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ZEROS AND LOCAL EXTREME POINTS OF FABER POLYNOMIALS ASSOCIATED WITH HYPOCYCLOIDAL DOMAINS*

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Abstract. Faber polynomials play an important role in different areas of constructive complex analysis. Here, the zeros and local extreme points of Faber polynomials for hypocycloidal domains are studied. For this task, we use tools from linear algebra, namely, the Perron-Frobenius theory of nonnegative matrices, the Gantmacher-Krein theory of oscillation matrices, and the Schmidt-Spitzer theory for the asymptotic spectral behavior of banded Toeplitz matrices.

Key words. Faber polynomials, cyclic of index p matrices, oscillation matrices.

AMS subject classifications. 30C15, 15A48, 15A57.

1. The Problem. Faber polynomials, introduced by Faber in [5] and [6], have been a mainstay for analysts interested in the approximation of analytic functions, and there is a rich mathematical literature (cf. [3], [9, §I.6], [17, Chapter 2], [19]) describing Faber polynomials, their properties, and their applications. Recently, applications of Faber polynomials, both in a theoretical as well as in a practical sense, have been made to the iterative solution of large nonsymmetric systems of linear equations, and the use of Faber polynomials has brought new analysis tools to this area of linear algebra (cf. [4], [7], [18]).

Our goal in this paper is just the *opposite*: we wish to show here that linear algebra techniques, especially the application of the theory of oscillation matrices to certain Hessenberg matrices, can provide new tools for the classical complex analysis problem of determining the zeros of Faber polynomials for special domains.

We briefly recall the definition of Faber polynomials. Let $\Omega \subset \mathbb{C}$ be a compact set, not a single point, whose complement $\mathbb{C}_\infty \setminus \Omega$ (with respect to the extended plane) is simply connected. By $z = \psi(w)$, we denote the conformal map from $|w| > 1$ onto $\mathbb{C}_\infty \setminus \Omega$, which is normalized by $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. The Faber polynomials $\{F_m\}_{m\geq 0}$ for Ω are then defined (cf. [17, p. 130]) from the following generating function:

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{m=0}^{\infty} F_m(z) w^{-m-1} \qquad (|w| > 1,\, z \in \Omega). \tag{1.1}$$

Here, we investigate the zeros and the local extreme points of Faber polynomials for a special class of compact sets.

With $\mathbb{N}(\mathbb{N}_0)$ denoting the set of positive (nonnegative) integers, consider the mapping

$$\psi(w) := \alpha w + \beta w^{1-p} \qquad (p \in \mathbb{N},\, p \geq 2,\, \alpha > 0,\, \beta \in \mathbb{C},\, \beta \neq 0)\,, \tag{1.2}$$

which is conformal in the exterior of the unit circle if and only if $\rho := (p-1)|\beta|/\alpha \leq 1$. The boundary of the associated compact set

$$\Omega = \mathrm{H}(p, \alpha, \beta) := \mathbb{C}_\infty \setminus \{z \in \mathbb{C} : z = \psi(w) \text{ with } |w| > 1\} \tag{1.3}$$

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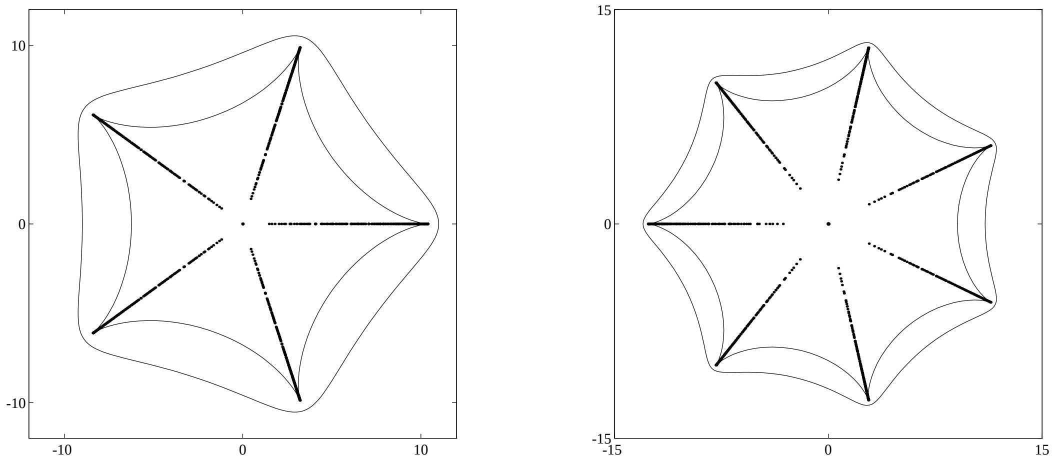

FIG. 1.1. *Hypocycloidal domains* $\mathrm{H}(p,\alpha,\beta)$ *for* $p = 5$ *and* $p = 7$*, together with the zeros of the associated Faber polynomials* $\{F_m\}_{m=1}^{50}$.

is a *hypocycloid*; more precisely, it is a *cusped hypocycloid* if $\rho = 1$ and a *blunted hypocycloid* if $\rho < 1$ (see Fig. 1.1)[1] Note that for $p = 2$, the Faber polynomials are well known: $\mathrm{H}(2,\alpha,\beta)$ is either an interval (if $\rho = 1$) or an ellipse together with its interior (if $\rho < 1$) and, for those sets, F_m is a suitably scaled mth Chebyshev polynomial T_m of the first kind (cf. Rivlin [13, p. 1]).

We show in this note that all zeros $\{\xi_{m,k}\}_{k=1}^{m}$ and all local extreme points $\{\zeta_{m-1,k}\}_{k=1}^{m-1}$ of the Faber polynomials F_m for the sets $\mathrm{H}(p,\alpha,\beta)$ are located on certain *stars*. This is illustrated in Fig. 1.1, where, for example, in the figure on the left, the boundary of $\mathrm{H}(5,10,1)$ is the outer closed curve (which is a blunted hypocycloid), and the boundary of $\mathrm{H}(5,10\tau,\tau^{-4})$ (with $\tau := (2/5)^{1/5}$) is shown as the inner closed curve (which is a cusped hypocycloid), along with all the zeros of the associated Faber polynomials $\{F_m\}_{m=1}^{50}$. (Up to a constant multiplicative factor, the Faber polynomials for these two sets are identical.[2]) These zeros lie in $p = 5$ equally spaced (in angle) intervals which emanate from the origin, thereby forming a star. We shall prove that the zeros $\{\xi_{m,k}\}_{k=1}^{m}$, as well as the local extreme points $\{\zeta_{m-1,k}\}_{k=1}^{m-1}$, are dense on these stars, as $m \to \infty$. In addition, we prove that the zeros $\{\xi_{m,k}\}_{k=1}^{m}$ *interlace* on these stars in a certain precise sense. Similar results are derived for the local extreme points $\{\zeta_{m-1,k}\}_{k=1}^{m-1}$.

[1] A *hypocycloid* is the curve traced by a point connected to a circle rolling on the interior of the circumference of another (fixed) circle. When the point is on the circumference of the rolling circle (which is equivalent to $\rho = 1$ in our above notation), the curve is called a *cusped* hypocycloid. When the point is not on circumference of the rolling circle, the curve is often called a *hypotrochoid*. We prefer the more suggestive notations of a *blunted* hypocycloid, for the case that the point is interior to the rolling circle (i.e., $\rho < 1$), and of a *looped* hypocycloid, for the case that the point is exterior to the rolling circle (i.e., $\rho > 1$). A detailed discussion of those curves is contained, e.g., in [15, p. 278].

[2] In general, the Faber polynomials F_m for $\Omega = \mathbb{C}_\infty \setminus \psi(\{w : |w| > 1\})$ and the Faber polynomials $\tilde{F}_m$ for $\tilde{\Omega} := \mathbb{C}_\infty \setminus \psi(\{w : |w| > \tau\})$, $\tau > 1$, are related by $\tilde{F}_m(z) = \tau^{-m} F_m(z)$ and therefore, have the same zeros. Consequently, the zeros of the Faber polynomials for $\mathrm{H}(p,\alpha,\beta)$ and for $\mathrm{H}(p,\alpha\tau,\beta\tau^{1-p})$ are identical.

All these results can be considered as generalizations of well-known properties of the zeros and local extreme points of Chebyshev polynomials. For the hypocycloidal domains $\mathrm{H}(p,\alpha,\beta)$, some of these results have been recently obtained by He and Saff [12]. As previously mentioned, our approach, which is completely different from the one used by He and Saff, is based merely on tools from linear algebra. Specifically, we apply basic facts from i) the Frobenius theory of nonnegative matrices (cf. [21, Chapter 2]), from ii) the Gantmacher-Krein theory of oscillation matrices (cf. [10, Chapter 2]), and from iii) the Schmidt-Spitzer theory of the asymptotic spectral behavior of finite sections of Toeplitz matrices (cf. [16]). We remark that a difference between our results here and those of [12] is the interlacing of the zeros (and the local extreme points) on their stars, which is a nice bonus from the theory of oscillation matrices!

