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POSITIVE AND RE-POSITIVE SOLUTIONS TO SOME SYSTEMS OF ADJOINTABLE OPERATOR EQUATIONS OVER HILBERT C^*– MODULES[*]

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Abstract. A necessary and sufficient condition for the existence of the general common positive solution to equations

$$A_1 X = C_1, \ \ XB_2 = C_2, \ \ A_3 X A_3^* = C_3, \ \ A_4 X A_4^* = C_4$$

for operators between Hilbert C^*-modules is established, and an expression for the common positive solution to the equations is derived when the solvability conditions are satisfied. As an application, a new necessary and sufficient condition for the system of adjointable operator equations

$$AX = C, \ \ XB = D$$

over Hilbert C^*-modules to have a common Re-positive solution is proved. Moreover, an expression of the general Re-positive solution is derived when the consistent conditions are met. The results of this paper extend some known results in the literature.

Key words. Hilbert C^*-module, Moore-Penrose inverse, Operator equation, Positive solution, Re-positive solution.

AMS subject classifications. 47A62, 47B15, 47B65, 15A09, 15A24.

1. Introduction. Hermitian positive semidefinite solutions to some matrix equations or some operator equations were investigated by many authors. For finite matrices, Khatri and Mitra [12] gave necessary and sufficient conditions for the existence of a common positive semidefinite solution to equations

$$AX = C, \ \ XB = D \tag{1.1}$$

over the complex field $\mathbb{C}$, and presented an explicit expression for the general Hermitian positive semidefinite solutions to system (1.1) by generalized inverses, when the

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solvability conditions were satisfied. Baksalary [1] and Groß [9] studied the nonnegative definite and positive definite solutions to matrix equation

$$AXA^* = B, \tag{1.2}$$

respectively. In 2007, Cvetković-Ilić et al. [2] investigated the positive solution to equation (1.2) in C^*-algebras. As a generation of equation (1.2), Zhang [26] derived a expression of the general nonnegative definite solution to system of matrix equations

$$A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4 \tag{1.3}$$

over the complex field $\mathbb{C}$. Some other results can be found in [4–6,15,16,19,24,25].

Meanwhile, the Re-positive solution to some matrix equations or some operator equations is also active. Wu [22] studied Re-positive solutions of

$$AX = C. \tag{1.4}$$

Wu and Cain [23] found the set of all complex Re-nnd (Re-nonnegative definite) matrices X satisfied

$$XB = D$$

and presented a criterion for Re-nndness. Groß [8] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [23]. Beside these papers, many other papers have dealt with the problem of finding the Re-nnd and Re-pd (Re-positive definite) solutions of some other forms of equations (see e.g. [3,20]). Dajić and Koliha [5] reviewed system (1.1) from a new perspective by studying them in the setting of associative rings with or without involution and derived a general form of Re-positive solutions of equation (1.4) over the Hilbert spaces. Unfortunately, the investigation of the common positive solutions cannot be applied to the common Re-positive solutions to system (1.1), since that method does not work for solutions which are not necessarily Hermitian.

Recently, Wang and Wu [18] presented an expression for general Hermitian solution to system of equations

$$A_1X = C_1, \quad XB_1 = C_2, \quad A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4 \tag{1.5}$$

over Hilbert C^*-modules. To our knowledge, so far there has been little information on either the common positive solution to (1.5) for operators in the framework of Hilbert C^*-modules, or the Re-positive solution to system (1.1) of matrix equations over the complex field and adjointable operator equations over Hilbert C^*-modules.

Motivated by the work mentioned above, we in this paper aim to consider the necessary and sufficient conditions for a system of adjointable operator equations (1.5)

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(or system (1.1)) to have a positive solution (Re-positive solution), as well as present an expression for the general positive solutions (Re-positive solutions) to this system when the consistent conditions are satisfied.

The paper is organized as follows. In Section 2, we begin with some basic concepts and results about adjointable operators and generalized inverse of adjointable operators over Hilbert C^*-modules. In Section 3, we give a necessary and sufficient condition for the existence of a positive solution to system (1.5) of adjointable operator equations over Hilbert C^*-modules. When the solvability conditions are met, we present an expression for the general positive solution to system (1.1). As an application, in Section 4, we show a necessary and sufficient condition for system (1.1) of adjointable operator equations over Hilbert C^*-modules to have a common Re-positive solution, and derive an expression of the general Re-positive solution when the consistent conditions are met. To conclude this paper, in Section 5, we propose some further research topics.

2. Preliminaries. Hilbert C^*-modules arose as generalizations of the notion of Hilbert space. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in a C^*-algebra. The structure was first used by Kaplansky [11] in 1952. For more details of C^*-algebra and Hilbert C^*-modules, we refer the readers to [13] and [21].

