{(\circ n)}(\overline{s})^T \leq \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{r}^T - \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{s}^T.$$

By Theorem 12.1 we have the identity

$$f_{\overline{a}}^{(\circ n)}(\overline{s})^T = \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{s}^T,$$

so we can cancel this quantity on both sides of the matrix inequality to obtain

$$f_{\overline{a}}^{(\circ n)}(\overline{r})^T \leq \left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{r}^T. \qquad \square$$

Here is another useful fact on iterations of $f_{\overline{a}}$. It generalizes a part of what was proved in Theorem 11.10(ii).

THEOREM 12.3. $\forall \overline{r} \in \mathbb{R}^{+|V|}, \forall n \in \mathbb{Z}^+$, $(f_{\overline{a}}^{(\circ n)})'(\overline{r})$ *is a nonnegative and irreducible matrix.*

Proof. In Theorem 11.10(ii) we proved the case $n = 1$. The proof of the irreducibility of $(f_{\overline{a}}^{(\circ n)})'(\overline{r})$ for $n > 1$ follows similarly to the case $n = 1$ by noting that $f_{\overline{a}}^{(\circ n)}$ corresponds to paths of length n between vertices of the graph.

The nonnegativity of $(f_{\overline{a}}^{(\circ n)})'(\overline{r})$ is a consequence of the fact that the mapping $f_{\overline{a}}^{(\circ n)}$ is isotone $\forall n \in \mathbb{Z}^+$. This is so because it is an n-folded composition of isotone mappings. $\square$

We quote a version of the min-max theorem for eigenvalues of symmetric matrices. It appears on page 394 of [5].

THEOREM 12.4. *(Courant–Fischer min-max theorem.) If* $A \in \mathbb{R}^{n\times n}$ *is symmetric, then*

$$\lambda_k(A) = \max_{\dim(S)=k} \min_{\overline{0}\neq\overline{y}\in S} \frac{\overline{y}A\overline{y}^T}{\overline{y}\overline{y}^T}, \;\; k = 1, \ldots, n,$$

where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ *are the eigenvalues of* A *and where, as usual,* $\overline{y} \in \mathbb{R}^n$ *is a row vector.*

We can now derive a useful estimate on the largest eigenvalue of $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$.

THEOREM 12.5. *Let* $\overline{s} \in \mathbb{R}^{+|V|}$, $n \in \mathbb{Z}^+ \cup \{0\}$ *and let us denote by* $\lambda^{(n)}(\overline{s})$ *the largest eigenvalue of* $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$. *Then*

$$\lambda^{(n)}(\overline{s}) \geq \max_{\overline{r}\in\mathbb{R}^{+|V|}, \overline{r}\cdot\overline{r}^T=1} \overline{r} \cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T.$$

Proof. We remark that $\lambda^{(n)}(\overline{s})$ is positive because $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$ is a nonnegative matrix. Next, by the min-max theorem, we have

$$\lambda^{(n)}(\overline{s}) = \max_{\dim(S)=1} \min_{\overline{r}\in S-\{\overline{0}\}} \frac{\overline{r}\left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s}) \cdot \overline{r}^T}{\overline{r}\cdot\overline{r}^T}.$$

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Hence, since $\dim(S) = 1$, we have

$$\lambda^{(n)}(\overline{s}) = \max_{\overline{r}\cdot\overline{r}^T=1} \overline{r}\left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s})\cdot\overline{r}^T.$$

By the fact that $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$ is nonnegative (Theorem 12.3) it follows that in fact

$$\lambda^{(n)}(\overline{s}) = \max_{\overline{r}\in\mathbb{R}^{+|V|},\overline{r}\cdot\overline{r}^T=1} \overline{r}\left(f_{\overline{a}}^{(\circ n)}\right)'(\overline{s})\cdot\overline{r}^T.$$

By Theorem 12.2 it follows that

$$\lambda^{(n)}(\overline{s}) \geq \max_{\overline{r}\in\mathbb{R}^{+|V|},\overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T. \qquad \square$$

We can now give an alternative proof for the rigidity theorem (Theorem 5.1). This proof is algebraic in nature. In fact we prove much more.

Theorem 12.6. *(The general rigidity theorem.)* $\forall\, n \in \mathbb{Z}^+$, $f_{\overline{a}}^{(\circ n)}$ *has at most one eigenvalue* $\lambda^{(n)}$ *(as a mapping* $\mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$*). If* $\lambda^{(n)}$ *exists, then any pair of corresponding eigenpoints are proportional to one another. We have the representation*

$$\lambda^{(n)} = \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(n)}(\overline{s}),$$

where $\lambda^{(n)}(\overline{s})$ *is the largest eigenvalue of the nonnegative matrix* $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$.

Proof. Let λ be a (positive) eigenvalue of $f_{\overline{a}}^{(\circ n)}$. Let $\overline{r_0} \in \mathbb{R}^{+|V|}$, $\overline{r_0}\cdot\overline{r_0}^T = 1$, be a corresponding eigenpoint, i.e.,

$$f_{\overline{a}}^{(\circ n)}(\overline{r_0}) = \lambda\overline{r_0}.$$

As in Theorem 11.11 it follows that λ is the unique positive eigenvalue of the matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{r_0})$. In fact we have $\lambda = \rho((f_{\overline{a}}^{(\circ n)})'(\overline{r_0}))$, and $\overline{r_0}^T$ is a positive eigenvector of the matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{r_0})$. Let $\overline{s} \in \mathbb{R}^{+|V|}$. By Theorem 12.4 we have

$$\lambda^{(n)}(\overline{s}) \geq \max_{\overline{r}\in\mathbb{R}^{+|V|},\overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T \geq \overline{r_0}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r_0})^T = \overline{r_0}\cdot(\lambda\overline{r_0}^T) = \lambda.$$

This proves that $\lambda \leq \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(n)}(\overline{s})$. By the above consequence of Theorem 11.11 we have $\lambda = \lambda^{(n)}(\overline{r_0})$. This proves that we actually have the equality

$$\lambda = \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(n)}(\overline{s}).$$

Since λ was any eigenvalue of $f_{\overline{a}}^{(\circ n)}$, the formula above shows that $f_{\overline{a}}^{(\circ n)}$ has at most one eigenvalue and hence rigidity.

Remark 12.7. If $\lambda^{(n)}$ exists and if $\overline{r_0}$ is a corresponding eigenpoint, then, by the above consequence of Theorem 11.11, $\overline{r_0}^T$ is a positive eigenvector of the nonnegative

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and irreducible matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{r_0})$ and hence by Perron–Frobenius it is projectively unique.

However, it seems as if $\lambda^{(n)}$ may have another (projectively different) eigenpoint $\overline{r_1}$. It is true that, again by Perron–Frobenius, $\overline{r_1}$ is the projectively unique eigenvector of the matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{r_1})$ but this does not imply the uniqueness of the eigenpoint for $f_{\overline{a}}^{(\circ n)}$. We will prove that part of Theorem 12.5 and much more in the next section. □

REMARK 12.8. We proved the existence of a positive eigenvalue of $f_{\overline{a}}$ (Theorem 9.2). Combining this with the general rigidity theorem (Theorem 12.5) proves the existence and the uniqueness of the positive eigenvalue and the corresponding projective eigenpoint of any iteration $f_{\overline{a}}^{(\circ n)}$.

We translate the results for $f_{\overline{a}}^{(\circ n)}$ to corresponding results for iterations of the normalization of the second kind $G_{\overline{a}}^{(\circ n)}$.

THEOREM 12.9. *$\forall\, n \in \mathbb{Z}^+$, $G_{\overline{a}}^{(\circ n)}$ has a unique positive fixed point.*

Proof. If $f_{\overline{a}}(\overline{r}) = (R_0, \ldots, R_{|V|-1})$, then by the definition we have

$$G_{\overline{a}}(\overline{r}) = \frac{1}{R_0} f_{\overline{a}}(\overline{r}).$$

This implies that every positive eigenpoint of $f_{\overline{a}}$ is a fixed point of $G_{\overline{a}}$ and vice versa. For we may assume (by the fact that $f_{\overline{a}}$ is homogeneous) that $\overline{r} = (1, \ldots)$ and so

$$f_{\overline{a}}(\overline{r}) = \lambda \overline{r} \Leftrightarrow \overline{r} = \frac{1}{\lambda} f_{\overline{a}}(\overline{r}) = G_{\overline{a}}(\overline{r}).$$

As for iterations, we use induction on n

$$\begin{aligned} G_{\overline{a}}^{(\circ(n+1))}(\overline{r}) &= G_{\overline{a}}^{(\circ n)}\left(G_{\overline{a}}(\overline{r})\right) = G_{\overline{a}}^{(\circ n)}\left(\frac{1}{R_0} f_{\overline{a}}(\overline{r})\right) \\ &= \frac{1}{R_0^{(n)}} f_{\overline{a}}^{(\circ n)}\left(\frac{1}{R_0} f_{\overline{a}}(\overline{r})\right) = \frac{1}{R_0 R_0^{(n)}} f_{\overline{a}}^{(\circ(n+1))}(\overline{r}), \end{aligned}$$

where $f_{\overline{a}}^{(\circ(n+1))}(\overline{r}) = (R_0 R_0^{(n)}, \ldots)$. So the theorem holds true for all $n \in \mathbb{Z}^+$. □

We can now prove our main result.