We shall briefly summarize in Section 2 the tools to be used in this paper. Our results, concerning the zeros and local extreme points of Faber polynomials for hypocycloidal domains, will be formulated in Section 3 and proved in Section 4. Finally in Section 5, we describe the properties of Faber polynomials for another class of compact sets which are closely related to hypocycloidal domains.

2. The Tools. The previously mentioned exterior conformal map ψ from $|w| > 1$, associated with an arbitrary compact set Ω (not a single point) whose complement $\mathbb{C}_\infty \setminus \Omega$ is simply connected, has a Laurent expansion of the form

$$\psi(w) = \alpha w + \sum_{k=0}^{\infty} \alpha_k w^{-k} \qquad (\alpha > 0,\ \alpha_k \in \mathbb{C} \text{ for } k \in \mathbb{N}_0)\,, \tag{2.1}$$

which converges for all $|w| > 1$. Substituting this expansion of ψ into (1.1) and comparing equal powers of w leads to the recurrence relation

$$\begin{aligned} &zF_k(z) = \alpha F_{k+1}(z) + \left(\sum_{j=0}^{k} \alpha_j F_{k-j}(z)\right) + k\alpha_k F_0(z) \qquad (k \in \mathbb{N}_0) \\ &\text{with } F_0(z) := 1 \end{aligned} \tag{2.2}$$

(cf. Curtiss [3]). But, if we rewrite (2.2) for $k = 0, 1, \ldots, m-1$ in matrix-vector form, we have

$$\begin{aligned} &z\,[F_0(z), F_1(z), \ldots, F_{m-1}(z)] \\ &= \ [F_0(z), F_1(z), \ldots, F_{m-1}(z)]\,\mathcal{F}_m + [0, \ldots, 0, \alpha F_m(z)]\,, \end{aligned} \tag{2.3}$$

where $\mathcal{F}_m$ denotes the mth section (i.e., the leading $m \times m$ principal submatrix) of the infinite upper Hessenberg matrix

$$\mathcal{F} := \begin{bmatrix} \alpha_0 & 2\alpha_1 & 3\alpha_2 & 4\alpha_3 & \cdots \\ \alpha & \alpha_0 & \alpha_1 & \alpha_2 & \\ & \alpha & \alpha_0 & \alpha_1 & \ddots \\ & & \alpha & \alpha_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{2.4}$$

(Note that $\mathcal{F}$ has a nearly Toeplitz structure, i.e., if we discard its first row, we obtain a Toeplitz matrix.) It is well known (and easy to see from (2.3)) that $\lambda \in \mathbb{C}$ is a zero

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of F_m if and only if λ is an eigenvalue of $\mathcal{F}_m$ (with corresponding left eigenvector $[F_0(\lambda), F_1(\lambda), \ldots, F_{m-1}(\lambda)]$). To find the zeros of F_m, it thus suffices to locate the eigenvalues of $\mathcal{F}_m$, the mth section of $\mathcal{F}$.

To determine the local extreme points of F_{m+1}, we consider another sequence $\{G_m\}_{m\geq 0}$ of polynomials defined by

$$\frac{1}{\psi(w)-z} = \sum_{m=0}^{\infty} G_m(z) w^{-m-1} \qquad (|w|>1,\, z\in\Omega)\,. \tag{2.5}$$

These are the *generalized* Faber polynomials for Ω with respect to the weight function $1/\psi'$ (cf. [17, §2.2]). Following [7], we call G_m the mth Faber polynomial of the *second kind*[3] for Ω. On differentiating (1.1) with respect to z and on differentiating (2.5) with respect to w, it easily follows (cf. [20]) that

$$G_m(z) = \frac{F'_{m+1}(z)}{m+1} \qquad (m\in\mathbb{N}_0)\,, \tag{2.6}$$

and hence, the local extreme points of F_{m+1} are the zeros of G_m.

In analogy with (2.2), the polynomials G_k satisfy the recurrence relations

$$\begin{aligned} &zG_k(z) = \alpha G_{k+1}(z) + \sum_{j=0}^{k} \alpha_j G_{k-j}(z) \qquad (k\in\mathbb{N}_0) \\ &\text{with } G_0(z) := 1/\alpha\,, \end{aligned} \tag{2.7}$$

or, in matrix-vector form,

$$\begin{aligned} & z\,[G_0(z), G_1(z), \ldots, G_{m-1}(z)] \\ =\; & [G_0(z), G_1(z), \ldots, G_{m-1}(z)]\,\mathcal{G}_m + [0,\ldots,0,\alpha G_m(z)]\,, \end{aligned} \tag{2.8}$$

where $\mathcal{G}_m$ is now the mth section of the infinite upper Hessenberg Toeplitz matrix

$$\mathcal{G} := \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \\ \alpha & \alpha_0 & \alpha_1 & \alpha_2 & & \\ & \alpha & \alpha_0 & \alpha_1 & \ddots & \\ & & \alpha & \alpha_0 & \ddots & \\ & & & \ddots & \ddots & \end{bmatrix}. \tag{2.9}$$

Thus, from (2.6) and (2.8), the local extreme points of F_{m+1} are nothing but the eigenvalues of $\mathcal{G}_m$. This connection, together with the asymptotic spectral properties of finite Toeplitz matrices, can be used to derive results on the *asymptotic behavior* of the local extreme points of the classical Faber polynomials (of the first kind) (cf. [20]) and also of the zeros of these polynomials (cf. [2]). Here, we are interested in *transient* (i.e., nonasymptotic) properties of the zeros and local extreme points of classical Faber polynomials for special sets.

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[3] This notation is motivated by the special cases when Ω is either an interval or an ellipse, together with its interior. As previously mentioned here, the Faber polynomials F_m (of the first kind) are suitably scaled Chebyshev polynomials T_m of the first kind. As shown in the beginning of Section 3, the generalized Faber polynomials G_m are then suitably scaled Chebyshev polynomials U_m of the second kind (cf. [13, p. 7]).

For the hypocycloidal domains $\mathrm{H}(p, \alpha, \beta)$ defined by (1.2) and (1.3), $\mathcal{G}$ of (2.9) reduces to the particularly simple infinite matrix

$$\mathcal{G} = \mathcal{G}_h = \begin{bmatrix} 0 & \cdots & 0 & \beta & & & & \\ \alpha & 0 & \cdots & 0 & \beta & & & \\ & \alpha & 0 & & 0 & \beta & & \\ & & \ddots & & & & \ddots & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \tag{2.10}$$

(where the subscript h refers to "hypocycloid"), and similarly, from (2.4), we have

$$\mathcal{F} = \mathcal{F}_h = \mathcal{G}_h + (p-1)\beta\, \mathbf{u}_1 \mathbf{u}_p^T \,, \tag{2.11}$$

where $\mathbf{u}_k$ denotes the kth unit column vector in $\mathbb{R}^\infty (k = 1, 2, \ldots)$. Note that $\mathcal{F}_h$ and $\mathcal{G}_h$ are banded matrices (i.e., only their first lower and their $(p-1)$st upper diagonals contain nonzero entries), and $\mathcal{F}_h$ and $\mathcal{G}_h$ differ only in the pth element of their first rows.

After a reduction to the case $\alpha = \beta = 1$, we shall show in Section 4 that $\mathcal{G}_h$ and $\mathcal{F}_h$ of (2.10) and (2.11) are cyclic of index p matrices, so that, with a suitable permutation, the pth powers of $\mathcal{G}_n$ and $\mathcal{F}_h$ are, respectively, the direct sum of p infinite matrices $\{\mathcal{H}^{(k)}\}_{k=0}^{p-1}$, and $\{\mathcal{K}^{(k)}\}_{k=0}^{p-1}$, where $\mathcal{H}^{(k)}$ and $\mathcal{K}^{(k)}$ are infinite banded upper Hessenberg matrices $(k = 0, 1, \cdots, p-1)$. The main part of our investigation is then a study of the spectral properties of their j-th sections $\mathcal{K}_j^{(k)}$ and $\mathcal{H}_j^{(k)}$, in $\mathbb{R}^{j\times j}$, for every $j \in \mathbb{N}$. It turns out all these sections are nonnegative and irreducible matrices, and, more importantly, these sections are also oscillation matrices. Thus, the full power of the Perron-Frobenius theory of nonnegative matrices and the Gantmacher-Krein theory of oscillation matrices are applicable to the study of the spectra of these sections.

The Perron-Frobenius theory of irreducible nonnegative matrices is an essential part of the tool box of anyone interested in matrix theory and numerical linear algebra. We therefore assume that the reader is familiar with this theory. (The facts we shall use in the sequel are described, e.g., in [21, Chapter 2], [1], and [8, Chapter 4].) Unfortunately, it seems that the theory of oscillation matrices is less well known to workers in numerical linear algebra. One of the objects of this paper is to also show that the theory of oscillation matrices is in fact a powerful tool even for problems which, at first glance, have little connections with matrices. For the reader's convenience, we therefore recall below the main definitions and results which we use from the theory of oscillation matrices.