Let $\mathfrak{A}$ be a C^*-algebra. An inner-product $\mathfrak{A}$-module is a linear space E which is a right $\mathfrak{A}$-module (with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in E, a \in \mathfrak{A}, \lambda \in \mathbb{C}$), together with a map $E \times E \to \mathfrak{A}, (x, y) \to \langle x, y \rangle$ such that

(1) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$;

(2) $\langle x, ya \rangle = \langle x, y \rangle a$;

(3) $\langle x, y \rangle = \langle y, x \rangle^*$;

(4) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

for any $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$. An inner-product $\mathfrak{A}$-module E is called a *(right) Hilbert $\mathfrak{A}$-module* if it is complete with respect to the induced norm $||x|| = |\langle x, x \rangle|^{1/2}$.

Assume that $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert $\mathfrak{A}$-modules, and $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is the set of all maps $T: \mathcal{H} \to \mathcal{K}$ for which there is a map $T^*: \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We know that any element T of $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ is a bounded linear operator. We call $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ the set of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$. In case $\mathcal{H} = \mathcal{K}$, $\mathfrak{B}(\mathcal{H}, \mathcal{H})$, which we abbreviate to $\mathfrak{B}(\mathcal{H})$, is a C^*-algebra and we use the notation $I_{\mathcal{H}}$ to denote the identity operator on $\mathcal{H}$. For any $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, the notation $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the range of A and the null space of A, respectively. An

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operator $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ is regular if there is an operator $A^- \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that $AA^-A = A$, A^- is called an inner inverse of A. It is well known that A is regular if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, are closed and complemented subspaces of $\mathcal{K}$ and $\mathcal{H}$.

An operator $A \in \mathfrak{B}(\mathcal{H})$ is called Hermitian (or self-adjoint) if $A^* = A$, and positive if $\langle Ax, x \rangle \geq 0$ for all $x \in H$, we write $A \geq 0$ if A is positive. The set $\mathfrak{B}(\mathcal{H})^+$ of the positive operators is a subset of the Hermitian operators. For $A \geq 0$, $A^{\frac{1}{2}}$ denotes the positive operator satisfies $X^2 = A$. An operator $A \in \mathfrak{U}(\mathcal{H})$ is called unitary if $A^*A = AA^* = I_{\mathcal{H}}$. The real part (or Hermitian part) of $A \in \mathfrak{B}(\mathcal{H})$ is defined by $Re\,(A) = \frac{1}{2}\,(A + A^*)$, $Re\,(A)$ is always Hermitian, $Re\,(A) = A$ if A is Hermitian, and $Re\,(A) = 0$ if A is skew-Hermitian. Further, $Re\,(A \pm B) = Re\,(A) \pm Re\,(B)$, $Re\,(X^*AX) = X^*Re\,(A)\,X$. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called Re-positive if $Re\,(A) \geq 0$.

Let $\mathcal{H}, \mathcal{K}$ be two Hilbert $\mathfrak{A}$-modules, $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$. The Moore-Penrose inverse $A^\dagger$ of A (if it exists) is defined as the unique element of $\mathfrak{B}(\mathcal{K}, \mathcal{H})$ which satisfies the following four Penrose equations

$$AXA = A, \ \ XAX = X, \ \ (AX)^* = AX, \ \ (XA)^* = XA.$$

For any $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, the Moore-Penrose inverse $A^\dagger$ of A exists if and only if A has closed range. In this case, $A^\dagger$ exists uniquely and $(A^*)^\dagger = (A^\dagger)^*$. If a regular operator A is positive, then $A^\dagger \geq 0$ and $AA^\dagger = A^\dagger A$. Moreover, both $P_A = A^\dagger A$ and $Q_A = AA^\dagger$ are idempotent and self-adjoint. For convenience, we use notations $\mathcal{L}_A$ and $\mathcal{R}_A$ to stand for $I_{\mathcal{H}} - A^\dagger A$ and $I_{\mathcal{K}} - AA^\dagger$ induced by A, respectively. Obviously, $\mathcal{L}_A$ and $\mathcal{R}_A$ are also idempotent and self-adjoint and $\mathcal{L}_A = \mathcal{R}_{A^*}$. For other important properties of operators and generalized inverses of operators, see [10] and [14].

3. Positive solution to system (1.5) of adjointable operator equations. In this Section, we mainly study some necessary and sufficient conditions for system (1.5) of adjointable operator equations to have a positive solution over the Hilbert C^*-modules. We begin this section with the following lemmas, which can be deduced from [24].

LEMMA 3.1. *(Theorem 2.1 in [24]) Let $A, C \in \mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$, A and CA^* have closed ranges. Then the adjointable operator equation $AX = C$ has a positive solution $X \in \mathfrak{B}(\mathcal{H}_1)$ if and only if $CA^* \geq 0, R(C) \subseteq R(CA^*)$. In this case, the general positive solution is given by*

$$X = C^*(CA^*)^-C + \mathcal{L}_A S \mathcal{L}_A^*,$$

where $S \in \mathfrak{B}(\mathcal{H}_1)^+$ is arbitrary, $C^*(CA^*)^-C$ is a particular positive solution to $AX = C$, independent of the choice of the inner inverse $(CA^*)^-$.