THEOREM 12.10. *$\forall\, \overline{r} \in \mathbb{R}^{+|V|}$, the limit $\lim_{n\to\infty} G_{\overline{a}}^{(\circ n)}(\overline{r})$ exists and equals the unique fixed point $(1, \ldots)$ of $G_{\overline{a}}$.*

Proof. By Theorem 12.9 $G_{\overline{a}}^{(\circ n)}$ has a unique fixed point $\forall\, n \in \mathbb{Z}^+$. Hence the assumptions of the theorem of C. Bessaga [3] hold. The conclusion is that for any $s \in (0, 1)$ there exists a metric d_s on $\mathbb{R}^{+|V|}$ (more accurately, on the set D that is defined in Definition 15.1) so that $(\mathbb{R}^{+|V|}, d_s)$ is a complete metric space and $G_{\overline{a}}$ is an s-contraction on $\mathbb{R}^{+|V|}$, i.e.,

$$\forall\, \overline{r}, \overline{s} \in \mathbb{R}^{+|V|}, \quad d_s\left(G_{\overline{a}}(\overline{r}), G_{\overline{a}}(\overline{s})\right) \le s d_s(\overline{r}, \overline{s}).$$

Moreover, d_s dominates locally the Euclidean metric in the sense that for any Euclidean neighborhood $U \subseteq \mathbb{R}^{+|V|}$ there exists a constant $C = C(U)$ such that

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$\forall\, \overline{r}, \overline{s} \in U$ we have the inequality

$$||\overline{r} - \overline{s}||_2 \leq C d_s(\overline{r}, \overline{s}).$$

Now the result follows by the theorem of Banach on contractions. □

We need to prove now the uniqueness of the eigenpoint of $f_{\overline{a}}^{(\circ n)}$ of Theorem 12.5. This will complete our theory so far. We do that in the next section.

13. Uniqueness of the eigenpoint and more formulas for the eigenvalue of $f_{\overline{a}}^{(\circ n)}$. We start with the following structure theorem. Its proof gives as a bonus a formula for $\lambda^{(n)}$, different from the one of Theorem 12.5.

THEOREM 13.1. *Suppose that $\overline{r}, \overline{s} \in \mathbb{R}^{+|V|}$ satisfy the following two assumptions: $f_{\overline{a}}^{(\circ n)}(\overline{r})^T = \lambda^{(n)}\overline{r}^T$ and $f_{\overline{a}}^{(\circ n)}(\overline{s})^T \geq \lambda^{(n)}\overline{s}^T$. Then $\forall\, \alpha, \beta \in \mathbb{R}^{+|V|} \cup \{0\}$, $\alpha^2 + \beta^2 > 0$, we have*

$$f_{\overline{a}}^{(\circ n)}(\alpha\overline{r} + \beta\overline{s})^T = \lambda^{(n)}(\alpha\overline{r} + \beta\overline{s})^T.$$

In particular $\overline{s}$ must also be an eigenpoint of $f_{\overline{a}}^{(\circ n)}$. We also have the following formula:

$$\lambda^{(n)} = \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} \overline{r_1} \cdot f_{\overline{a}}^{(\circ n)}(\overline{r_1})^T.$$

Proof. $\lambda^{(n)}$ is an eigenvalue of the matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{r})$, so by the min-max theorem for nonnegative matrices and by Theorem 12.2 we obtain

$$\lambda^{(n)} = \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} \overline{r_1} \left(f_{\overline{a}}^{(\circ n)} \right)'(\overline{r}) \cdot \overline{r_1}^T \geq \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} \overline{r_1} \cdot f_{\overline{a}}^{(\circ n)}(\overline{r_1})^T. \tag{13.1}$$

We will denote $\overline{r_1} = (r_0, \ldots, r_{|V|-1})$ and $f_{\overline{a}}^{(\circ n)}(\overline{r_1}) = (R_0(\overline{r_1}), \ldots, R_{|V|-1}(\overline{r_1}))$. Then we can rewrite (13.1) as follows:

$$\lambda^{(n)} \geq \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} \sum_{i=0}^{|V|-1} r_i R_i(\overline{r_1}). \tag{13.2}$$

By superadditivity (see the proof of Theorem 12.2) we have

$$f_{\overline{a}}^{(\circ n)}(\alpha\overline{r} + \beta\overline{s})^T \geq f_{\overline{a}}^{(\circ n)}(\alpha\overline{r})^T + f_{\overline{a}}^{(\circ n)}(\beta\overline{s})^T. \tag{13.3}$$

By the fact that $f_{\overline{a}}^{(\circ n)}$ is homogeneous and by the assumptions on $\overline{r}$ and on $\overline{s}$ we get

$$f_{\overline{a}}^{(\circ n)}(\alpha\overline{r})^T + f_{\overline{a}}^{(\circ n)}(\beta\overline{s})^T \geq \lambda^{(n)}(\alpha\overline{r} + \beta\overline{s})^T. \tag{13.4}$$

So (13.3) and (13.4) give us

$$f_{\overline{a}}^{(\circ n)}(\alpha\overline{r} + \beta\overline{s})^T \geq \lambda^{(n)}(\alpha\overline{r} + \beta\overline{s})^T. \tag{13.5}$$

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If $\overline{u} = \alpha\overline{r} + \beta\overline{s}$, (11) implies that, $\forall\, 0 \le i < |V|$, $R_i(\overline{u}) \ge \lambda^{(n)} u_i$. So for such a $\overline{u}$, which is normalized to satisfy $||\overline{u}||_2 = 1$, we get

$$\sum_{i=0}^{|V|-1} u_i R_i(\overline{u}) \ge \sum_{i=0}^{|V|-1} u_i(\lambda^{(n)} u_i) = \lambda^{(n)}.$$

Combining this with (13.2) gives us

$$\lambda^{(n)} \ge \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} r_i R_i(\overline{r_1}) \ge \sum_{i=0}^{|V|-1} u_i R_i(\overline{u}) \ge \lambda^{(n)}.$$

Thus we must have the equality sign all along and hence

$$\lambda^{(n)} = \max_{\overline{r_1} \in \mathbb{R}^{+|V|}, ||\overline{r_1}||_2 = 1} \overline{r_1} \cdot f_{\overline{a}}^{(\circ n)}(\overline{r_1})^T = \overline{u} \cdot f_{\overline{a}}^{(\circ n)}(\overline{u})^T.$$

This proves the formula for $\lambda^{(n)}$ and also proves the equalities $R_i(\overline{u}) = \lambda^{(n)} u_i \;\forall\, 0 \le i < |V|$. These equalities are equivalent to $f_{\overline{a}}^{(\circ n)}(\overline{u}) = \lambda^{(n)}\overline{u}$ and so also the first part of the theorem follows. □

Here is a conclusion of Theorem 13.1.

THEOREM 13.2. *$\forall \overline{v} \in \mathbb{R}^{+|V|}$ either $\overline{v}$ is an eigenpoint of $f_{\overline{a}}^{(\circ n)}$, i.e., $f_{\overline{a}}^{(\circ n)}(\overline{v}) = \lambda^{(n)}\overline{v}$, or $f_{\overline{a}}^{(\circ n)}(\overline{v})^T \not\geq \lambda^{(n)}\overline{v}^T$. This gives one more formula for $\lambda^{(n)}$:*

$$\lambda^{(n)} = \max\left\{\alpha \in \mathbb{R}^+ \mid \exists \overline{v} \in \mathbb{R}^{+|V|} \text{ s.t. } f_{\overline{a}}^{(\circ n)}(\overline{v})^T \ge \alpha\overline{v}^T\right\}.$$

Proof. If $\overline{v} \in \mathbb{R}^{+|V|}$ does not satisfy $f_{\overline{a}}^{(\circ n)}(\overline{v})^T \not\geq \lambda^{(n)}\overline{v}^T$, then $f_{\overline{a}}^{(\circ n)}(\overline{v})^T \ge \lambda^{(n)}\overline{v}^T$, and by Theorem 13.1 this implies that $\overline{v}$ is an eigenpoint of $f_{\overline{a}}^{(\circ n)}$.