Let $\mathcal{M} = [\mu_{i,j}]_{1\le i,j\le m}$ be an arbitrary matrix in $\mathbb{R}^{m\times m}$. The set of multi-indices $\mathbf{i}$, with k (for $1 \le k \le m$) elements from $\{1, 2, \ldots, m\}$, is defined by

$$\Delta_{m,k} := \{\mathbf{i} = (i_1, i_2, \ldots, i_k) \,:\, 1 \le i_1 < i_2 < \cdots < i_k \le m\}.$$

For $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_k)$ in $\Delta_{m,k}$, the $k \times k$ submatrix of $\mathcal{M}$ with rows i_l and columns j_n will be denoted by $\mathcal{M}(\mathbf{i}, \mathbf{j}) := [\mu_{i,j}]_{i\in\mathbf{i}, j\in\mathbf{j}}$. Then, $\mathcal{M}$ is *totally nonnegative* (*totally positive*) if $\det \mathcal{M}(\mathbf{i}, \mathbf{j})$ is nonnegative (positive) for all $\mathbf{i}, \mathbf{j} \in \Delta_{m,k}$ and all $k = 1, 2, \cdots, m$.

The best known examples of totally nonnegative matrices arise from the class of

Jacobi matrices, i.e., tridiagonal matrices of the form

$$\mathcal{J} = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & & \\ & \gamma_2 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \beta_{m-2} & \\ & & & \gamma_{m-2} & \alpha_{m-1} & \beta_{m-1} \\ & & & & \gamma_{m-1} & \alpha_m \end{bmatrix}.$$

A nonsingular Jacobi matrix $\mathcal{J}$ is totally nonnegative if and only if $\beta_i \geq 0$ and $\gamma_i \geq 0$ ($i = 1, 2, \ldots, m-1$), and, in addition, all sections of $\mathcal{J}$ have positive determinants, i.e.,

$$\alpha_1 > 0,\ \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \alpha_2 \end{bmatrix} > 0,\ \det \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 \\ 0 & \gamma_2 & \alpha_3 \end{bmatrix} > 0, \ldots,\ \det \mathcal{J} > 0$$

(cf. [10, p. 94]).

By definition, an arbitrary square matrix $\mathcal{M}$ is an *oscillation matrix* if $\mathcal{M}$ is totally nonnegative and if some $k \in \mathbb{N}$ is such that $\mathcal{M}^k$ is totally positive. A totally nonnegative Jacobi matrix $\mathcal{J}$ is an oscillation matrix if and only if its sub- and super-diagonal entries β_i and γ_i ($i = 1, 2, \ldots, m-1$) are all positive (cf. [10, p. 119]). A main result of the Gantmacher-Krein theory is that this oscillation matrix criterion is valid for any totally nonnegative matrix:

THEOREM A ([10, p. 115]). *A totally nonnegative matrix $\mathcal{M} = [\mu_{i,j}]_{1\leq i,j\leq m}$ is an oscillation matrix if and only if $\mathcal{M}$ is nonsingular and $\mu_{i,i+1} > 0$ and $\mu_{i+1,i} > 0$ hold for all $i = 1, 2, \ldots, m-1$.*

For our subsequent use, we quote two other sufficient criteria for oscillation matrices.

THEOREM B ([10, p. 112 and p. 118]). *If $\mathcal{M} \in \mathbb{R}^{m\times m}$ is an oscillation matrix, then each principal submatrix of $\mathcal{M}$ is also an oscillation matrix.*

$\mathcal{M} \in \mathbb{R}^{m\times m}$ is an oscillation matrix if $\mathcal{M}$ can be expressed as a product of an oscillation matrix with a nonsingular totally nonnegative matrix.

The theory of oscillation matrices turns out to be extremely useful for determining properties of the zeros of special Faber polynomials because oscillation matrices, although they are not necessarily symmetric, are a natural generalization of positive definite Hermitian Jacobi matrices.

THEOREM C ([10, p. 100]). *The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of an oscillation matrix $\mathcal{M} = [\mu_{i,j}]_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m}$ are simple and positive, i.e., they can be arranged so that*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m .$$

We shall also make use of the following two results concerning the behavior of the eigenvalues of oscillation matrices.

THEOREM D ([10, p. 124]). *Let $\mathcal{M} = [\mu_{i,j}]_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m}$ be an oscillation matrix, and let $\{\lambda_{k,l}\}_{l=1}^{k}$ be the eigenvalues of $\mathcal{M}_k$, the kth section of $\mathcal{M}$*

($k = 1, 2, \ldots, m$), where these eigenvalues are arranged as

$$0 < \lambda_{k,1} < \lambda_{k,2} < \cdots < \lambda_{k,k} \,.$$

Then, the eigenvalues of $\mathcal{M}_{j+1}$ interlace with those of $\mathcal{M}_j$, i.e.,

$$0 < \lambda_{j+1,1} < \lambda_{j,1} < \lambda_{j+1,2} < \lambda_{j,2} < \cdots < \lambda_{j,j} < \lambda_{j+1,j+1}$$

for every $j = 1, 2, \ldots, m-1$.

Theorem E ([10, p. 127]). *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ denote the eigenvalues of an oscillation matrix $\mathcal{M} = [\mu_{i,j}]_{1 \le i,j \le m} \in \mathbb{R}^{m \times m}$. Then,*

$$\frac{\partial \lambda_j}{\partial \mu_{1,1}} > 0 \text{ and } \frac{\partial \lambda_j}{\partial \mu_{m,m}} > 0 \text{ for every } j \in \{1, 2, \ldots, m\} \,.$$

Finally, we shall apply a main result of the Schmidt-Spitzer theory [16] concerning the asymptotic spectral behavior of finite Toeplitz matrices. We briefly recall this result, but only for the special case which we shall need in the sequel. Let

$$\mathcal{T} = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_m & & & & \\ \alpha & \alpha_0 & \alpha_1 & \cdots & \alpha_m & & & \\ & \alpha & \alpha_0 & \alpha_1 & \cdots & \alpha_m & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & & & & & \end{bmatrix} \tag{2.12}$$

be a (semi-infinite) banded upper Hessenberg Toeplitz matrix, whose *symbol* is defined by

$$t(w) := \alpha w + \alpha_0 + \alpha_1 w^{-1} + \cdots + \alpha_m w^{-m}.$$

If $\mathcal{T}_m$ denotes the mth section (i.e., the leading principal $m \times m$ submatrix) of $\mathcal{T}$, and if $\Lambda(\mathcal{T}_m)$ denotes its spectrum, then we set

$$\Lambda_\infty(\mathcal{T}) := \left\{ \lambda \in \mathbb{C} : \lambda = \lim_{m \to \infty} \lambda_m, \text{ where } \lambda_m \in \Lambda\left(\mathcal{T}_{i_m}\right) \text{ and } \lim_{m \to \infty} i_m = \infty \right\}, \tag{2.13}$$

so that $\Lambda_\infty(\mathcal{T})$ is the set in $\mathbb{C}$ of all accumulation points of $\{\Lambda(\mathcal{T}_m)\}_{m \ge 1}$.

To describe $\Lambda_\infty(\mathcal{T})$ in a different way, more terminology is needed. For any real number ρ with $\rho > 0$, let Γ_ρ be the image of the circle $|w| = \rho$ under the mapping $w \mapsto t(w)$ and set

$$\Lambda_\rho(\mathcal{T}) := \{\lambda \in \mathbb{C} \,:\, n\left(\Gamma_\rho, \lambda\right) \ne 0\} \,, \tag{2.14}$$

where $n\left(\Gamma_\rho, \lambda\right)$ is the winding number of Γ_ρ with respect to the point λ. For $\lambda \in \Gamma_\rho$, we follow the usual convention of defining $n\left(\Gamma_\rho, \lambda\right)$ to be different from zero.