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LEMMA 3.2. (Theorem 3.7 in [24]) *Let* $\mathcal{H}, \mathcal{K}, \mathcal{L}$ *be Hilbert* C^**-modules, and let* $A_1, C_1 \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, $B_2, C_2 \in \mathfrak{B}(\mathcal{L}, \mathcal{H})$,

$$D = \left[\begin{array}{c} A_1 \\ B_2^* \end{array} \right], \ \ E = \left[\begin{array}{c} C_1 \\ C_2^* \end{array} \right], \ \ F = \left[\begin{array}{cc} C_1 A_1^* & C_1 B_2 \\ (A_1 C_2)^* & C_2^* B_2 \end{array} \right]$$

such that D, F *are regular. Then system (1.1) has a positive solution* $X \in \mathfrak{B}(\mathcal{H})$ *if and only if* F *is positive and* $\mathcal{R}(E) \subseteq \mathcal{R}(F)$. *In this case, the general positive solution of system (1.1) can be expressed as*

$$X = E^* F^- E + \mathcal{L}_D T \mathcal{L}_D^*,$$

where $T \in \mathfrak{B}(\mathcal{H})^+$ *is arbitrary.*

In 2000, Groß in [9] presented a solvability condition for matrix equation (1.2), and derived an expression for the general positive solution to (1.2), which can be generalized into Hilbert C^*-module.

LEMMA 3.3. (Corollary 2.3 in [9]) *Let* H, K *be Hilbert* C^**-modules. Assume that* $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, $C \in \mathfrak{B}(\mathcal{K})$ *such that* A *has closed range,* C *is Hermitian and* $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. *Then equation (1.2) has a positive solution* $X \in \mathfrak{B}(\mathcal{H})$ *if and only if* C *is positive. If, in addition,* C *has closed range, then the general positive solution of (1.2) can be expressed as*

$$X = \left[A^- B + \left(I_n - A^- A \right) Y \right] \left[A^- B + \left(I_n - A^- A \right) Y \right]^*$$

where Y *is an arbitrary operator in* $\mathfrak{B}(\mathcal{H})$, B *is an arbitrary operator with* $C = BB^*$.

It is easy to verify the following Lemma.

LEMMA 3.4. *Given operators* $A, C \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$. *Then* $A^*A = C^*C$ *if and only if* $A = CT$ *for some unitary operator* T.

THEOREM 3.5. *Let* $A_1, C_1 \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$, $B_2, C_2 \in \mathfrak{B}(\mathcal{K}_2, \mathcal{H})$, $A_3 \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_3)$, $A_4 \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_4)$, $C_3 \in \mathfrak{B}(\mathcal{K}_3)$, $C_4 \in \mathfrak{B}(\mathcal{K}_4)$ *be given such that system of adjointable operator equations*

$$A_1 X = C_1, \ \ X B_2 = C_2, \ \ A_3 X A_3^* = C_3, \ \ A_4 X A_4^* = C_4 \qquad (3.1)$$

is consistent. Denote

$$M = \left[\begin{array}{cc} C_1 A_1^* & C_1 B_2 \\ C_2^* A_1^* & C_2^* B_2 \end{array} \right], \ \ X_0 = \left[\begin{array}{cc} C_1^* & C_2 \end{array} \right] M^- \left[\begin{array}{c} C_1 \\ C_2^* \end{array} \right], \ \ A_{11} = \left[\begin{array}{c} A_1 \\ B_2^* \end{array} \right],$$

$$C_{33} = C_3 - A_3 X_0 A_3^*, \ \ C_{44} = C_4 - A_4 X_0 A_4^*, \ \ A_{33} = A_3 L_{A_{11}}, \ \ A_{44} = A_4 L_{A_{11}}.$$

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Suppose that $A_1, B_2, A_3, A_4, A_{33}, A_{44}, C_{33}, C_{44}$ and M have closed ranges. Then the following statements are equivalent:

(1) the consistent system (3.1) of adjointable operator equations has a positive solution in $\mathfrak{B}(\mathcal{H})^+$,

(2) $M, C_3, C_4, C_{33}, C_{44}$ are positive,

$$\mathcal{R}\left(\begin{bmatrix} C_1 \\ C_2^* \end{bmatrix}\right) \subseteq \mathcal{R}(M), \;\; \mathcal{R}(C_{33}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{33}), \;\; \mathcal{R}(C_{44}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{44}), \qquad (3.2)$$

and there exist an unitary operator T such that

$$A_{44}\mathcal{L}_{A_{33}}(A_{44}\mathcal{L}_{A_{33}})^- \left(C_{44}^{\frac{1}{2}}T - A_{44}A_{33}^- C_{33}^{\frac{1}{2}}\right) = \left(C_{44}^{\frac{1}{2}}T - A_{44}A_{33}^- C_{33}^{\frac{1}{2}}\right). \qquad (3.3)$$

In this case, the general positive solutions can be expressed as

$$X = X_0 + \mathcal{L}_{A_{11}}\left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}}Y\right)\left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}}Y\right)^*(\mathcal{L}_{A_{11}})^*, \qquad (3.4)$$

where

$$Y = (A_{44}\mathcal{L}_{A_{33}})^- \left(C_{44}^{\frac{1}{2}}T - A_{44}A_{33}^- C_{33}^{\frac{1}{2}}\right) + W - (A_{44}\mathcal{L}_{A_{33}})^-(A_{44}\mathcal{L}_{A_{33}})W, \qquad (3.5)$$

with W is free to vary over $\mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$.