Next we take any $\overline{v} \in \mathbb{R}^{+|V|}$. If $\overline{v}$ is not an eigenpoint of $f_{\overline{a}}^{(\circ n)}$, then by the above we have $f_{\overline{a}}^{(\circ n)}(\overline{v})^T \not\geq \lambda^{(n)}\overline{v}^T$ so that there exists an index i, $0 \le i < |V|$, such that $R_i(\overline{v}) < \lambda^{(n)} v_i$ (where, as usual, $\overline{v} = (v_0, \ldots, v_{|V|-1})$ and $f_{\overline{a}}^{(\circ n)}(\overline{v}) = (R_0(\overline{v}), \ldots, R_{|V|-1}(\overline{v})))$. Hence for such a $\overline{v}$ if $f_{\overline{a}}^{(\circ n)}(\overline{v})^T \ge \alpha\overline{v}^T$ then necessarily $\alpha < \lambda^{(n)}$. This proves our formula. □

We can now conclude the proof of Theorem 12.5. Let us define $V^{(n)} = \{\overline{r} \in \mathbb{R}^{+|V|} \mid f_{\overline{a}}^{(\circ n)}(\overline{r}) = \lambda^{(n)}\overline{r}\}$. Then Theorem 13.1 implies that $V^{(n)}$ is closed with respect to nonnegative linear combinations. $\forall\, \alpha, \beta \in \mathbb{R}^+ \cup \{0\}$, $\alpha^2 + \beta^2 \ne 0$, $\forall\, \overline{r}, \overline{s} \in V^{(n)}$, also $\alpha\overline{r} + \beta\overline{s} \in V^{(n)}$. Moreover, $f_{\overline{a}}^{(\circ n)}$ acts linearly on $V^{(n)}$ for

$$\begin{aligned} f_{\overline{a}}^{(\circ n)}(\alpha\overline{r} + \beta\overline{s})^T &\ge f_{\overline{a}}^{(\circ n)}(\alpha\overline{r})^T + f_{\overline{a}}^{(\circ n)}(\beta\overline{s})^T \\ &= \alpha f_{\overline{a}}^{(\circ n)}(\overline{r})^T + \beta f_{\overline{a}}^{(\circ n)}(\overline{s})^T = \alpha(\lambda^{(n)}\overline{r}^T) + \beta(\lambda^{(n)}\overline{s}^T) \\ &= \lambda^{(n)}(\alpha\overline{r} + \beta\overline{s})^T \end{aligned}$$

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and Theorem 13.2 implies that there must be an equality sign:

$$f_{\overline{a}}^{(\circ n)}(\alpha\overline{r}+\beta\overline{s})^T=\lambda^{(n)}(\alpha\overline{r}+\beta\overline{s})^T=\alpha f_{\overline{a}}^{(\circ n)}(\overline{r})^T+\beta f_{\overline{a}}^{(\circ n)}(\overline{s})^T.$$

Finally, the defining equations for $f_{\overline{a}}$ are of the form

$$\sum_{k=1}^{l-1}\cos^{-1}\left(1-\frac{2r_kr_{k+1}}{(R+r_k)(R+r_{k+1})}\right)=a.$$

These show that $f_{\overline{a}}$ cannot be linear on $V^{(1)}$ if $\dim V^{(1)}\geq 2$ (for example, one can differentiate with respect to α,β in $\alpha\overline{r}+\beta\overline{s}$ and observe that the derivatives cannot be constants if $\overline{r},\overline{s}$ are linearly independent). Similarly $f_{\overline{a}}^{(\circ n)}$ cannot be linear on $V^{(n)}$ if $\dim V^{(n)}\geq 2$.

14. The packing property. Suppose that G is a graph embedding, $\overline{a}\in\mathbb{R}^{|V|}$ is an angles-parameter vector, and $f_{\overline{a}}:\mathbb{R}^{+|V|}\to\mathbb{R}^{+|V|}$ is the corresponding $\overline{a}$-mapping. We proved that $f_{\overline{a}}$ has a unique eigenvalue $\lambda^{(1)}$ and a projectively unique corresponding eigenvector, say $\overline{r}$. Thus the equation

$$f_{\overline{a}}(\overline{r})=\lambda^{(1)}\overline{r}$$

is satisfied. We recall that our motivation for the study of $f_{\overline{a}}$ is geometric. Specifically we are interested in a circle-packing realization of the graph embedding G that corresponds to the angles-parameter vector $\overline{a}$. By the very definition of $f_{\overline{a}}$ this implies that we are interested in geometric configurations $(G,\overline{a})$ for which

$$\lambda^{(1)}=\lambda^{(1)}(G,\overline{a})=1.$$

Thus we are led to the following definition.

Definition 14.1. We will say that *the geometric configuration* $(G,\overline{a})$, which consists of the graph embedding G and the angles-parameter vector $\overline{a}\in\mathbb{R}^{|V|}$, *has the property* P (for packing) if $\lambda^{(1)}=1$.

An interesting problem will be to characterize the geometric configurations that have the property P. At this stage, we can give some partial results for this geometrical problem. It will be convenient to list some of the relations we proved so far for eigenvalues of $f_{\overline{a}}$ and of its iterates. We recall our notation.

$\lambda^{(n)}=\lambda^{(n)}(G,\overline{a})$ is the unique eigenvalue of the mapping $f_{\overline{a}}^{(\circ n)}:\mathbb{R}^{+|V|}\to\mathbb{R}^{+|V|}$.

$\lambda^{(n)}(\overline{s})=\lambda^{(n)}(G,\overline{a})(\overline{s})$ is the largest eigenvalue of the symmetric, nonnegative, and irreducible matrix $(f_{\overline{a}}^{(\circ n)})'(\overline{s})$.

Here is a list of some facts that we proved.

(1) $f_{\overline{a}}^{(\circ n)}(\overline{r})^T\leq(f_{\overline{a}}^{\circ n})'(\overline{s})\cdot\overline{r}^T$ and $f_{\overline{a}}^{(\circ n)}(\overline{r})^T=(f_{\overline{a}}^{\circ n})'(\overline{r})\cdot\overline{r}^T$ $\forall\, n\in\mathbb{Z}^+$, $\forall\,\overline{r},\overline{s}\in\mathbb{R}^{+|V|}$. (These follow by Theorem 12.2 and Theorem 12.1, respectively.)

(2) If $\overline{r}\in\mathbb{R}^{+|V|}$ is the eigenpoint of $f_{\overline{a}}^{(\circ n)}$, i.e., $f_{\overline{a}}^{(\circ n)}(\overline{r})^T=\lambda^{(n)}\overline{r}^T$, then $\lambda^{(n)}=\rho((f_{\overline{a}}^{(\circ n)})'(\overline{r}))=\lambda^{(n)}(\overline{r})$. (This follows by the proof of Theorem 12.5.)

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(3) $\lambda^{(n)}(\overline{s}) \geq \max_{\overline{r}\in\mathbb{R}^{+|V|},\,\overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T$ and $\lambda^n(\overline{s}) = \max_{\overline{r}\in\mathbb{R}^{+|V|},\,\overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot (f_{\overline{a}}^{(\circ n)})'(\overline{s})\cdot\overline{r}^T$. (These follow by Theorem 12.4 and by its proof, respectively.)

(4) $\lambda^{(n)} = \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(n)}(\overline{s})$. (This follows by Theorem 12.5.)

(5) $\lambda^{(n)} = \max_{\overline{r}\in\mathbb{R}^{+|V|},\,\overline{r}\cdot\overline{r}^T=1} \overline{r}\cdot f_{\overline{a}}^{(\circ n)}(\overline{r})^T$. (This follows by Theorem 13.1.)

(6)
$\lambda^{(n)} = \max\{\alpha\in\mathbb{R}^+ \,|\, \exists\,\overline{v}\in\mathbb{R}^{+|V|}\ \text{such that}\ f_{\overline{a}}^{(\circ n)}(\overline{v}) \geq \alpha\overline{v}\}$. (This follows by Theorem 13.2.)

Using fact (4) we obtain immediately that

$$\lambda^{(1)} < 1 \iff \exists\,\overline{s}\in\mathbb{R}^{+|V|}\ \text{such that}\ \lambda^{(1)}(\overline{s}) = \rho(f_{\overline{a}}{}'(\overline{s})) < 1,$$

so that the existence of an $\overline{s}\in\mathbb{R}^{+|V|}$ for which $\lambda^{(1)}(\overline{s}) < 1$ implies that the configuration $(G,\overline{a})$ does not have the property P.

Using the three representations we have for $\lambda^{(1)}$ in facts (4), (5), and (6) we can give three characterizations of geometric configurations that have the property P. All we have to do is to take $n = 1$ and substitute $\lambda^{(1)} = 1$ in those formulas. We obtain the following theorem.

Theorem 14.2. *The following four conditions are equivalent.*

(i) *The geometric configuration $(G,\overline{a})$ has the property P.*

(ii) $\forall\,\overline{s}\in\mathbb{R}^{+|V|}$, $\lambda^{(1)}(\overline{s}) \geq 1$ *and* $\exists\,\overline{r}\in\mathbb{R}^{+|V|}$ *such that* $\lambda^{(1)}(\overline{r}) = 1$.