With these notations, we now can formulate the fundamental theorem of Schmidt and Spitzer:

Theorem F ([16, Theorem 1]). *The set $\Lambda_\infty(\mathcal{T})$ of (2.13), associated with the Toeplitz matrix $\mathcal{T}$ of (2.12), has the following characterization:*

$$\Lambda_\infty(\mathcal{T}) = \bigcap_{\rho > 0} \Lambda_\rho(\mathcal{T}) \,.$$

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3. The Results. To motivate our results, we first consider the case $p = 2$ of (1.2), where, as mentioned previously, the Faber polynomials are closely related to the familiar Chebyshev polynomials. It is not difficult to see from (2.2) that, for $\Omega = \mathrm{H}(2, \alpha, \beta)$ of (1.3) with $|\beta| \le \alpha$,

$$F_0(z) = T_0(z) \text{ and } F_m(z) = 2\left(\frac{\beta}{\alpha}\right)^{m/2} T_m\left(\frac{z}{2\sqrt{\alpha\beta}}\right) \quad (m \in \mathbb{N}),$$

where $T_m(z) := \cos(m \arccos z)$ $(-1 \le z \le 1)$, and from (2.7) that

$$G_m(z) = \frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{m/2} U_m\left(\frac{z}{2\sqrt{\alpha\beta}}\right) \quad (m \in \mathbb{N}_0),$$

where $U_m(z) := \sin((m+1)\arccos z)/\sin(\arccos z)$ $(-1 < z < 1)$. The zeros $\{\xi_{m,j}\}_{j=1}^{m}$ of F_m are thus given by

$$\xi_{m,j} = 2\sqrt{\alpha\beta}\,\cos\left(\frac{(2j-1)\pi}{2m}\right) \quad (j = 1, 2, \ldots, m),$$

whereas the zeros $\{\zeta_{m,j}\}_{j=1}^{m}$ of G_m are given by

$$\zeta_{m,j} = 2\sqrt{\alpha\beta}\,\cos\left(\frac{j\pi}{m+1}\right) \quad (j = 1, 2, \ldots, m).$$

This explicit knowledge of these Faber polynomials and their zeros shows that the following results (of Theorems 3.1-3.4) are well known for the case of $p = 2$.

We begin with a structural property of the polynomials F_m and G_m for general hypocycloidal domains.

THEOREM 3.1. *For $\Omega = \mathrm{H}(p, \alpha, \beta)$, the Faber polynomials of the first kind, $\{F_m\}_{m\ge 0}$, and the Faber polynomials of the second kind, $\{G_m\}_{m\ge 0}$, have the following form:*

If $m = jp+k$, where $j \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, p-1\}$, then there exist polynomials $K_j^{(k)}$ and $H_j^{(k)}$ of exact degree j such that

$$F_m(z) = \left(\frac{\beta}{\alpha}\right)^{j}\left(\frac{z}{\alpha}\right)^{k} K_j^{(k)}\left(\frac{z^p}{\alpha^{p-1}\beta}\right) \text{ and } G_m(z) = \frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{j}\left(\frac{z}{\alpha}\right)^{k} H_j^{(k)}\left(\frac{z^p}{\alpha^{p-1}\beta}\right).$$

The polynomials $K_j^{(k)}$ and $H_j^{(k)}$ are independent of α and β. They fulfill identical $(p+1)$-term recurrence relations. More precisely, for every $k \in \{0, 1, \ldots, p-1\}$, there holds

$$\begin{aligned} zK_j^{(k)}(z) &= \sum_{l=0}^{p}\binom{p}{l}K_{j+1-l}^{(k)}(z) \quad (j = p, p+1, \ldots), \quad \textit{and} \\ zH_j^{(k)}(z) &= \sum_{l=0}^{p}\binom{p}{l}H_{j+1-l}^{(k)}(z) \quad (j = p-1, p, \ldots). \end{aligned}$$

We remark that the polynomials $K_j^{(k)}$ and $H_j^{(k)}$ will be treated in more detail in Section 4.

To determine the zeros of F_m and G_m, it is therefore sufficient to investigate the zeros of $K_j^{(k)}$ and $H_j^{(k)}$, respectively.

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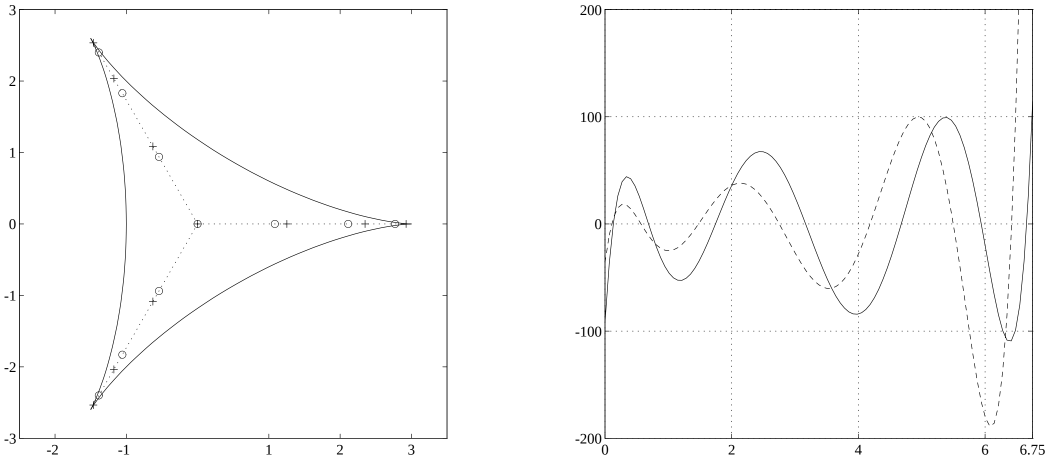

FIG. 3.1. *Zeros of* F_{10} $(+)$ *and of* G_{10} $(\circ)$ *for* $\mathrm{H}(3,2,1)$ *(on the left). Graphs of* $K_7^{(0)}$ *(solid line) and* $H_7^{(0)}$ *(dashed line) for* $p=3$ *(on the right).*

THEOREM 3.2. *For every* $j \in \mathbb{N}$ *and every* $k \in \{0,1,\ldots,p-1\}$*, the zeros* $\{\lambda_{j,l}^{(k)}\}_{\ell=1}^{j}$ *of* $K_j^{(k)}$ *and the zeros* $\{\eta_{j,l}^{(k)}\}_{\ell=1}^{j}$ *of* $H_j^{(k)}$ *are all simple, positive, and strictly less than* $\kappa_p := p^p/(p-1)^{p-1}$*:*

$$0 < \lambda_{j,1}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \lambda_{j,j}^{(k)} < \kappa_p \quad \text{and} \quad 0 < \eta_{j,1}^{(k)} < \eta_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)} < \kappa_p.$$

Moreover, the zeros of each of these $2p$ *polynomial sequences,* $\{K_j^{(k)}\}_{j\geq 1}$ *and* $\{H_j^{(k)}\}_{j\geq 1}$*, are dense in* $[0,\kappa_p]$*, for* $j \to \infty$*.*

From Theorems 3.1 and 3.2, it is clear that the zeros $\{\xi_{m,l}\}_{l=1}^{m}$ of F_m, as well as the zeros $\{\zeta_{m,l}\}_{l=1}^{m}$ of G_m, are located on the stars

$$\begin{aligned} \mathcal{S}(p,\alpha,\beta) \quad := \quad & \Big\{ z = re^{i[\arg(\beta)+2\pi k]/p} : \\ & k = 0,1,\ldots,p-1 \text{ and } 0 \leq r \leq \alpha \frac{p}{p-1}[(p-1)|\beta|/\alpha]^{1/p} \Big\}, \end{aligned} \tag{3.1}$$

and that $\{\xi_{m,l}\}_{l=1}^{m}$, as well as $\{\zeta_{m,l}\}_{l=1}^{m}$, are dense on $\mathcal{S}(p,\alpha,\beta)$ as $m \to \infty$. As an illustration, the zeros of F_{10} and G_{10} for $\mathrm{H}(3,2,1)$ are plotted on the left in Fig. 3.1.

Our next result describes the interlacing properties of these zeros.

THEOREM 3.3. *For every* $k \in \{0,1,\ldots,p-1\}$*, the zeros of* $K_{j+1}^{(k)}$ *interlace with the zeros of* $K_j^{(k)}$*, i.e.,*

$$\lambda_{j+1,1}^{(k)} < \lambda_{j,1}^{(k)} < \lambda_{j+1,2}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \lambda_{j,j}^{(k)} < \lambda_{j+1,j+1}^{(k)},$$

and the zeros of $H_{j+1}^{(k)}$ *interlace with the zeros of* $H_j^{(k)}$*, i.e.,*

$$\eta_{j+1,1}^{(k)} < \eta_{j,1}^{(k)} < \eta_{j+1,2}^{(k)} < \eta_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)} < \eta_{j+1,j+1}^{(k)},$$

for all $j \in \mathbb{N}$*.*

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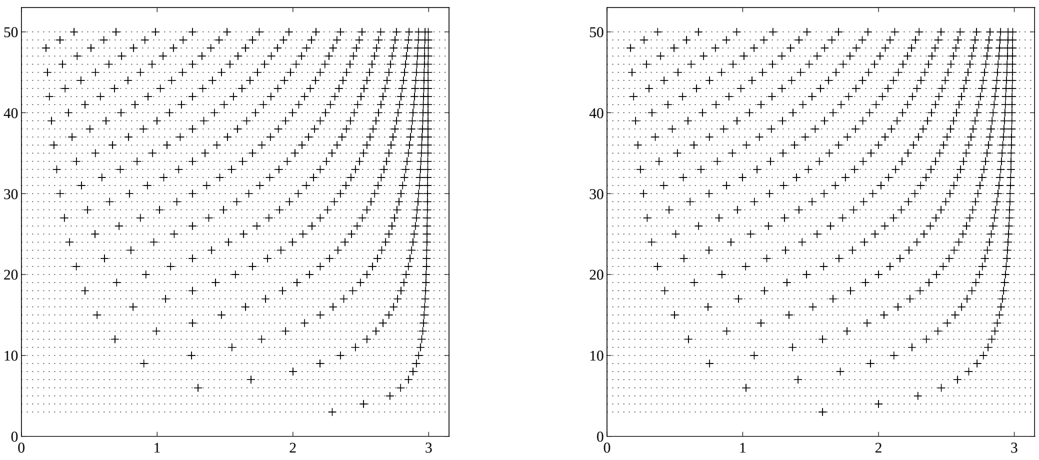

FIG. 3.2. *Positive zeros of* $\{F_m\}_{m=3}^{50}$ *(on the left) and of* $\{G_m\}_{m=3}^{50}$ *(on the right) for* H$(3,2,1)$ *plotted versus* m.