Proof. $(1) \Rightarrow (2)$. Suppose that system (3.1) of adjointable operator equations has a positive solution X, then it follows from Lemma 3.2-3.3 that C_3, C_4, M are positive, and

$$\mathcal{R}\left(\begin{bmatrix} C_1 \\ C_2^* \end{bmatrix}\right) \subseteq \mathcal{R}(M).$$

Obviously, X satisfies system of adjointable operator equations

$$A_1 X = C_1, \;\; X B_2 = C_2, \qquad (3.6)$$

thus there exist an operator $V \in \mathfrak{B}(H_1)^+$ such that

$$X = X_0 + \mathcal{L}_D V \mathcal{L}_D^*. \qquad (3.7)$$

Taking (3.7) into $A_3 X A_3^* = C_3$ yields that

$$A_3 \mathcal{L}_{A_{11}} V (A_3 \mathcal{L}_{A_{11}})^* = C_3 - A_3 X_0 A_3^* \qquad (3.8)$$

has a positive solution with respect to V. By Lemma 3.3, we have

$$C_{33} = C_3 - A_3 X_0 A_3^* \geq 0, \;\; \mathcal{R}(C_{33}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_3 \mathcal{L}_{A_{11}}) = \mathcal{R}(A_{33}),$$

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and V can be expressed as

$$V = \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right)^*, \tag{3.9}$$

with some $Y \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$. Taking (3.7) with (3.9) into $A_4 X A_4^* = C_4$ gives

$$A_4 \left(X_0 + \mathcal{L}_{A_{11}} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right)^* (\mathcal{L}_{A_{11}})^* \right) A_4^* = C_4.$$

Then $C_{44} = C_4 - A_4 X_0 A_4^* \geq 0$ and $\mathcal{R}(C_{44}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_4 \mathcal{L}_{A_{11}}) = \mathcal{R}(A_{44})$. Moreover, it follows Lemma 3.4 that there exist an unitary operator T such that

$$A_{44} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) = C_{44}^{\frac{1}{2}} T.$$

Then equation

$$A_{44} \mathcal{L}_{A_{33}} Y = C_{44}^{\frac{1}{2}} T - A_{44} A_{33}^- C_{33}^{\frac{1}{2}}$$

is consistent with respect to Y, from which we can get (3.3).

(2) $\Rightarrow$ (1). Since M is positive and $\mathcal{R}(M) \subseteq \mathcal{R}\left(\begin{bmatrix} C_1 \\ C_2^* \end{bmatrix} \right)$, then system (3.6) has a positive solution which can be expressed as (3.7). Taking (3.7) into $A_3 X A_3^* = C_3$, and combining C_{33}, C_{44} are positive, $\mathcal{R}(C_{33}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{33})$, we have equation (3.8) has a positive solution with respect to V, which can be written as

$$V = \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right)^*.$$

Thus, equations

$$A_1 X = C_1, \quad X B_2 = C_2, \quad A_3 X A_3^* = C_3$$

have a common positive solution

$$X = \begin{bmatrix} C_1^* & C_2 \end{bmatrix} M^- \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix} \tag{3.10}$$
$$+ \mathcal{L}_{A_{11}} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right)^* \mathcal{L}_{A_{11}}^*,$$

where Y free to vary over $\mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$. Next we will show that (3.10) with (3.5) satisfies $A_4 X A_4^* = C_4$. Choosing Y as (3.5) and by direct computation we have

$$A_4 \left(X_0 + \mathcal{L}_{A_{11}} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right)^* (\mathcal{L}_{A_{11}})^* \right) A_4^*$$
$$= A_4 X_0 A_4^* + \left(A_{44} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \right) \left(A_{44} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} Y \right) \right)^*$$
$$= A_4 X_0 A_4^* + C_{44}^{\frac{1}{2}} T \left(C_{44}^{\frac{1}{2}} T \right)^* = C_4.$$

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Therefore, (3.10) is a positive solution to system (3.1).

From the above proofs, we get that (3.4) with (3.5) is a positive solution to system (3.1). Next we show that every positive solution of system (3.1) can be expressed as (3.4) with (3.5). Suppose that X_1 is an arbitrary positive solution of system (3.1), then X_1 is a positive solution of system of adjointable operator equations (3.6). It follows from Lemma 3.2 that there exists a positive V_0 such that

$$X_1 = \begin{bmatrix} C_1^* & C_2 \end{bmatrix} M^- \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix} + \mathcal{L}_{A_{11}} V_0 \mathcal{L}_{A_{11}}^*.$$