(iii) $\forall\,\overline{r}\in\mathbb{R}^{+|V|}$ *such that* $||\overline{r}||_2 = 1$ *we have* $\overline{r}\cdot f_{\overline{a}}(\overline{r})^T \leq 1$ *and* $\exists\,\overline{s}\in\mathbb{R}^{+|V|}$ *such that* $||\overline{s}||_2 = 1$ *and* $\overline{s}\cdot f_{\overline{a}}(\overline{s})^T = 1$.

(iv) *(If $\overline{v}\in\mathbb{R}^{+|V|}$ and $\alpha\in\mathbb{R}^+$ satisfy $f_{\overline{a}}(\overline{v})^T \geq \alpha\overline{v}^T$ then $\alpha \leq 1$) and ($\exists\,\overline{v}\in\mathbb{R}^{+|V|}$ such that $f_{\overline{a}}(\overline{v})^T = \overline{v}^T$).*

Proof. The equivalence between (i) and (ii) follows by fact (4). The equivalence between (i) and (iii) follows by fact (5). The equivalence between (i) and (iv) follows by fact (6). ∎

Remark 14.3. Clearly the second part of condition (iv) is enough for the equivalence with (i), however, by the first part, it follows, for example, that if we can find a $\overline{v}\in\mathbb{R}^{+|V|}$ and an $\alpha > 1$ such that $f_{\overline{a}}(\overline{v})^T \geq \alpha\overline{v}^T$, then $(G,\overline{a})$ cannot have the property P.

In fact the last remark gives rise to several necessary conditions on the geometric configuration $(G,\overline{a})$ to have the property P. These are interesting because they have a geometrical nature.

We recall our notation from section 7:

$$\begin{aligned} A_{i,1} = \{&(r_0,\ldots,r_{|V|-1})\in\mathbb{R}^{+|V|} \,|\, R_i \geq r_i,\\ &\text{where } f_{\overline{a}}(r_0,\ldots,r_{|V|-1}) = (R_0,\ldots,R_{|V|-1})\}.\end{aligned}$$

These are closed cones in $\overline{\mathbb{R}^{+|V|}}$ with the properties that are specified in section 7. In some sense the intersection of those $|V|$ cones is the set that is responsible for having the property P.

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THEOREM 14.4. *We will denote* $\overline{v} = (r_0, \ldots, r_{|V|-1}) \in \mathbb{R}^{+|V|}$ *and* $f_{\overline{a}}(\overline{v}) = (R_0, \ldots, R_{|V|-1})$*. If the geometric configuration* $(G, \overline{a})$ *has the property* P*, then each of the following is true.*

(i) *For all* $\overline{v} \in \mathbb{R}^{+|V|}$ *there exists an index* $0 \le i \le |V| - 1$ *such that* $R_i \le r_i$*.*

(ii) *For every choice of radii* $r_0, \ldots, r_{|V|-1}$ *at the vertices of* G *there must exist at least one that shrinks after an application of* $f_{\overline{a}}$*. Shrinks here means does not grow.*

(iii)

$$\text{closure}\left(\left(\cap_{i=0}^{|V|-1} A_{i,1}\right)^c\right) = \overline{\mathbb{R}^{+|V|}}.$$

Here the complement is taken with respect to the set $\mathbb{R}^{+|V|} \cup \{\overline{0}\}$*.*

Proof. It is clear that the optimal α in the condition $f_{\overline{a}}(\overline{v})^T \ge \alpha \overline{v}^T$ of Theorem 14.2(iv), i.e., the largest α, is

$$\alpha = \min_{0 \le i < |V|} \frac{R_i}{r_i}.$$

Thus the necessary condition of Theorem 14.2(iv), $\alpha \le 1$, translates to

$$\min_{0 \le i < |V|} \frac{R_i}{r_i} \le 1,$$

which proves the validity of (i) and (ii) above.

But these are equivalent to $\forall \overline{v} \in \mathbb{R}^{+|V|}\ \exists 0 \le i < |V|$ such that $\overline{v} \in \overline{A_{i,1}^c}$. This is the same as

$$\cup_{i=0}^{|V|-1} \overline{A_{i,1}^c} \supseteq \mathbb{R}^{+|V|},$$

which is

$$\overline{\cup_{i=0}^{|V|-1} A_{i,1}^c} \supseteq \mathbb{R}^{+|V|},$$

so

$$\text{closure}\left(\left(\cap_{i=0}^{|V|-1} A_{i,1}\right)^c\right) \supseteq \mathbb{R}^{+|V|},$$

which proves (iii). □

In fact we can sharpen the condition of Theorem 14.4(iii) in order to obtain the following characterization.

THEOREM 14.5. *The geometric configuration* $(G, \overline{a})$ *has the property* P *iff the intersection set* $\cap_{i=0}^{|V|-1} A_{i,1}$ *is a straight ray starting at the origin. In other words,*

$$\dim\left(\cap_{i=0}^{|V|-1} A_{i,1}\right) = 1.$$

In this case any point on $\cap_{i=0}^{|V|-1} A_{i,1}$ *is an eigenpoint of* $f_{\overline{a}}$ *and any eigenpoint of* $f_{\overline{a}}$ *is a point on* $\cap_{i=0}^{|V|-1} A_{i,1}$*.*

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Proof. First let us suppose that $(G, \overline{a})$ has the property P. Then we know that projectively $f_{\overline{a}}$ has a unique eigenpoint, say $\overline{r_0}$. Thus $f_{\overline{a}}(\overline{r_0}) = \overline{r_0}$ and for any $\overline{r}$ that satisfies $f_{\overline{a}}(\overline{r}) = \overline{r}$ there exists a $t > 0$ such that $\overline{r} = t\overline{r_0}$. In particular the straight ray

$$\{t\overline{r_0} \,|\, t > 0\}$$

is contained in $\cap_{i=0}^{|V|-1} A_{i,1}$. We claim that in fact we have the equality

$$\{t\overline{r_0} \,|\, t > 0\} = \cap_{i=0}^{|V|-1} A_{i,1}.$$

To see this, let $\overline{s} \in \cap_{i=0}^{|V|-1} A_{i,1}$. Then by the definition of $A_{i,1}$ we have $f_{\overline{a}}(\overline{s}) \geq \overline{s}$. By Theorem 13.1 it follows that $\overline{s}$ must be an eigenpoint of $f_{\overline{a}}$, so that $f_{\overline{a}}(\overline{s}) = \overline{s}$ and $\overline{s} \in \{t\overline{r_0} \,|\, t > 0\}$.

To prove the converse, let us assume that $\cap_{i=0}^{|V|-1} A_{i,1}$ is a straight ray starting at the origin. Say $\cap_{i=0}^{|V|-1} A_{i,1} = \{t\overline{r_0} \,|\, t > 0\}$ for some $\overline{r_0} \in \mathbb{R}^{+|V|}$. (We remark here that the intersection $\cap_{i=0}^{|V|-1} A_{i,1}$ is invariant with respect to $f_{\overline{a}}$, i.e., $f_{\overline{a}}(\cap_{i=0}^{|V|-1} A_{i,1}) \subseteq \cap_{i=0}^{|V|-1} A_{i,1}$. This is so because $f_{\overline{a}}(\overline{r}) \geq \overline{r} \Rightarrow f_{\overline{a}}(f_{\overline{a}}(\overline{r})) \geq f_{\overline{a}}(\overline{r})$.) In this case every point of $\cap_{i=0}^{|V|-1} A_{i,1}$ must be a boundary point of every $A_{i,1}$, i.e., we must have the following equality:

$$\cap_{i=0}^{|V|-1} A_{i,1} = \cap_{i=0}^{|V|-1} \partial A_{i,1} = \{t\overline{r_0} \,|\; t > 0\}.$$

We noted that $\forall\, 0 \leq i < |V|$ we have

$$\partial A_{i,1} = \{(r_0, \ldots, r_{|V|-1}) \in \mathbb{R}^{+|V|} \,|\, R_i = r_i\},$$

where, as usual, $f_{\overline{a}}(r_0, \ldots, r_{|V|-1}) = (R_0, \ldots, R_{|V|-1})$. Hence for every

$$\overline{s} \in \cap_{i=0}^{|V|-1} \partial A_{i,1}$$

we have the equation

$$f_{\overline{a}}(\overline{s}) = \overline{s}.$$

This proves that $\lambda^{(1)} = 1$ and so $(G, \overline{a})$ has the property P. ∎

15. The λ-packing property. This section extends in a straightforward manner the results of section 14. It turns out that the packing property is merely the special case of λ-packing where $\lambda = 1$. We will omit the proofs of Theorems 15.2, 15.3, and 15.4, which correspond to the proofs of Theorems 14.2, 14.4, and 14.5, respectively, with the value $\lambda^{(1)} = 1$ switched to $\lambda^{(1)} = \lambda$. First we give the following definition.

Definition 15.1. We will say that the geometric configuration $(G, \overline{a})$ has the property λ-P (for λ-packing) if $\lambda^{(1)} = \lambda$.