Moreover, for every $j \in \mathbb{N}$ *and every* $k \in \{0, 1, \dots, p-1\}$*, the following relation holds between the zeros of* $K_j^{(k)}$ *and the zeros of* $H_j^{(k)}$*:*

$$\eta_{j,1}^{(k)} < \lambda_{j,1}^{(k)} < \eta_{j,2}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)} < \lambda_{j,j}^{(k)} .$$

The last assertion above of Theorem 3.3 implies that the successive zeros of $K_j^{(k)}$ are always *larger* than the corresponding zeros of $H_j^{(i)}$, which can be directly seen in Figure 3.1 (right-hand side). In view of Theorem 3.1, this implies that the zeros of F_m are always radially *larger* than the corresponding zeros of G_m on *each* of the p intervals of the star $\mathcal{S}(p, \alpha, \beta)$; this can also be seen in Figure 3.1 (left-hand side). The first assertion relates the zeros of F_m to the zeros of F_{m+p}, and the zeros of G_m to the zeros of G_{m+p}.

But, we can prove an even stronger result:

THEOREM 3.4. *For every* $j \in \mathbb{N}$*, the zeros of* $K_j^{(k+1)}$ *are strictly larger than the zeros of* $K_j^{(k)}$ *if* $k \in \{0, 1, \dots, p-2\}$*, i.e.,*

$$\lambda_{j,1}^{(k)} < \lambda_{j,1}^{(k+1)} < \lambda_{j,2}^{(k)} < \lambda_{j,2}^{(k+1)} < \cdots < \lambda_{j,j}^{(k)} < \lambda_{j,j}^{(k+1)},$$

and the zeros of $K_{j+1}^{(0)}$ *interlace with the zeros of* $K_j^{(p-1)}$*, i.e.,*

$$\lambda_{j+1,1}^{(0)} < \lambda_{j,1}^{(p-1)} < \lambda_{j+1,2}^{(0)} < \lambda_{j,2}^{(p-1)} < \cdots < \lambda_{j,j}^{(p-1)} < \lambda_{j+1,j+1}^{(0)} .$$

Similar inequalities hold true for the zeros of the polynomials $H_j^{(k)}$*, i.e., for* $j \in \mathbb{N}$*,*

$$\eta_{j,1}^{(k)} < \eta_{j,1}^{(k+1)} < \eta_{j,2}^{(k)} < \eta_{j,2}^{(k+1)} < \cdots < \eta_{j,j}^{(k)} < \eta_{j,j}^{(k+1)}$$

for $k \in \{0, 1, \dots, p-2\}$*, and*

$$\eta_{j+1,1}^{(0)} < \eta_{j,1}^{(p-1)} < \eta_{j+1,2}^{(0)} < \eta_{j,2}^{(p-1)} < \cdots < \eta_{j,j}^{(p-1)} < \eta_{j+1,j+1}^{(0)} .$$

Note, as a consequence of Theorems 3.3 and 3.4, that

$$\lambda_{j+1,l}^{(0)} < \lambda_{j,l}^{(0)} < \lambda_{j,l}^{(1)} < \lambda_{j,l}^{(2)} < \cdots < \lambda_{j,l}^{(p-1)} < \lambda_{j+1,l+1}^{(0)}$$

and

$$\eta_{j+1,l}^{(0)} < \eta_{j,l}^{(0)} < \eta_{j,l}^{(1)} < \eta_{j,l}^{(2)} < \cdots < \eta_{j,l}^{(p-1)} < \eta_{j+1,l+1}^{(0)}$$

hold for every $j \in \mathbb{N}$. The obvious consequences from Theorem 3.4 for the zeros of F_m and G_m are illustrated in Fig. 3.2.

4. The Proofs. We begin by recalling two previous notational conventions: For an arbitrary (semi-)infinite matrix $\mathcal{M} = [\mu_{i,j}]_{1\le i,j<\infty}$, $\mathcal{M}_m := [\mu_{i,j}]_{1\le i,j\le m}$ denotes its mth section, i.e., the leading $m \times m$ principal submatrix of $\mathcal{M}$. Further, $\mathbf{u}_m$ is always the mth unit column vector, of finite or infinite dimension, whatever the context dictates.

As a first step, we apply a diagonal similarity transformation to the matrices $\mathcal{G}_h$ of (2.10) and $\mathcal{F}_h$ of (2.11). With $\delta := (\alpha/\beta)^{1/p}$ (where it makes no difference which branch of the pth root is selected), we define an infinite diagonal matrix $\mathcal{D}$ by

$$\mathcal{D} := \operatorname{diag}\left(\delta^0, \delta^1, \delta^2, \ldots\right)$$

and observe that

$$\mathcal{D}^{-1}\mathcal{G}_h\mathcal{D} = \left(\alpha^{p-1}\beta\right)^{1/p} \mathcal{H} \quad \text{and} \quad \mathcal{D}^{-1}\mathcal{F}_h\mathcal{D} = \left(\alpha^{p-1}\beta\right)^{1/p} \mathcal{K}, \tag{4.1}$$

where

$$\mathcal{H} := \begin{bmatrix} 0 & \cdots & 0 & 1 & & & & \\ 1 & 0 & \cdots & 0 & 1 & & & \\ & 1 & 0 & & 0 & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & & & & & \end{bmatrix} \quad \text{and} \quad \mathcal{K} := \mathcal{H} + (p-1)\,\mathbf{u}_1\mathbf{u}_p^T. \tag{4.2}$$

Note that $\mathcal{H}$ and $\mathcal{K}$ possess the same sparsity pattern as $\mathcal{G}_h$ and $\mathcal{F}_h$.

Observation 4.1. *Without loss of generality, we may assume that the parameters α and β of $\mathcal{G}_h$ and $\mathcal{F}_h$ are both equal to 1. We therefore need only investigate the eigenvalues of the sections of $\mathcal{H}$ and $\mathcal{K}$.*

In terms of the polynomial sequences $\{F_m\}_{m\ge0}$ and $\{G_m\}_{m\ge0}$ associated with the matrices $\mathcal{F}$ of (2.11) and $\mathcal{G}$ of (2.10), this is nothing but a linear change of variables and a rescaling. In other words, after the transformation $F_m(z) \mapsto \delta^m F_m(\alpha z/\delta)$, the following recurrence relations for the Faber polynomials (of the first and second kind) are valid for the set $\mathrm{H}(p,1,1)$ (cf. (1.3)):

$$\begin{aligned} &F_m(z) = z^m \quad (m = 0,1,\ldots,p-1), \;\; F_p(z) = z^p - p, \\ &F_m(z) = zF_{m-1}(z) - F_{m-p}(z) \quad (m = p+1, p+2, \ldots), \end{aligned} \tag{4.3}$$

and, similarly,

$$\begin{aligned} &G_m(z) = z^m \quad (m = 0,1,\ldots,p-1), \\ &G_m(z) = zG_{m-1}(z) - G_{m-p}(z) \quad (m = p, p+1, \ldots). \end{aligned} \tag{4.4}$$

As a second step, we note that the mth sections of $\mathcal{H}$ and $\mathcal{K}$ are nonnegative matrices which are, for all $m \geq p$, moreover irreducible matrices which are cyclic of index p. This follows by inspecting their directed graphs (cf. [21, p. 19]) which, e.g. for $p = 3$, have the following structure:

$$\begin{array}{ccccccccc}
\nu_1 & & \nu_3 & & \nu_5 & & \nu_7 & & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & & \cdots \\
\uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & & \cdots \\
\nu_2 & & \nu_4 & & \nu_6 & & \nu_8 & &
\end{array} .$$

It is evident that the above directed graphs are strongly connected (for $m \geq p$) which gives the irreducibility of these sections. In addition, as any closed path in the above directed graphs has a length which is always a multiple of p, these sections are (cf. [21, p. 49]) then cyclic of index p. This latter property allows us to assign one of p different colors to each vertex of these directed graphs (for any $m \geq 1$) arising from the sections of the matrices $\mathcal{H}$ and $\mathcal{K}$.