Setting $W = V_0^{\frac{1}{2}}$, we have

$$Y = (A_{44}\mathcal{L}_{A_{33}})^- \left(C_{44}^{\frac{1}{2}} T - A_{44} A_{33}^- C_{33}^{\frac{1}{2}} \right) + V_0^{\frac{1}{2}} - (A_{44}\mathcal{L}_{A_{33}})^- (A_{44}\mathcal{L}_{A_{33}}) V_0^{\frac{1}{2}} = V_0^{\frac{1}{2}}.$$

Taking it into (3.4), we can get

$$X_1 = \begin{bmatrix} C_1^* & C_2 \end{bmatrix} M^- \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix} + \mathcal{L}_{A_{11}} \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} V_0^{\frac{1}{2}} \right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + \mathcal{L}_{A_{33}} V_0^{\frac{1}{2}} \right)^* (\mathcal{L}_{A_{11}})^*.$$

Thus, (3.4) with (3.5) is the general positive solution of system (3.1). □

We now can consider a special case of Theorem 3.5.

COROLLARY 3.6. *Let $A_3 \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_3)$, $A_4 \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_4)$, $C_3 \in \mathfrak{B}(\mathcal{K}_3)$ and $C_4 \in \mathfrak{B}(\mathcal{K}_4)$ be given such that system of adjointable operator equations*

$$A_3 X A_3^* = C_3, \quad A_4 X A_4^* = C_4 \tag{3.11}$$

is consistent. Suppose A_3 and A_4 have closed range. Then the following statements are equivalent:
(1) System (3.11) has a positive solution.
(2) C_3, C_4 are positive

$$\mathcal{R}(C_3^{\frac{1}{2}}) \subseteq \mathcal{R}(A_3), \quad \mathcal{R}(C_3^{\frac{1}{2}}) \subseteq \mathcal{R}(A_4),$$

and there exist a $T \in \mathfrak{U}(\mathcal{H})$ such that

$$(A_4 L_{A_3})^- A_4 L_{A_3} \left(C_4^{\frac{1}{2}} T - A_4 A_3^- C_3^{\frac{1}{2}} \right) = \left(C_4^{\frac{1}{2}} T - A_4 A_3^- C_3^{\frac{1}{2}} \right). \tag{3.12}$$

In this case, the general positive solution of system (3.11) can be expressed as

$$X = \left(A_3^- C_3^{\frac{1}{2}} + \mathcal{L}_{A_3} Y \right) \left(A_3^- C_3^{\frac{1}{2}} + \mathcal{L}_{A_3} Y \right)^* \tag{3.13}$$

with

$$Y = (A_4 \mathcal{L}_{A_3})^- \left(C_4^{\frac{1}{2}} T - A_4 A_3^- C_3^{\frac{1}{2}} \right) + W - (A_4 \mathcal{L}_{A_3})^- A_4 \mathcal{L}_{A_3} W,$$

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where W is free to vary over $\mathfrak{B}(\mathcal{H})$.

REMARK 3.7. For finite complex matrices, Theorem 3.5 can be seen as a complementarity of Zhang [26] by adding the proof that every positive solution of system (3.11) can be expressed as the form (3.13) with proper choice of W.

4. Re-positive solution to system (1.1) of adjointable operator equations. In order to establish some necessary and sufficient conditions for system (1.1) of adjointable operator equations to have a Re-positive solution over the Hilbert C^*-modules, we need the following lemma which is due to Wang et al. [17].

LEMMA 4.1. *Let $A \in \mathfrak{B}(\mathcal{K}_1, \mathcal{H}), B \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_2), C \in \mathfrak{B}(\mathcal{K}_3, \mathcal{H})$ and $D \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_4)$ be given. Denote $M = R_A C, N = DL_B$ and $S = CL_M$. Suppose that $A, B, C, D, E,$ $M, N, S, R_A E$ and EL_B have closed ranges. Then*
(a) there exist $X \in \mathfrak{B}(\mathcal{K}_2, \mathcal{K}_1), Y \in \mathfrak{B}(\mathcal{K}_3, \mathcal{K}_4)$ such that

$$AXB + CYD = E$$

if and only if

$$\mathcal{R}(E) \subseteq \mathcal{R}\begin{bmatrix} A & C \end{bmatrix}, \ \ \mathcal{R}(E^*) \subseteq \mathcal{R}\left(\begin{bmatrix} B^* & D^* \end{bmatrix}\right)^*,$$

$$\mathcal{R}(E) \subseteq \mathcal{R}(A) \oplus \mathcal{R}(D^*), \ \ \mathcal{R}(E) \subseteq \mathcal{R}(C) \oplus \mathcal{R}(B^*),$$

or equivalently,

$$R_M R_A E = 0, \ \ R_A E L_D = 0, \ \ E L_B L_N = 0, \ \ R_C E L_B = 0.$$

In this case, the general solution can be expressed as

$$X = A^\dagger E B^\dagger - A^\dagger C M^\dagger R_A E B^\dagger - A^\dagger S C^\dagger E L_B N^\dagger D B^\dagger - A^\dagger S V R_N D B^\dagger + L_A U + Z R_B,$$
$$Y = M^\dagger R_A E D^\dagger + L_M S^\dagger S C^\dagger E L_B N^\dagger + L_M \left(V - S^\dagger S V N N^\dagger\right) + W R_D,$$

where U, V, W, Z are arbitrary.