Theorem 15.2. *The following four conditions are equivalent.*

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(i) *The geometric configuration* $(G, \overline{a})$ *has the property* λ-P.

(ii) *For all* $\overline{s} \in \mathbb{R}^{+|V|}$, $\lambda^{(1)}(\overline{s}) \geq \lambda$, *and there exists an* $\overline{r} \in \mathbb{R}^{+|V|}$ *such that* $\lambda^{(1)} = \lambda$.

(iii) *For all* $\overline{s} \in \mathbb{R}^{+|V|}$ *such that* $||\overline{s}||_2 = 1$ *we have* $\overline{s} \cdot f_{\overline{a}}(\overline{s})^T \leq \lambda$ *and there exists an* $\overline{r} \in \mathbb{R}^{+|V|}$ *such that* $||\overline{r}||_2 = 1$ *and* $\overline{r} \cdot f_{\overline{a}}(\overline{r})^T = \lambda$.

(iv) *If* $\overline{v} \in \mathbb{R}^{+|V|}$ *and* $\alpha \in \mathbb{R}^{+}$ *satisfy* $f_{\overline{a}}(\overline{v})^T \geq \alpha \overline{v}^T$, *then* $\alpha \leq \lambda$, *and there exists a* $\overline{v} \in \mathbb{R}^{+|V|}$ *such that* $f_{\overline{a}}(\overline{v})^T = \lambda \overline{v}^T$.

THEOREM 15.3. *We will denote* $\overline{v} = (r_0, \ldots, r_{|V|-1}) \in \mathbb{R}^{+|V|}$ *and* $f_{\overline{a}}(\overline{v}) = (R_0, \ldots, R_{|V|-1})$. *If the geometric configuration* $(G, \overline{a})$ *has the property* λ-P *then each of the following is true.*

(i) *For all* $\overline{v} \in \mathbb{R}^{+|V|}$ *there exists an index* $0 \leq i < |V|$ *such that* $R_i \leq \lambda r_i$.

(ii) *For every choice of radii* $r_0, \ldots, r_{|V|-1}$ *at the vertices of* G *there must exist at least one that does not grow more than* λ *times its original size after an application of* $f_{\overline{a}}$.

(iii)

$$\text{closure}\left(\left(\cap_{i=0}^{|V|-1} A_{i,\lambda}\right)^c\right) = \overline{\mathbb{R}^{+|V|}}.$$

THEOREM 15.4. *The geometric configuration* $(G, \overline{a})$ *has the property* λ-P *iff the intersection set* $\cap_{i=0}^{|V|-1} A_{i,\lambda}$ *is a straight ray starting at the origin. In other words,*

$$\dim\left(\cap_{i=0}^{|V|-1} A_{i,\lambda}\right) = 1.$$

In this case any point on $\cap_{i=0}^{|V|-1} A_{i,\lambda}$ *is an eigenpoint of* $f_{\overline{a}}$ *and any eigenpoint of* $f_{\overline{a}}$ *is a point on* $\cap_{i=0}^{|V|-1} A_{i,\lambda}$.

16. The range of the radii and of the eigenvalue. Let $(G, \overline{a})$ be a geometric configuration and let $f_{\overline{a}} : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$ be the corresponding $\overline{a}$-mapping. Let $\lambda^{(1)}$ be the unique eigenvalue of the mapping $f_{\overline{a}}$. We recall that, if $\overline{r} \in \mathbb{R}^{+|V|}$ is an eigenpoint of $f_{\overline{a}}$, i.e., if the following equation holds true:

$$f_{\overline{a}}(\overline{r}) = \lambda^{(1)} \overline{r},$$

then the set of all the eigenpoints of $f_{\overline{a}}$ is given by the ray

$$\{t\overline{r} \,|\, t > 0\} = \cap_{i=0}^{|V|-1} A_{i,\lambda^{(1)}}.$$

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Notation.

We will denote by $\overline{r}_{\overline{a}}$ the eigenpoint of $f_{\overline{a}}$ whose first coordinate equals the number 1. Thus $\overline{r}_{\overline{a}}$ is characterized by the equations

$$f_{\overline{a}}(\overline{r}_{\overline{a}}) = \lambda^{(1)}\overline{r}_{\overline{a}} \text{ and } \overline{r}_{\overline{a}} = (1, \ldots, r_{|V|-1}).$$

Clearly $\lambda^{(1)} = \lambda^{(1)}(G, \overline{a})$ and $\overline{r}_{\overline{a}} = \overline{r}(G, \overline{a})$ are both functions of the geometric configuration $(G, \overline{a})$. The first is a scalar-valued function with positive values and the second is a vector-valued function whose entries are positive and whose first entry equals 1. The problem we would like to address in this section is the following: What are the possible values of the functions $\lambda^{(1)}(G, \overline{a})$ and $\overline{r}(G, \overline{a})$?

The answer to this question is arithmetical in nature. We recall that the defining system of equations for $\overline{r}(G, \overline{a})$ and $\lambda^{(1)}(G, \overline{a})$ is given by

$$\sum_{k=1}^{l_i-1} \cos^{-1}\left(1 - \frac{2r_{j_i,k}r_{j_i,k+1}}{(\lambda^{(1)}r_i + r_{j_i,k})(\lambda^{(1)}r_i + r_{j_i,k+1})}\right) = a_i, \;\; 0 \le i < |V|.$$

As noted in the proof of Proposition 7.9, these equations are algebraic in $\lambda^{(1)}$ in $\overline{r}(G, \overline{a})$ and in $\cos\overline{a} = (\cos a_0, \ldots, \cos a_{|V|-1})$. One way to see this is to note the elementary fact that $\cos^{-1} X + \cos^{-1} Y = \cos^{-1} Z$ implies that up to a sign we have $XY - \sqrt{1-X^2}\sqrt{1-Y^2} = Z$ and to iterate this $l_i - 2$ times in order to get

$$\text{ALGEBRAIC EXPRESSION}\,(\lambda^{(1)}, \overline{r}(G, \overline{a})) = \cos a_i.$$

It is interesting to note that the left-hand side of this equation involves, except for the four arithmetic operations (of addition, subtraction, multiplication, and division), only the extractions of square roots. Let us discuss the form of the above algebraic expressions more carefully. The defining algebraic system is a finite set of equations that can be written in one of the following equivalent forms:

$$\sum_{j=1}^{d_k} \cos^{-1}\left\{1 - \frac{2r_j r_{j+1}}{(R + r_j)(R + r_{j+1})}\right\} = Const.$$

or equivalently

$$\sum_{j=1}^{d_k} \sin^{-1}\left\{\sqrt{\frac{r_j r_{j+1}}{(R + r_j)(R + r_{j+1})}}\right\} = Const.$$

We have used the short notation R for $\lambda^{(1)}r$. We note that, if

$$X = 1 - \frac{2r_j r_{j+1}}{(R + r_j)(R + r_{j+1})}, \qquad *$$

then

$$\sqrt{1 - X^2} = \frac{2\sqrt{r_j r_{j+1} R(R + r_{j+1} + r_j)}}{(R + r_j)(R + r_{j+1})}, \qquad **$$

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and, if

$$X = \sqrt{\frac{r_j r_{j+1}}{(R+r_j)(R+r_{j+1})}},$$

then

$$\sqrt{1-X^2} = \sqrt{\frac{R(R+r_{j+1}+r_j)}{(R+r_j)(R+r_{j+1})}}.$$

It will be convenient to use these computations in what follows. In order to obtain the exact form of the algebraic system we carry out the following computation. We fix the following notation:

$$X_j = \cos\alpha_j,\ 0 \le \alpha_j < \pi,\ 1 \le j \le m.$$

We assume that $\sum_{j=1}^m \cos^{-1} X_j = C$ and we would like to write down algebraic formulas for $\cos C$ and for $\sin C$ in terms of the X_j's. If we define two sequences as follows:

$$C_m = \cos\left(\alpha_1 + \ldots + \alpha_m\right),$$

$$S_m = \sin\left(\alpha_1 + \ldots + \alpha_m\right),$$

then we obtain the recursion relations

$$\begin{cases} C_1 = X_1, & S_1 = \sqrt{1-X_1^2}, \\ C_{m+1} = C_m X_{m+1} & - \quad S_m\sqrt{1-X_{m+1}^2}, \\ S_{m+1} = S_m X_{m+1} & + \quad C_m\sqrt{1-X_{m+1}^2}. \end{cases}$$

We can get closed forms for the sequences C_m and S_m by using the Euler formula

$$C_m + iS_m = e^{i(\alpha_1+\cdots+\alpha_m)} = \prod_{j=1}^m e^{i\alpha_j} = \prod_{j=1}^m \left(X_j + i\sqrt{1-X_j^2}\right).$$

The expansion of this last product proves the following theorem.