By a result of Romanovsky [14], the spectra of $\mathcal{H}$ and $\mathcal{K}$ and *all* their mth sections (including $1 \leq m < p$) are therefore invariant with respect to rotations of the complex plane about the origin through the angles $2\pi k/p$ ($k = 0, 1, \ldots, p-1$). This could also be deduced from the p-fold symmetry of the hypocycloid domains $\mathrm{H}(p, \alpha, \beta)$.

Following from the fact (cf. [21, p. 39]) that a nonnegative irreducible cyclic of index p matrix has a normal form, it is not surprising that the infinite matrix $\mathcal{H}$ also has a a cyclic normal form given by

$$\mathcal{P}^T \mathcal{H} \mathcal{P} = \begin{bmatrix} & & & \mathcal{E}^T \\ \mathcal{E} & & & \\ & \ddots & & \\ & & \mathcal{E} & \end{bmatrix}, \quad \text{where } \mathcal{E} := \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \ddots \\ & & & \ddots \end{bmatrix}, \tag{4.5}$$

i.e., $\mathcal{P}^T \mathcal{H} \mathcal{P}$ is a $p \times p$ block matrix whose blocks are infinite bidiagonal Toeplitz matrices. The (infinite) permutation matrix $\mathcal{P} = \left[\mathbf{u}_{\pi(1)}, \mathbf{u}_{\pi(2)}, \mathbf{u}_{\pi(3)}, \ldots\right]$ in (4.5) gathers successive vertices of the *same* color in the directed graph above.

Next, since the pth power of a cyclic of index p matrix is a block-diagonal matrix (cf. [21, p. 43]), it similarly follows that the pth power of $\mathcal{P}^T \mathcal{H} \mathcal{P}$ has the block-diagonal form

$$\left[\mathcal{P}^T \mathcal{H} \mathcal{P}\right]^p = \begin{bmatrix} \mathcal{H}^{(0)} & & & \\ & \mathcal{H}^{(1)} & & \\ & & \ddots & \\ & & & \mathcal{H}^{(p-1)} \end{bmatrix}, \tag{4.6}$$

where the (infinite) diagonal blocks are given by

$$\mathcal{H}^{(k)} := \mathcal{E}^k \mathcal{E}^T \mathcal{E}^{p-k-1} \qquad (k = 0, 1, \ldots, p-1). \tag{4.7}$$

Similarly, $\mathcal{K}$ has the cyclic normal form

$$\mathcal{P}^T \mathcal{K} \mathcal{P} = \begin{bmatrix} & & & \tilde{\mathcal{E}}^T \\ \mathcal{E} & & & \\ & \ddots & & \\ & & \mathcal{E} & \end{bmatrix}, \quad \text{where } \tilde{\mathcal{E}} := \mathcal{E} + (p-1)\mathbf{u}_1 \mathbf{u}_1^T, \tag{4.8}$$

with the same permutation matrix $\mathcal{P}$ of (4.5), and

$$\left[\mathcal{P}^T\mathcal{K}\,\mathcal{P}\right]^p = \begin{bmatrix} \mathcal{K}^{(0)} & & & \\ & \mathcal{K}^{(1)} & & \\ & & \ddots & \\ & & & \mathcal{K}^{(p-1)} \end{bmatrix} \tag{4.9}$$

with

$$\mathcal{K}^{(k)} := \mathcal{E}^k\tilde{\mathcal{E}}^T\mathcal{E}^{p-k-1} = \mathcal{E}^k\left(\mathcal{E}^T + (p-1)\mathbf{u}_1\mathbf{u}_1^T\right)\mathcal{E}^{p-k-1} \tag{4.10}$$

$(k = 0, 1, \ldots, p-1)$.

What is the polynomial interpretation of these matrix manipulations? In matrix-vector form, the recurrence relations (2.3) for the polynomials F_m become

$$z\,[F_0(z), F_1(z), F_2(z), \ldots] = [F_0(z), F_1(z), F_2(z), \ldots]\,\mathcal{K},$$

or, equivalently,

$$z\,[F_0(z), F_1(z), F_2(z), \ldots]\,\mathcal{P} = [F_0(z), F_1(z), F_2(z), \ldots]\,\mathcal{P}\left(\mathcal{P}^T\mathcal{K}\,\mathcal{P}\right).$$

If we take into account the structure of the permutation matrix $\mathcal{P}$ (cf. (4.5)), as well as the structure of $\mathcal{P}^T\mathcal{K}\,\mathcal{P}$ (cf. (4.8)), we see that the last identity is equivalent to the p equations

$$\begin{aligned}
z\,[F_0(z), F_p(z), F_{2p}(z), \ldots] &= [F_1(z), F_{p+1}(z), F_{2p+1}(z), \ldots]\,\mathcal{E},\\
z\,[F_1(z), F_{p+1}(z), F_{2p+1}(z), \ldots] &= [F_2(z), F_{p+2}(z), F_{2p+2}(z), \ldots]\,\mathcal{E},\\
\vdots\qquad &= \qquad\vdots\\
z\,[F_{p-2}(z), F_{2p-2}(z), F_{3p-2}(z), \ldots] &= [F_{p-1}(z), F_{2p-1}(z), F_{3p-1}(z), \ldots]\,\mathcal{E},\\
z\,[F_{p-1}(z), F_{2p-1}(z), F_{3p-1}(z), \ldots] &= [F_0(z), F_p(z), F_{2p}(z), \ldots]\,\tilde{\mathcal{E}}^T.
\end{aligned}$$

Using the above equations in succession gives

$$\begin{aligned}
z^p\,[F_0(z), F_p(z), F_{2p}(z), \ldots] &= z^{p-1}\,[F_1(z), F_{p+1}(z), F_{2p+1}(z), \ldots]\,\mathcal{E}\\
&= z^{p-2}\,[F_2(z), F_{p+2}(z), F_{2p+2}(z), \ldots]\,\mathcal{E}^2\\
&= \qquad\vdots\\
&= z\,[F_{p-1}(z), F_{2p-1}(z), F_{3p-1}(z), \ldots]\,\mathcal{E}^{p-1}\\
&= [F_0(z), F_p(z), F_{2p}(z), \ldots]\,\tilde{\mathcal{E}}^T\mathcal{E}^{p-1}\\
&= [F_0(z), F_p(z), F_{2p}(z), \ldots]\,\mathcal{K}^{(0)},
\end{aligned}$$

where the last equation above makes use of the case $k = 0$ of (4.10). The same procedure shows that

$$z^p\,[F_k(z), F_{p+k}(z), F_{2p+k}(z), \ldots] = [F_k(z), F_{p+k}(z), F_{2p+k}(z), \ldots]\,\mathcal{K}^{(k)}$$

for $k = 0, 1, \ldots p-1$. Together with $F_k(z) = z^k$ for $k \in \{0, 1, \ldots, p-1\}$ from (4.3), this implies by induction that F_{jp+k} has the form

$$F_{jp+k}(z) = z^k K_j^{(k)}(z^p) \quad (j \in \mathbf{N}_0 \text{ and } k = 0, 1, \ldots, p-1).$$

Here, $K_j^{(k)}$ is a polynomial of exact degree j. Moreover, using the last two displays, each of the p polynomial sequences $\{K_j^{(k)}\}_{j\geq 0}$ can be computed recursively, namely, from $K_0^{(k)}(z) = 1$ and

$$z\left[K_0^{(k)}(z), K_1^{(k)}(z), K_2^{(k)}(z), \ldots\right] = \left[K_0^{(k)}(z), K_1^{(k)}(z), K_2^{(k)}(z), \ldots\right]\mathcal{K}^{(k)}$$

(k = 0, 1, …, p-1). Analogously for the polynomials G_k of (4.4), there holds

$$G_{jp+k}(z) = z^k H_j^{(k)}(z^p) \quad (j \in \mathbb{N}_0 \text{ and } k = 0, 1, \ldots, p-1)$$

with polynomials $H_j^{(k)}$ of exact degree j satisfying $H_0^{(k)}(z) = 1$ and

$$z\left[H_0^{(k)}(z), H_1^{(k)}(z), H_2^{(k)}(z), \ldots\right] = \left[H_0^{(k)}(z), H_1^{(k)}(z), H_2^{(k)}(z), \ldots\right]\mathcal{H}^{(k)}$$

$(k = 0, 1, \ldots, p-1)$. The above recurrence relations can then be used to derive the explicit recurrence relations for $K_j^{(k)}(z)$ and $H_j^{(k)}(z)$, appearing at the end of Theorem 3.1.