LEMMA 4.2. *Let $A \in \mathfrak{B}(\mathcal{K}_1, \mathcal{H}), B \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_2)$ and $C \in \mathfrak{B}(\mathcal{H})$ be given. Denote $M = R_A B^*, N = A^* L_B$ and $S = B^* L_M$. Suppose that A, B, C, M, N and S have closed ranges, then*
(a) there exist an $X \in \mathfrak{B}(\mathcal{K}_2, \mathcal{K}_1)$ such that

$$AXB + (AXB)^* = C, \tag{4.1}$$

if and only if

$$C^* = C, \ \ \mathcal{R}(C) \subseteq \mathcal{R}\begin{bmatrix} A & B^* \end{bmatrix}, \tag{4.2}$$

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$$\mathcal{R}(C) \subseteq \mathcal{R}(A) \oplus \mathcal{R}(A^*), \;\; \mathcal{R}(C) \subseteq \mathcal{R}(B) \oplus \mathcal{R}(B^*), \tag{4.3}$$

or equivalently,

$$C^* = C, \;\; \begin{bmatrix} A & B^* \end{bmatrix} \begin{bmatrix} A & B^* \end{bmatrix}^\dagger C = C$$

and $R_A C R_A = 0$, $L_B C L_B = 0$.

(b) In this case, the general solution of (4.1) can be expressed as

$$X = \frac{1}{2}\left(U + V^*\right), \tag{4.4}$$

where U and V are the general solutions of equation

$$AUB + B^*VA^* = C.$$

Written in an explicit form

$$U = A^\dagger C B^\dagger - (B^*)^\dagger M^\dagger R_A C B^\dagger \tag{4.5}$$
$$- A^\dagger S (B^*)^\dagger C L_B N^\dagger A^* B^\dagger - A^\dagger S V R_N A^* B^\dagger + L_A U + Z R_B,$$

$$V = M^\dagger R_A C (A^*)^\dagger + L_M S^\dagger S (B^*)^\dagger C L_B N^\dagger \tag{4.6}$$
$$+ L_M \left(V - S^\dagger S V N N^\dagger\right) + W R_{A^*},$$

where U, V, W, Z are arbitrary.

Proof. Suppose X satisfies (4.1). Then it is easy to see that C is Hermitian. On the other hand, if there is an X satisfying (4.1), then there are X, Y satisfying $AXB + B^*YA^* = C$. Hence by Lemma 4.1, we have

$$\mathcal{R}(C) \subseteq \mathcal{R}\begin{bmatrix} A & B^* \end{bmatrix}, \;\; \mathcal{R}(C) \subseteq \mathcal{R}(A) \oplus \mathcal{R}(A^*), \;\; \mathcal{R}(C) \subseteq \mathcal{R}(B) \oplus \mathcal{R}(B^*).$$

Conversely, if (4.2) and (4.3) hold, then there exist U and V such that

$$AUB + B^*VA^* = C.$$

Taking conjugate transpose for this equality gives

$$B^*U^*A^* + AV^*B = C.$$

Adding these two equalities and dividing by 2 gives

$$A\left(\frac{U + V^*}{2}\right)B + B^*\left(\frac{U + V^*}{2}\right)^* A^* = C.$$

This equality implies that for any pair of solutions of $AUB + B^*VA^* = C$ the expression (4.4) is a solution to (4.1). Suppose X_0 is any solution to (4.1). Then $AUB + B^*VA^* = C$ has solutions $U = V^* = X_0$. Thus, X_0 can be expressed as

$$X_0 = \frac{1}{2}\left(X_0 + X_0\right) = \frac{1}{2}\left(U + V^*\right).$$

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From Lemma 4.1, the general solutions of can be written as (4.4) with (4.5) and (4.6). □

THEOREM 4.3. *Let $A, C \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$, $B, D \in \mathfrak{B}(\mathcal{K}_2, \mathcal{H})$ be given such that system (1.1) is consistent. For simplicity, put*

$$X_0 = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger, \quad J = \begin{bmatrix} L_A & R_B \end{bmatrix}, \quad M = P_A R_B,$$

$$N = L_A Q_B, \quad S = R_B L_M A_{33} = P_A P_J, \quad A_{44} = Q_B P_J,$$

$$H_0 = (X_0 + X_0^*) R_J (R_J (X_0 + X_0^*) R_J)^\dagger R_J (X_0 + X_0^*),$$

$$C_3 = P_A (X_0 + X_0^*) P_A, \quad C_4 = Q_B (X_0 + X_0^*) Q_B, \quad G = R_J (X_0 + X_0^*) R_J,$$

$$Z = R_J (X_0 + X_0^*) R_J, \quad C_{33} = C_3 - P_A R_J Z^\dagger R_J P_A, \quad C_{44} = C_4 - Q_B R_J Z^\dagger R_J Q_B,$$