Theorem 16.1. *Let* $X_j = \cos\alpha_j,\ 0 \le \alpha_j < \pi,\ 1 \le j \le m.$ *Then*

$$\begin{aligned} \cos(\alpha_1+\cdots+\alpha_m) &= X_1X_2\cdots X_m \\ &- \sum_{1\le i<j\le m} (X_1\cdots X_{i-1}\sqrt{1-X_i^2}X_{i+1}\cdots X_{j-1}\sqrt{1-X_j^2}X_{j+1}\cdots X_m) \\ &+ \sum_{1\le i<j<k<l\le m} (X_1\cdots\sqrt{1-X_i^2}X_{i+1}\cdots\sqrt{1-X_j^2}\cdots\sqrt{1-X_l^2}X_{l+1}\cdots X_m) - \cdots, \\ \sin(\alpha_1+\cdots+\alpha_m) &= \sum_{1\le i\le m} (X_1\cdots X_{i-1}\sqrt{1-X_i^2}X_{i+1}\cdots X_m) \\ &- \sum_{1\le i<j<k\le m} (X_1\cdots X_{i-1}\sqrt{1-X_i^2}\cdots X_{j-1}\sqrt{1-X_j^2}\cdots X_{k-1}\sqrt{1-X_k^2}\cdots X_m) + \cdots. \end{aligned}$$

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We can now answer the question of the possible values of the functions $\lambda^{(1)}(G,\overline{a})$ and $\overline{r}(G,\overline{a})$.

THEOREM 16.2. *If $(G,\overline{a})$ is a geometric configuration, then the corresponding eigenvalue $\lambda^{(1)}$ and the set of radii $(r_0,\ldots,r_{|V|-1})$ are solutions of the following algebraic system:*

$$\begin{aligned}
&\prod_{k=1}^{l_i-1}\left(1-\frac{2r_{j_i,k}r_{j_i,k+1}}{(\lambda^{(1)}r_i+r_{j_i,k})(\lambda^{(1)}r_i+r_{j_i,k+1})}\right)\\
&\quad\times\left\{1-\sum_{1\le k<l\le l_i-1}\left(\frac{2\sqrt{r_{j_i,k}r_{j_i,k+1}\lambda^{(1)}r_i(\lambda^{(1)}r_i+r_{j_i,k}+r_{j_i,k+1})}}{(\lambda^{(1)}r_i+r_{j_i,k})(\lambda^{(1)}r_i+r_{j_i,k+1})-2r_{j_i,k}r_{j_i,k+1}}\right)\right.\\
&\quad\left.\cdot\left(\frac{2\sqrt{r_{j_i,l}r_{j_i,l+1}\lambda^{(1)}r_i(\lambda^{(1)}r_i+r_{j_i,l}+r_{j_i,l+1})}}{(\lambda^{(1)}r_i+r_{j_i,l})(\lambda^{(1)}r_i+r_{j_i,l+1})-2r_{j_i,l}r_{j_i,l+1}}\right)+\cdots\right\}\\
&\quad=\cos a_i,\quad 0\le i<|V|,\\
&\prod_{k=1}^{l_i-1}\left(1-\frac{2r_{j_i,k}r_{j_i,k+1}}{(\lambda^{(1)}r_i+r_{j_i,k})(\lambda^{(1)}r_i+r_{j_i,k+1})}\right)\\
&\quad\times\left\{\sum_{1\le k\le l_i-1}\left(\frac{2\sqrt{r_{j_i,k}r_{j_i,k+1}\lambda^{(1)}r_i(\lambda^{(1)}r_i+r_{j_i,k}+r_{j_i,k+1})}}{(\lambda^{(1)}r_i+r_{j_i,k})(\lambda^{(1)}r_i+r_{j_i,k+1})-2r_{j_i,k}r_{j_i,k+1}}\right)-\cdots\right\}\\
&\quad=\sin a_i,\quad 0\le i<|V|.
\end{aligned}$$

Proof. This follows by the defining system of equations for $\lambda^{(1)}(G,\overline{a})$ and $\overline{r}(G,\overline{a})$, by Theorem 16.1, and by equations (*) and (**) above in this section, which imply that if

$$X=1-\frac{2r_kr_{k+1}}{(R+r_k)(R+r_{k+1})}$$

then

$$\frac{\sqrt{1-X^2}}{X}=\frac{2\sqrt{r_kr_{k+1}R(R+r_k+r_{k+1})}}{(R+r_k)(R+r_{k+1})-2r_kr_{k+1}}. \qquad \square$$

An immediate consequence is the following.

COROLLARY 16.3. *If $(G,\overline{a})$ is a geometric configuration, then the corresponding eigenvalue $\lambda^{(1)}$ and the set of the normalized radii $\overline{r}(G,\overline{a})=(1,\ldots,r_{|V|-1})$ belong to the finite-field extension of the rationals $\mathbb{Q}(\cos\overline{a},\sin\overline{a})$.*

Proof. By Theorem 16.2 those numbers belong to the smallest field that contains $\mathbb{Q}$ and $\{\cos a_0,\sin a_0,\ldots,\cos a_{|V|-1},\sin a_{|V|-1}\}$. This field is $\mathbb{Q}(\cos\overline{a},\sin\overline{a})$. $\square$

REMARK 16.4. We note that the field $\mathbb{Q}(\cos\overline{a},\sin\overline{a})$ depends only on $\overline{a}$ and not on the graph embedding G.

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17. The packing property and geometry. In section 14 we gave the definition of the packing property. The geometric configuration $(G, \overline{a})$ was said to have this property iff $\lambda^{(1)}(G, \overline{a}) = 1$. We would like to point out the fact that our graph embeddings are completely arbitrary. This may result in configurations that have the packing property but for which the interpretation of a circle-packing realization is not at all clear. The origin of these nonintuitive situations lies in the fact that a completely arbitrary graph embedding cannot, in general, be embedded (in the natural sense) on a reasonable 2-manifold. To demonstrate this we will pick the family of embeddings of the complete graph on $|V|$ vertices.

Notation.

We will denote by $K_{|V|}$ any embedding of the complete graph on $|V|$ vertices.

Thus the structure of any $K_{|V|}$ is as follows. It contains $|V|$ rows. Row number i, $0 \le i < |V|$, is any permutation of the $|V|-1$ numbers $\{0, 1, \ldots, i-1, i+1, \ldots, |V|-1\}$.

REMARK 17.1. K_3 and K_4 are planar; however, by a theorem of Kuratowskii on planar graphs, $K_{|V|}$ is not planar for $|V| \ge 5$. Clearly, as $|V|$ grows it is harder to imagine how $K_{|V|}$ could be embedded on a reasonable 2-manifold.

In spite of the above remark we will prove that any $K_{|V|}$ has the packing property for some choice of the angles-parameter vector. We start with the following simple proposition.

PROPOSITION 17.2. *Let $(G, \overline{a})$ be a geometric configuration. Let us denote $\overline{a} = (a_0, \ldots, a_{|V|-1})$ and the corresponding vector of valences by $(d_0, \ldots, d_{|V|-1})$. Then the following inequality is true:*

$$\lambda^{(1)}(G, \overline{a}) \ge \frac{1}{|V|} \left\{ \sum_{a_i-\text{closed}} \frac{1}{|\sin(a_i/(2d_i))|} + \sum_{a_i-\text{open}} \frac{1}{|\sin(a_i/(2(d_i-1)))|} \right\} - 1.$$

Proof. By Theorem 13.1 we have $\lambda^{(1)} = \max_{\overline{r} \in \mathbb{R}^{+|V|}, \overline{r} \cdot \overline{r}^T = 1} \overline{r} \cdot f_{\overline{a}}(\overline{r})^T$. Hence for any $\overline{r_0} \in \mathbb{R}^{+|V|}$ such that $\overline{r_0} \cdot \overline{r_0}^T = 1$ we have the following estimate:

$$\lambda^{(1)} \ge \overline{r_0} \cdot f_{\overline{a}}(\overline{r_0})^T.$$

Let us take

$$\overline{r_0} = \frac{1}{\sqrt{|V|}}(1, 1, \ldots, 1).$$

Then to compute $f_{\overline{a}}(\overline{r_0})$ we need to find radii R_i such that at a closed angle a_i we will obtain exactly d_i equal angles, each equal to a_i/d_i, and at an open angle a_i we will obtain exactly $d_i - 1$ equal angles. The law of cosines gives us, after scaling by the factor $\sqrt{|V|}$, the equation

$$(1+1)^2 = 2(1+R_i)^2 - 2(1+R_i)^2 \cos\left(\frac{a_i}{d_i}\right)$$

or

$$R_i = \frac{1}{|\sin(a_i/(2d_i))|} - 1$$

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or

$$(1+1)^2 = 2(1+R_i)^2 - 2(1+R_i)^2 \cos\left(\frac{a_i}{d_i - 1}\right),$$

i.e.,

$$R_i = \frac{1}{|\sin(a_i/(2(d_i-1)))|} - 1.$$

Hence

$$\begin{aligned} \overline{r_0} \cdot f_{\overline{a}}(\overline{r_0}) &= \frac{1}{\sqrt{|V|}}(1,1,\dots,1) \cdot \frac{1}{\sqrt{|V|}}(\dots, \frac{1}{|\sin(a_i/(2d_i))|} - 1, \dots) \\ &= \frac{1}{|V|}\left\{ \sum_{a_i-\text{closed}} \left(\frac{1}{|\sin(a_i/(2d_i))|} - 1\right) + \sum_{a_i-\text{open}} \left(\frac{1}{|\sin(a_i/(2(d_i-1)))|} - 1\right)\right\} \\ &= \frac{1}{|V|}\left\{ \sum_{a_i-\text{closed}} \frac{1}{|\sin(a_i/(2d_i))|} + \sum_{a_i-\text{open}} \frac{1}{|\sin(a_i/(2(d_i-1)))|}\right\} - 1. \end{aligned}$$

This proves the result. □

A calculation similar to the computation made above for closed angles gives the following proposition.