OBSERVATION 4.2. *If $m = jp+k$, where $j \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, p-1\}$, then the spectrum of $\mathcal{H}_m$ consists of the pth roots of the eigenvalues of $\mathcal{H}_j^{(k)}$ (which is defined as the jth section of $\mathcal{H}^{(k)}$) and the eigenvalue $\lambda = 0$ with multiplicity k. Similarly, the spectrum of $\mathcal{K}_m$ consists of the pth roots of the eigenvalues of $\mathcal{K}_j^{(k)}$ (which is defined as the jth section of $\mathcal{K}^{(k)}$) and the eigenvalue $\lambda = 0$ with multiplicity k. To determine the eigenvalues of finite sections $\mathcal{H}_m$ and $\mathcal{K}_m$ of $\mathcal{H}$ and $\mathcal{K}$, it suffices to investigate the finite sections of $\mathcal{H}^{(k)}$ and $\mathcal{K}^{(k)}$ of* (4.7) *and* (4.10).

Taking into account the multiplicative factor $(\alpha^{p-1}\beta)^{1/p}$ in (4.1) between the eigenvalues of $\mathcal{G}_h$ and $\mathcal{H}$, and $\mathcal{F}_h$ and $\mathcal{K}$, we note that the remainder of Theorem 3.1 then follows from Observation 4.2.

As a third step, we have a closer look at the infinite matrices $\mathcal{H}^{(k)}$ and $\mathcal{K}^{(k)}$, and also at their finite sections $\mathcal{H}_j^{(k)}$ and $\mathcal{K}_j^{(k)}$.

LEMMA 4.3. *For every $k \in \{0, 1, \ldots, p-1\}$, the infinite matrices $\mathcal{H}^{(k)}$ and $\mathcal{K}^{(k)}$ are banded upper Hessenberg matrices with nonzero entries only along the diagonals $-1, 0, \ldots, p-1$. Moreover, they have nearly a Toeplitz structure, i.e., if we discard their first rows, then the resulting matrices are all equal to the Toeplitz matrix with symbol $(1 + 1/w)^p$.*

The first row of $\mathcal{H}^{(k)}$ equals $\sum_{j=p-1-k}^{p-1} \mathbf{b}_j^T$, where the "binomial vectors" $\mathbf{b}_k$ are defined by

$$\mathbf{b}_k^T := \left[\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k}, 0, \ldots\quad\right], \tag{4.11}$$

and the first row of $\mathcal{K}^{(k)}$ equals $\left(\sum_{j=p-1-k}^{p-1} \mathbf{b}_j^T\right) + (p-1)\,\mathbf{b}_{p-k-1}^T$.

Proof. Both matrices, $\mathcal{H}^{(k)}$ and $\mathcal{K}^{(k)}$, are, by definition (cf. (4.7) and (4.10)), products of $p-1$ upper bidiagonal matrices and one lower bidiagonal matrix, and they have thus banded upper Hessenberg form with nonzero entries only on the diagonals

$-1, 0, 1, \ldots, p-1$. We note that

$$(4.12)\quad \mathcal{H}^{(p-1)} = \mathcal{E}^{p-1}\mathcal{E}^T = \begin{bmatrix} \binom{p}{1} & \binom{p}{2} & \cdots & & \binom{p}{p} & & & \\ \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & & & \binom{p}{p} & & \\ & \binom{p}{0} & \binom{p}{1} & \ddots & & & \binom{p}{p} & \\ & & \ddots & \ddots & & & & \ddots \\ & & & & & & & \end{bmatrix}$$

is clearly a Toeplitz matrix with symbol $w(1+1/w)^p$. Note further from (4.7) that, for $k = 0, 1, \ldots p-2$,

$$\mathcal{H}^{(k+1)} - \mathcal{H}^{(k)} = \mathcal{E}^k \left(\mathcal{E}\mathcal{E}^T - \mathcal{E}^T\mathcal{E}\right)\mathcal{E}^{p-k-2} = \mathcal{E}^k \left(\mathbf{u}_1\mathbf{u}_1^T\right)\mathcal{E}^{p-k-2},$$

which, together with $\mathcal{E}\mathbf{u}_1 = \mathbf{u}_1$ and $\mathbf{u}_1^T\mathcal{E}^k = \mathbf{b}_k^T$, leads to

$$(4.13)\qquad \mathcal{H}^{(k+1)} = \mathcal{H}^{(k)} + \mathbf{u}_1\mathbf{b}_{p-k-2}^T \qquad (k = 0, 1, \ldots p-2).$$

Similarly, $\mathcal{K}^{(k)}$ and $\mathcal{H}^{(k)}$ are connected by the relation

$$(4.14)\qquad \mathcal{K}^{(k)} = \mathcal{H}^{(k)} + (p-1)\,\mathbf{u}_1\mathbf{b}_{p-k-1}^T \qquad (k = 0, 1, \ldots p-1).$$

Thus, all assertions of Lemma 4.3 follow easily from (4.12) – (4.14). □

Our next aim is to show that the finite sections $\mathcal{H}_j^{(k)}$ of $\mathcal{H}^{(k)}$ and $\mathcal{K}_j^{(k)}$ of $\mathcal{K}^{(k)}$ are oscillation matrices. To this end, we introduce, for $k = 0, 1, \ldots, p-1$, the matrices

$$(4.15)\quad \mathcal{A}_j^{(k)} := \mathcal{E}_j^k\mathcal{E}_j^T\mathcal{E}_j^{p-k-1} \quad \text{and} \quad \mathcal{B}_j^{(k)} := \mathcal{E}_j^k[\mathcal{E}^T + (p-1)\mathbf{u}_1\mathbf{u}_1^T]_j\mathcal{E}_j^{p-k-1}.$$

LEMMA 4.4. *For every $k \in \{0, 1, \ldots, p-1\}$ and for every $j \in \mathbb{N}$, the finite matrices $\mathcal{H}_j^{(k)}$ and $\mathcal{K}_j^{(k)}$ are oscillation matrices.*

Proof. First, we show that $\mathcal{A}_j^{(k)}$ and $\mathcal{B}_j^{(k)}$ of (4.15) are oscillation matrices: The upper bidiagonal matrix $\mathcal{E}_j$ and the lower bidiagonal matrix $\mathcal{E}_j^T$ are, from (4.5), clearly totally nonnegative, and, as their determinants are unity, they are both nonsingular. Similarly, $\tilde{\mathcal{E}}_j^T$ is also totally nonnegative and nonsingular. As a consequence of the Cauchy-Binet formula (cf. [10, p. 86] or [11, Theorem 2.3]), any product of these matrices, e.g., $\mathcal{A}_j^{(k)}$ or $\mathcal{B}_j^{(k)}$, is also nonsingular and totally nonnegative.

Next, it can be verified that $[\mathcal{E}^k\mathcal{E}^T]_j = [\mathcal{E}_j]^k\mathcal{E}_j^T + \mathbf{c}_k\mathbf{u}_j^T$ for $k = 1, 2, \ldots$, where $\mathbf{u}_j$ denotes the jth unit vector in $\mathbb{R}^j$ and where $\mathbf{c}_k$ in $\mathbb{R}^j$ is defined by

$$\begin{aligned} \mathbf{c}_k \;&:=\; \left[0, \ldots, 0, \tbinom{k}{0}, \tbinom{k}{1}, \ldots, \tbinom{k}{k-1}\right]^T, \quad \text{if } k < j, \text{ and} \\ \mathbf{c}_k \;&:=\; \left[\tbinom{k}{k-j}, \tbinom{k}{k-j+1}, \cdots, \tbinom{k}{k-1}\right]^T \qquad \text{for } k \geq j. \end{aligned}$$

Further, there holds $[\mathcal{M}\mathcal{E}]_j = \mathcal{M}_j\mathcal{E}_j$ for every infinite matrix $\mathcal{M}$. It follows (cf. (4.7) and (4.10)) that

$$(4.16)\qquad \begin{aligned} \mathcal{H}_j^{(0)} &= \mathcal{A}_j^{(0)}, \qquad & \mathcal{H}_j^{(k)} &= \mathcal{A}_j^{(k)} + \mathbf{c}_k\mathbf{u}_j^T \quad \text{and} \\ \mathcal{K}_j^{(0)} &= \mathcal{B}_j^{(0)}, \qquad & \mathcal{K}_j^{(k)} &= \mathcal{B}_j^{(k)} + \mathbf{c}_k\mathbf{u}_j^T, \end{aligned}$$

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$(k = 1, 2, \ldots, p-1)$. This shows that $\mathcal{A}_j^{(k)}$ differs from $\mathcal{H}_j^{(k)}$ at most in their last column (and that between $\mathcal{B}_j^{(k)}$ and $\mathcal{K}_j^{(k)}$, the same relation holds true).