$$C_5 = C_3 - P_A H_0 P_A, \quad C_6 = C_4 - Q_B H_0 Q_B.$$

Suppose J, M, N, S, A_{33} and A_{44} have closed ranges. Then the following statements are equivalent:
(1) system (1.1) has a Re-positive solution,
(2) C_3, C_4, G, C_{33} and C_{44} are positive,

$$\mathcal{R}\left(R_J (X_0 + X_0^*) R_J\right) \subseteq \mathcal{R}\left(R_J (X_0 + X_0^*)\right), \ \mathcal{R}(C_{33}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{33}), \ \mathcal{R}(C_{44}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{44})$$

and there exist a $T \in \mathfrak{U}(\mathcal{H})$ such that

$$A_{44} L_{A_{33}} (A_{44} L_{A_{33}})^\dagger \left(C_{44}^{\frac{1}{2}} T - A_{44} A_{33}^\dagger C_{33}^{\frac{1}{2}}\right) = \left(C_{44}^{\frac{1}{2}} T - A_{44} (A_{33})^\dagger C_{33}^{\frac{1}{2}}\right).$$

In this case, the general Re-positive solution to system (1.1) can be expressed as

$$X = X_0 + \frac{1}{2} L_A (U + Z^*) R_B, \tag{4.7}$$

where

$$U = L_A (H - (X_0 + X_0^*)) R_B - R_B M^\dagger P_A (H - (X_0 + X_0^*)) R_B \tag{4.8}$$
$$\quad - L_A S R_B (H - (X_0 + X_0^*)) Q_B N^\dagger L_A R_B$$
$$\quad - L_A S W_1 R_N L_A R_B + P_A W_2 + W_3 Q_B,$$

$$Z = M^\dagger P_A (H - (X_0 + X_0^*)) L_A + L_M S^\dagger S R_B (H - (X_0 + X_0^*)) Q_B N^\dagger \tag{4.9}$$
$$\quad + L_M \left(W_1 - S^\dagger S W_1 N N^\dagger\right) + W_4 P_A,$$

with

$$H = H_0 + Q_J \left((P_A Q_J)^\dagger C_5^{\frac{1}{2}} + L_{P_A P_J} Y\right) \left((P_A Q_J)^\dagger C_5^{\frac{1}{2}} + L_{P_A P_J} Y\right)^* Q_J, \tag{4.10}$$

and

$$Y = (Q_B Q_J P_A Q_J)^- \left(C_6^{\frac{1}{2}} T - Q_B Q_J (P_A Q_J)^- C_5^{\frac{1}{2}}\right) + L_{Q_B Q_J P_A Q_J} W_5, \tag{4.11}$$

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W_i, $i = 1, \ldots, 5$, *be arbitrary.*

Proof. It is easy to verify that the general solution of (1.1) can be expressed as

$$X = X_0 + L_A V R_B,$$

where $X_0 = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger$, V is free to vary over $\mathfrak{B}(\mathcal{K}_2, \mathcal{H})$. Note that

$$X + X^* = X_0 + X_0^* + L_A V R_B + (L_A V R_B)^* = H.$$

Hence,

$$X + X^* \geq 0 \Leftrightarrow H \geq 0.$$

Fix H first and it is easy to get $H = H^*$. From Lemma 4.1, equation

$$L_A V R_B + (L_A V R_B)^* = H - (X_0 + X_0^*) \tag{4.12}$$

is consistent for V if and only if

$$\mathcal{R}(H - X_0 - X_0^*) \subseteq \mathcal{R} \begin{bmatrix} L_A & R_B \end{bmatrix}, \ \ \mathcal{R}(H - X_0 - X_0^*) \subseteq \mathcal{R}(L_A) \oplus \mathcal{R}((L_A)^*)$$

and

$$\mathcal{R}(H - X_0 - X_0^*) \subseteq \mathcal{R}(R_B) \oplus \mathcal{R}((R_B)^*),$$

which are equivalent to the following system

$$\begin{cases} R_J(H - X_0 - X_0^*) = 0 \\ P_A(H - X_0 - X_0^*)P_A = 0 \\ Q_B(H - X_0 - X_0^*)Q_B = 0 \end{cases} \Leftrightarrow \begin{cases} R_J H = R_J(X_0 + X_0^*) \\ P_A H P_A = P_A(X_0 + X_0^*)P_A \\ Q_B H Q_B = Q_B(X_0 + X_0^*)Q_B \end{cases} \tag{4.13}$$

has a positive solution H. Next we will show the equivalence between (1) and (2).