Proposition 17.3.

$$\lambda(K_{|V|}, (\alpha, \alpha, \dots, \alpha)) = \frac{1}{|\sin(\alpha/(2(|V|-1)))|} - 1$$

and

$$\overline{r}(K_{|V|}, (\alpha, \alpha, \dots, \alpha)) = (1,1,\dots,1).$$

Proof. Here all the angles are closed and all the valences are identical. In fact, $d_i = |V| - 1$, $0 \leq i < |V|$. Hence the computation made in the proof of the previous proposition for closed angles gives us

$$f_{(\alpha,\alpha,\dots,\alpha)}((1,1,\dots,1)) = \left\{\frac{1}{|\sin(\alpha/(2(|V|-1)))|} - 1\right\}(1,1,\dots,1),$$

which proves the result. □

Corollary 17.4. *The geometric configuration*

$$\left(K_{|V|}, \frac{\pi}{3}(|V|-1)(1,1,\dots,1)\right)$$

has the packing property.

Proof. By Proposition 17.3 we need to solve for α so that

$$\frac{1}{|\sin(\alpha/(2(|V|-1)))|} - 1 = 1.$$

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Hence

$$\left|\sin\left(\frac{\alpha}{2(|V|-1)}\right)\right| = \frac{1}{2}$$

and so a possible solution is

$$\alpha = \frac{\pi}{3}(|V|-1). \qquad \square$$

REMARK 17.5. Strangely enough, the configuration

$$(K_7, (2\pi, 2\pi, \ldots, 2\pi))$$

seems to be flat at every vertex, with a circle-packing realization of seven congruent (i.e., identical) circles. However, it is hard to imagine K_7 embedded flat on a 2-manifold.

18. The packing theorem. Going back to the packing property or more generally to the λ-packing property, it is of interest to answer the following question: Given a graph embedding G and a positive number $\lambda > 0$, is there an angles-parameter vector $\overline{a}$ such that the geometric configuration $(G, \overline{a})$ has the λ-packing property?

That the answer is positive is the content of the main result of this section.

THEOREM 18.1. *(the packing theorem) Let G be a graph embedding and let $\lambda > 0$. Then there exists an angles-parameter vector $\overline{a}$ such that the geometric configuration $(G, \overline{a})$ has the λ-packing property.*

To prove this theorem we will use Proposition 17.2, which gives us a lower bound on $\lambda^{(1)}(G, \overline{a})$, and the following fact.

THEOREM 18.2. *Let G be a graph embedding and let $\lambda > 0$. Then there exists an angles-parameter vector $\overline{a}$ such that*

$$\lambda^{(1)}(\overline{r_0}) = \rho(f_{\overline{a}}{}'(\overline{r_0})) = \lambda, \quad \overline{r_0} = (r, r, \ldots, r),\ r > 0.$$

Moreover, an eigenvector of the nonnegative matrix $f_{\overline{a}}{}'(\overline{r_0})$ that corresponds to λ is $(1, 1, \ldots, 1)$.

Proof. Let us consider the vector $\overline{r_0} = (r, r, \ldots, r)$ for some positive $r > 0$. Using the definition of the mapping $f_{\overline{a}}$ we deduce that at a closed angle a_i we have

$$(r+r)^2 = 2(r+R_i)^2 - 2(r+R_i)^2 \cos\left(\frac{a_i}{d_i}\right)$$

or

$$R_i(\overline{r_0}) = r\left(\frac{1}{|\sin(a_i/(2d_i))|} - 1\right)$$

and at any open angle a_i we have

$$R_i(\overline{r_0}) = r\left(\frac{1}{|\sin(a_i/(2(d_i-1)))|} - 1\right).$$

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Hence for any $0 \leq i < |V|$ we have

$$\frac{dR_i(\overline{r_0})}{dr} = \frac{1}{|\sin(a_i/(2d_i))|} - 1$$

at a closed a_i and

$$\frac{dR_i(\overline{r_0})}{dr} = \frac{1}{|\sin(a_i/(2(d_i-1)))|} - 1$$

at an open a_i. An elementary fact from calculus is the following: Suppose that the function $R = f(r_0, r_1, \ldots, r_{|V|-1})$ has first-order partial derivatives. Then the function of a single variable r

$$R(r) = f(r, r, \ldots, r)$$

has a derivative with respect to r (whenever it makes sense) and

$$\frac{dR(r)}{dr} = \sum_{i=0}^{|V|-1} \frac{\partial f}{\partial r_i}(r, r, \ldots, r).$$

Hence we have proved the following formula:

$$f_{\overline{a}}{}'(\overline{r_0}) \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \vdots \\ |\sin(a_i/(2d_i))|^{-1} - 1 \\ \vdots \\ |\sin(a_j/(2(d_j-1)))|^{-1} - 1 \\ \vdots \end{pmatrix}.$$

To conclude the proof of the theorem we define $\overline{a}$ to be the solution vector of the following system:

$$\begin{cases} |\sin(a_i/(2d_i))|^{-1} - 1 = \lambda & \text{if} \quad a_i \text{ is closed,} \\ |\sin(a_j/(2(d_j-1)))|^{-1} - 1 = \lambda & \text{if} \quad a_i \text{ is open.} \end{cases}$$

There are always solutions $\overline{a}$ of this system and for these we have

$$f_{\overline{a}}{}'(\overline{r_0}) \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Since the vector $(1, 1, \ldots, 1)^T$ is positive it follows by the Perron–Frobenius theory that λ is the largest eigenvalue of the nonnegative matrix $f_{\overline{a}}{}'(\overline{r_0})$. Hence

$$\lambda^{(1)}(\overline{r_0}) = \rho(f_{\overline{a}}{}'(\overline{r_0})) = \lambda$$

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and the proof is completed. □

REMARK 18.3. We note that for a given graph embedding G, the eigenvalue $\lambda^{(1)}(G,\overline{a})$ of $f_{\overline{a}}$ is a continuous function of $\overline{a}$.

We are now in a position to give a proof of the packing theorem.

Proof. By Theorem 12.5,

$\lambda^{(1)} = \min_{\overline{s}\in\mathbb{R}^{+|V|}} \lambda^{(1)}(\overline{s})$. Hence for any $\overline{r_0} \in \mathbb{R}^{+|V|}$ we have the estimate $\lambda^{(1)} \leq \lambda^{(1)}(\overline{r_0})$. Let us take an angles-parameter vector $\overline{a_0}$ such that

$$\lambda^{(1)}(\overline{r_0}) = \rho(f_{\overline{a}}{}'(\overline{r_0})) = \frac{1}{2}\lambda.$$

Such an $\overline{a_0}$ exists by Theorem 18.2. Then $\lambda^{(1)}(G,\overline{a_0}) \leq \lambda/2$. Now using the estimate of Proposition 17.2 it is seen that we can find a sequence of angles-parameter vectors $\overline{a_j}$, $j = 1, 2, 3, \ldots$, so that

$$\lim_{j\to\infty} \lambda^{(1)}(G,\overline{a_j}) = +\infty.$$

In fact this could be done by altering a single coordinate angle. In particular we may assume that

$$\lambda \leq \lambda^{(1)}(G,\overline{a_1}).$$

Thus we have the double inequality

$$\lambda^{(1)}(G,\overline{a_0}) < \lambda \leq \lambda^{(1)}(G,\overline{a_1}).$$

Using the remark above, we can find on any path from $\overline{a_0}$ to $\overline{a_1}$ a point $\overline{a_\lambda}$ such that

$$\lambda^{(1)}(G,\overline{a_\lambda}) = \lambda. \qquad \square$$

19. The expected value of $\lambda^{(1)}(G,\overline{a})$. Let G be a graph embedding. Any angles-parameter vector $\overline{a}$ determines the geometric configuration $(G,\overline{a})$ and all of its byproducts. In particular we obtain the positive-valued function

$$\lambda^{(1)}(G,\overline{a}) : \mathbb{R}^{|V|} \to \mathbb{R}^+.$$

By the packing theorem we know that $\lambda^{(1)}(G,\overline{a})$ is a surjective function.

