A p-adic Study of the Partial Sums of the Harmonic Series

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Abstract

Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be the n-th partial sum of the harmonic series. A classical result of Wolstenholme states that, if $p > 3$ is prime, the numerator of H_{p-1} is divisible by p^2. Here we consider, for a given prime p, the set J_p of n for which p divides the numerator of H_n. This set J_p had been previously determined for $p = 2, 3, 5, 7$. One of our results is that J_{11} contains exactly 638 integers, the largest of which is a number of 31 decimal digits. We determine J_p for all $p < 550$ with three exceptions: 83, 127 and 397.

The computation is based on a new p-adically convergent formula for the quantity $H_{pn} - H_n/p$. We describe a probabilistic model for the sets J_p, based on branching processes. The model predicts that $|J_p| = O(p^2(\log\log p)^{2+\epsilon})$, and that there are infinitely many p with $|J_p| \geq p^2(\log\log p)^2$. This strengthens an earlier conjecture of Eswarathasan and Levine that $|J_p|$ is finite for all p. Another prediction of the model is that there will be infinitely many pairs (n, p) for which p^3 divides the numerator of H_n, but only finitely many for which p^4 divides H_n.

It has been conjectured that there are infinitely many p for which $|J_p| = 3$. We give a probabilistic argument that suggests that such primes have a density $1/e$ in the set of all primes, and experimentally confirm this by a determination of all such $p \leq 10^5$.