(1) $\Rightarrow$ (2). Suppose that system (1.1) has a Re-positive solution X_1. Then it follows from the above proof that system (4.13) has a positive solution. Noticing Theorem 3.5, we have

$$R_J(X_0 + X_0^*)R_J, \ \ P_A(X_0 + X_0^*)P_A, \ \ Q_B(X_0 + X_0^*)Q_B, \tag{4.14}$$
$$C_3 - P_A R_J Z^\dagger R_J P_A, \ \ C_4 - Q_B R_J Z^\dagger R_J Q_B$$

are positive,

$$\mathcal{R}(R_J(X_0 + X_0^*)R_J) \subset \mathcal{R}(R_J(X_0 + X_0^*)), \mathcal{R}(C_{33}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{33}), \mathcal{R}(C_{44}^{\frac{1}{2}}) \subseteq \mathcal{R}(A_{44}), \tag{4.15}$$

and there exist a $T \in \mathfrak{U}(\mathcal{H})$ such that

$$A_{44} L_{A_{33}} (A_{44} L_{A_{33}})^\dagger \left(C_{44}^{\frac{1}{2}} T - A_{44} A_{33}^\dagger C_{33}^{\frac{1}{2}} \right) = \left(C_{44}^{\frac{1}{2}} T - A_{44}(A_{33})^\dagger C_{33}^{\frac{1}{2}} \right). \tag{4.16}$$

$(2) \Rightarrow (1)$. Note that the operators in (4.14) are all positive and equations (4.15)–(4.16) are satisfied. By Theorem 3.5, system (4.13) has a positive solution H which can be expressed as

$$H = H_0 + Q_J K \left(Q_J K \right)^*,$$

where

$$K = \left(P_A Q_J \right)^- \left(P_A \left(X_0 + X_0^* \right) P_A - P_A H_0 P_A \right)^{\frac{1}{2}} + L_{P_A P_J} Y$$

with

$$Y = \left(Q_B Q_J P_A Q_J \right)^- \left(\left(C_4 - Q_B H_0 Q_B \right)^{\frac{1}{2}} T - Q_B Q_J \left(P_A Q_J \right)^- \left(C_3 - P_A H_0 P_A \right)^{\frac{1}{2}} \right)$$
$$+ L_{Q_B Q_J P_A Q_J} W_5,$$

where W_5 is free to vary over $\mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$. For arbitrary H satisfies (4.13) and noting Lemma 4.1, we can get the general solution to equation (4.12) relate to V can be written as $V = \frac{1}{2} (U + Z^*)$ with

$$U = L_A \left(H - \left(X_0 + X_0^* \right) \right) R_B - R_B M^\dagger P_A \left(H - \left(X_0 + X_0^* \right) \right) R_B$$
$$- L_A S R_B \left(H - \left(X_0 + X_0^* \right) \right) Q_B N^\dagger L_A R_B - L_A S W_1 R_N L_A R_B + P_A W_2 + W_3 Q_B,$$
$$Z = M^\dagger P_A \left(H - \left(X_0 + X_0^* \right) \right) L_A + L_M S^\dagger S R_B \left(H - \left(X_0 + X_0^* \right) \right) Q_B N^\dagger$$
$$+ L_M \left(W_1 - S^\dagger S W_1 N N^\dagger \right) + W_4 P_A,$$

where $W_i, i = 1, \ldots, 4$ are arbitrary. Then the general Re-positive solution of system (1.1) can be expressed as (4.7) with (4.8)–(4.11). $\square$

COROLLARY 4.4. *Let $A, C \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_1)$ be given such that equation (1.4) is consistent. Then (1.4) has a real positive solution if and only if $P_A \left(A^\dagger C + \left(A^\dagger C \right)^* \right) P_A$ is positive. In this case, the general real positive solution to equation (1.4) can be expressed as*

$$X = A^\dagger C + \frac{1}{2} L_A \left(H - A^\dagger C - \left(A^\dagger C \right)^* \right) \left(I + A^\dagger A \right)$$

with

$$H = P_A^= \left(P_A \left(A^\dagger C + \left(A^\dagger C \right)^* \right) P_A \right) P_A^= + L_A U L_A,$$

where

$$P_A^= = P_A + L_A Z \left(P_A \left(A^\dagger C + \left(A^\dagger C \right)^* \right) P_A \right)^{\frac{1}{2}},$$

Z is free to vary over $\mathfrak{B}(\mathcal{K}_1, \mathcal{H})$ and $U \in \mathfrak{B}(\mathcal{H})^+$ is arbitrary.

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Remark 4.5. Groß [8] presented a necessary and sufficient condition for the existence of Re-positive solution to matrix equation (1.4) and established the expression of the general Re-positive solution in terms of generalized inverse of some matrices when the solvability conditions are satisfied. However, in 2008, Dajić and Koliha [5] pointed out that the general expression of Re-positive solution to matrix equation (1.4) in [8] did not involve all of the Re-positive solution. They also gave a correct expression of the general Re-positive solution to operator equation (1.4) in terms of generalized inverses of some adjointable operators [5, Theorem 7.3]. In Corollary 4.4, we also give a new expression of this general Re-positive solution which is different from one in [5].

5. Conclusion. In this paper, we have established some necessary and sufficient conditions for system (1.5) and system (1.1) to have a positive or Re-positive solution, respectively, and derived some expressions of the general positive or Re-positive solutions to these systems when the conditions are satisfied.

It is worthy to say that the approach and results in this paper are also true to the bounded operators between quaternionic Hilbert spaces, which plays an important role in certain physical problems (see, for example, [7]).

Motivated by the work in this paper, it would be of interest to investigate the common Re-positive solutions to equations (1.3) for Hilbert C^*-modules operators.

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