REMARK 19.1. In fact $\lambda^{(1)}(G,\overline{a})$ is defined only for admissible values of $\overline{a}$. Actually $(G,\overline{a})$ is a geometric configuration iff $\overline{a}$ is admissible.

We now recall what these admissible values are. As usual, we will denote by $\{0,1,\ldots,|V|-1\}$ the set of indices of the vertices of G. The vector of the corresponding valences will be denoted by $(d_0,d_1,\ldots,d_{|V|-1})$. Let us consider vertex number i, $0 \leq i < |V|$. It has d_i neighboring vertices arranged counterclockwise according to the embedding G. The angle made by any two subsequent neighbors and the vertex i itself is strictly less than π in absolute value. Hence the total angle about vertex i is strictly less than (in absolute value) $d_i\pi$ or $(d_i-1)\pi$ according to whether the

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angle at i is closed or open. Similarly, this angle is never 0. Hence, the set of all the admissible values for a_i is

$$0 < |a_i| < \begin{cases} d_i\pi, & \text{closed angle,} \\ (d_i - 1)\pi, & \text{open angle.} \end{cases}$$

If we use the notation

$$l_i = \left\{ \begin{array}{ll} d_i, & \text{closed angle} \\ d_i - 1, & \text{open angle} \end{array} \right\},$$

then we can write the set of admissible values for a_i as the punctured interval $(-l_i\pi, l_i\pi)^\times = (-l_i\pi, l_i\pi) - \{0\}$, $0 \le i < |V|$. Thus the set of all the admissible angles-parameter vectors $\overline{a}$ is the Cartesian product

$$\prod_{i=0}^{|V|-1} (-l_i\pi, l_i\pi)^\times.$$

Now we can write more accurately

$$\lambda^{(1)}(G, \overline{a}) \,:\, \prod_{i=0}^{|V|-1} (-l_i\pi, l_i\pi)^\times \to \mathbb{R}^+.$$

As mentioned above this is a surjective function. We will use the notation

$$S(G) = \prod_{i=0}^{|V|-1} (-l_i\pi, l_i\pi)^\times.$$

Computing experiments indicate that $\lambda^{(1)}(G, \overline{a})$ is not uniformly distributed in $\overline{a}$. In other words when one chooses $\overline{a}$ uniformly in $S(G)$, the values of $\lambda^{(1)}(G, \overline{a})$ are not uniformly distributed in $\mathbb{R}^+$. There will be some values that are more likely to be taken by $\lambda^{(1)}(G, \overline{a})$ than other values. If we think of $\lambda^{(1)}(G, \overline{a})$ as a random variable defined over the sample space $S(G)$, then we can adopt the following standard definition.

Definition 19.2. The *expected value of the eigenvalue* $\lambda^{(1)}(G)$ is defined by

$$E(\lambda^{(1)}(G)) = \int_{S(G)} \lambda^{(1)}(G, \overline{a}) d\overline{a}.$$

It turns out that this notion is not very interesting in our context. An immediate consequence of Proposition 17.2 is the following theorem.

Theorem 19.3. *For every graph embedding G, we have*

$$E(\lambda^{(1)}(G)) = +\infty.$$

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Proof. By Proposition 17.2 we have the inequality

$$\lambda^{(1)}(G,\overline{a}) \geq \frac{1}{|V|}\left\{\sum_{i=0}^{|V|-1}\frac{1}{|\sin(a_i/(2l_i))|}\right\} - 1.$$

Integrating this over the sample space gives us

$$\begin{aligned} E(\lambda^{(1)}(G)) &= \int_{S(G)} \lambda^{(1)}(G,\overline{a})d\overline{a} \\ &\geq \int_{S(G)}\left(\frac{1}{|V|}\left\{\sum_{i=0}^{|V|-1}\frac{1}{|\sin(a_i/(2l_i))|}\right\} - 1\right)d\overline{a} = +\infty. \qquad \square \end{aligned}$$

Clearly, Proposition 17.2 indicates that $\lambda^{(1)}(G,\overline{a})$ has singularities whenever $a_i \to 0^+$ for some $0 \leq i < |V|$. These singularities are at least as wild as

$$\frac{1}{\min_{0\leq i<|V|}|\sin(a_i/(2l_i))|} \approx \frac{c}{\min_{0\leq i<|V|}|a_i|}.$$

Naturally we would like to estimate the growth of $\lambda^{(1)}(G,\overline{a})$ near the faces $a_i = 0$. It turns out that the above order of magnitude is the correct one.

THEOREM 19.4. *Let $(G,\overline{a})$ be a geometric configuration. Let us denote $\overline{a} = (a_0,\ldots,a_{|V|-1})$ and the corresponding vector of valences by $(d_0,\ldots,d_{|V|-1})$. Let $(l_0,\ldots,l_{|V|-1})$ be the vector of the adjusted valences (i.e., $l_i = d_i$ or $d_i - 1$ according to whether the angle at vertex i is closed or open, respectively). Then the following estimate holds:*

$$\frac{1}{|V|}\sum_{i=0}^{|V|-1}\frac{1}{|\sin(a_i/(2l_i))|} - 1 \leq \lambda^{(1)}(G,\overline{a}) \leq \max_{0\leq i<|V|}\left\{\frac{1}{|\sin(a_i/(2l_i))|} - 1\right\}.$$

Proof. The lower bound for $\lambda^{(1)}(G,\overline{a})$ was evaluated in Proposition 17.2. Thus it remains to prove the right inequality (the upper bound). By Theorem 12.5 we have $\lambda^{(1)}(G,\overline{a}) = \min_{\overline{s}\in\mathbb{R}^{+|V|}}\lambda^{(1)}(\overline{s})$, where $\lambda^{(1)}(\overline{s})$ is the largest eigenvalue of the nonnegative, symmetric, and irreducible matrix $f_{\overline{a}}'(\overline{s})$. In fact by Perron–Frobenius we have $\lambda^{(1)}(\overline{s}) = \rho(f_{\overline{a}}'(\overline{s}))$, the spectral radius. Thus for any $\overline{s} \in \mathbb{R}^{+|V|}$ we have the inequality $\lambda^{(1)}(G,\overline{a}) \leq \lambda^{(1)}(\overline{s})$. It is well known that if A is any $n\times n$ complex matrix and if λ is any eigenvalue of A, then $\lambda \leq ||A||$ for any matrix norm $||\cdot||$ on A. The one-line proof is that, if $\overline{u} \neq \overline{0}$ is an eigenvector of A corresponding to λ, then $||A\overline{u}|| = |\lambda|\,||\overline{u}||$. In particular $|\lambda| \leq ||A||_\infty$. It is also well known that $||A||_\infty = \max_{1\leq i\leq n}\sum_{j=1}^n |a_{ij}|$ [10] (page 41 equation (12)). Combining the above inequalities we deduce that for any $\overline{s} \in \mathbb{R}^{+|V|}$ we have the estimate

$$\lambda^{(1)}(G,\overline{a}) \leq \max_{0\leq i<|V|}\sum_{j=0}^{|V|-1}\frac{\partial R_i(\overline{s})}{\partial r_j},$$

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where we denote $f_{\overline{a}}(\overline{r}) = (R_0(\overline{r}), \ldots, R_{|V|-1}(\overline{r}))$, $\overline{r} = (r_0, \ldots, r_{|V|-1})$. In particular, if we take $\overline{s} = \overline{r}_0 = (r, r, \ldots, r)$, then the computation in the proof of Theorem 18.2 shows that

$$\sum_{j=0}^{|V|-1} \frac{\partial R_i(\overline{r}_0)}{\partial r_j} = \frac{1}{|\sin(a_i/(2l_i))|} - 1$$

and hence we obtain the inequality

$$\lambda^{(1)}(G, \overline{a}) \leq \max_{0 \leq i < |V|} \left\{ \frac{1}{|\sin(a_i/(2l_i))|} - 1 \right\}. \qquad \square$$

An immediate consequence of the last theorem is the following tight estimate.

THEOREM 19.5.

$$\lambda^{(1)}(G, \overline{a}) = \Omega\left(\frac{1}{\min_{0 \leq i < |V|} |a_i|} \right).$$

The constants depend only on $|V|$ and on the valences $(d_0, \ldots, d_{|V|-1})$ but not on any particular embedding of G.

Proof. By Theorem 19.4 it follows that there are two positive constants $c_1 \leq c_2$ such that

$$\frac{c_1}{\min_{0 \leq i < |V|} |\sin(a_i/(2l_i))|} \leq \lambda^{(1)}(G, \overline{a}) \leq \frac{c_2}{\min_{0 \leq i < |V|} |\sin(a_i/(2l_i))|}.$$

For small $|x|$ we have

$$\sin\left(\frac{x}{2l}\right) \approx \frac{x}{2l}$$

from which we deduce the result. $\square$

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