Next, we claim that all subdiagonal and superdiagonal entries of $\mathcal{A}_j^{(k)}$ are positive for any $j \geq 2$. To see this, fix any $j \geq 2$. From (4.5), write $\mathcal{E}_j^k := \mathcal{I} + \mathcal{U}(k)$, where $\mathcal{U}(k)$ in $\mathbb{R}^{j \times j}$ is a nonnegative strictly upper triangular matrix with a positive superdiagonal for each k with $1 \leq k \leq p-1$. Similarly, write $\mathcal{E}_j^T := \mathcal{I} + \mathcal{L}$, where $\mathcal{L}$ in $\mathbb{R}^{j \times j}$ is a nonnegative lower triangular matrix consisting of a subdiagonal of all 1's and remaining entries zero. From (4.15), we have

$$\begin{aligned} \mathcal{A}_j^{(k)} &= (\mathcal{I} + \mathcal{U}(k))(\mathcal{I} + \mathcal{L})(\mathcal{I} + \mathcal{U}(p-k-1)) \\ &= \mathcal{I} + \mathcal{L} + \{\mathcal{U}(k) + \mathcal{U}(p-k-1)\} + \text{ a nonnegative matrix in } \mathbb{R}^{j \times j}. \end{aligned}$$

But for each k with $k = 0, 1, \ldots, p-1$, the sum $\{\mathcal{U}(k) + \mathcal{U}(p-k-1)\}$ always has a positive superdiagonal. Hence, $\mathcal{A}_j^{(k)}$ has all subdiagonal and superdiagonal entries positive for all $k = 0, 1, \ldots, p-1$ and all $j \geq 2$, with the same holding for $\mathcal{B}_j^{(k)}$ of (4.15).

To summarize, $\mathcal{A}_j^{(k)}$ and $\mathcal{B}_j^{(k)}$ are nonsingular, totally nonnegative, and have only nonzero entries along their sub- and super-diagonals. Consequently, they are oscillation matrices from Theorem A. Moreover, as $\mathcal{H}_j^{(k)}$ is a leading principal submatrix of $\mathcal{A}_{j+1}^{(k)}$, and similarly, as $\mathcal{K}_j^{(k)}$ is a leading principal submatrix of $\mathcal{B}_{j+1}^{(k)}$, they are therefore also oscillation matrices from Theorem B. $\square$

REMARK. *In Lemma* 4.4, *we proved that the leading principal submatrices of*

$$\mathcal{K}^{(k)} = \mathcal{E}^k \left(\mathcal{E}^T + (p-1)\mathbf{u}_1\mathbf{u}_1^T\right) \mathcal{E}^{p-k-1}$$

are oscillation matrices. The same argument shows that the leading principal submatrices of

$$\mathcal{E}^k \left(\mathcal{E}^T + p\mathbf{u}_1\mathbf{u}_1^T\right) \mathcal{E}^{p-k-1} \qquad (k = 0, 1, \ldots, p-1)$$

are also oscillation matrices.

Recall that the eigenvalues of $\mathcal{H}_j^{(k)}$ are the zeros of $H_j^{(k)}$ and that the eigenvalues of $\mathcal{K}_j^{(k)}$ are the zeros of $K_j^{(k)}$. This leads, from Theorems C and D, to

OBSERVATION 4.5. *The zeros* $\left\{\lambda_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $K_j^{(k)}$ *and the zeros* $\left\{\eta_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $H_j^{(k)}$ *satisfy*

$$0 < \lambda_{j,1}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \lambda_{j,j}^{(k)} \ \text{ and } \ 0 < \eta_{j,1}^{(k)} < \eta_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)},$$

for each $k \in \{0, 1, \ldots, p-1\}$.

Moreover, the following interlacing properties hold:

$$\begin{aligned} 0 < \lambda_{j+1,1}^{(k)} < \lambda_{j,1}^{(k)} < \lambda_{j+1,2}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \lambda_{j,j}^{(k)} < \lambda_{j+1,j+1}^{(k)}, \\ 0 < \eta_{j+1,1}^{(k)} < \eta_{j,1}^{(k)} < \eta_{j+1,2}^{(k)} < \eta_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)} < \eta_{j+1,j+1}^{(k)}. \end{aligned}$$

Note that this proves the main part of Theorem 3.2 and the first assertion of Theorem 3.3.

As a fourth step, we investigate the relation between the eigenvalues of $\mathcal{H}_j^{(k)}$ and the eigenvalues of $\mathcal{H}_j^{(k+1)}$. Similarly, we seek for a relation between the eigenvalues of $\mathcal{K}_j^{(k)}$ and the eigenvalues of $\mathcal{K}_j^{(k+1)}$. We need another set of auxiliary matrices, namely

$$\mathcal{C}_j^{(k)} := \mathcal{E}_j \mathcal{H}_j^{(k)} \mathcal{E}_j^{-1} \quad \text{and} \quad \mathcal{D}_j^{(k)} := \mathcal{E}_j \mathcal{K}_j^{(k)} \mathcal{E}_j^{-1}, \tag{4.17}$$

$(k = 0, 1, \ldots, p-2)$.

LEMMA 4.6. *For every $k \in \{0, 1, \ldots, p-2\}$ and for every $j \in \mathbb{N}$, the finite matrices $\mathcal{C}_j^{(k)}$ and $\mathcal{D}_j^{(k)}$ are oscillation matrices.*

Moreover, the following relations are valid:

$$\mathcal{H}_j^{(k+1)} = \mathcal{C}_j^{(k)} + \mathbf{u}_j \mathbf{u}_j^T \quad \textit{and} \quad \mathcal{K}_j^{(k+1)} = \mathcal{D}_j^{(k)} + \mathbf{u}_j \mathbf{u}_j^T\,,\ k \in \{0, 1, \ldots, p-2\}\,. \tag{4.18}$$

Proof. Note from (4.7) that, for $k \in \{0, 1, \ldots, p-2\}$,

$$\mathcal{H}_j^{(k)} \mathcal{E}_j^{-1} = [\mathcal{E}^k \mathcal{E}^T \mathcal{E}^{p-k-1}]_j \mathcal{E}_j^{-1} = [\mathcal{E}^k \mathcal{E}^T \mathcal{E}^{p-k-2}]_j$$

is the product of nonsingular totally nonnegative matrices which, using the method of proof of Lemma 4.4, can be verified to be an oscillation matrix. The matrix $\mathcal{C}_j^{(k)}$, as the product of the nonsingular totally nonnegative matrix $\mathcal{E}_j$ and the oscillation matrix $\mathcal{H}_j^{(k)} \mathcal{E}_j^{-1}$, is therefore an oscillation matrix from Theorem B.

Similarly,

$$\mathcal{K}_j^{(k)} \mathcal{E}_j^{-1} = [\mathcal{E}^k (\mathcal{E}^T + (p-1)\mathbf{u}_1 \mathbf{u}_1^T) \mathcal{E}^{p-k-1}]_j \mathcal{E}_j^{-1} = [\mathcal{E}^k (\mathcal{E}^T + (p-1)\mathbf{u}_1 \mathbf{u}_1^T) \mathcal{E}^{p-k-2}]_j$$

is an oscillation matrix (cf. the Remark following Lemma 4.4), and thus, $\mathcal{D}_j^{(k)}$, the product of a nonsingular totally nonnegative matrix and an oscillation matrix, is an oscillation matrix.

To show (4.18), we observe from (4.16) that

$$\begin{aligned} \mathcal{H}_j^{(k+1)} - \mathcal{C}_j^{(k)} &= \mathcal{A}_j^{(k+1)} + \mathbf{c}_{k+1} \mathbf{u}_j^T - \mathcal{E}_j [\mathcal{A}_j^{(k)} + \mathbf{c}_k \mathbf{u}_j^T] \mathcal{E}_j^{-1} \\ &= \mathbf{c}_{k+1} \mathbf{u}_j^T - \mathcal{E}_j \mathbf{c}_k \mathbf{u}_j^T \mathcal{E}_j^{-1}; \end{aligned}$$

(here, we set $\mathbf{c}_0 := \mathbf{0} \in \mathbb{R}^j$). Since $\mathbf{u}_j^T \mathcal{E}_j^{-1} = \mathbf{u}_j^T$ and $\mathbf{c}_{k+1} - \mathcal{E}_j \mathbf{c}_k = \mathbf{u}_j$, the first assertion of (4.18) is now proven, and the second follows exactly along the same lines. □

Note from (4.17) that the eigenvalues of the oscillation matrices $\mathcal{D}_j^{(k)}$ and $\mathcal{K}_j^{(k)}$ are obviously the same, and from (4.18) that $\mathcal{K}_j^{(k+1)}$ is obtained from $\mathcal{D}_j^{(k)}$ by adding 1 to the (j, j) entry, provided that $k \in \{0, 1, \ldots, p-2\}$. In addition,

$$\mathcal{E}_{j+1}^{p-1} \mathcal{K}_{j+1}^{(0)} \mathcal{E}_{j+1}^{1-p} = \mathcal{E}_{j+1}^{p-1} \mathcal{B}_{j+1}^{(0)} \mathcal{E}_{j+1}^{1-p} = \mathcal{E}_{j+1}^{p-1} [\mathcal{E}^T + (p-1)\mathbf{u}_1 \mathbf{u}_1^T]_{j+1} = \mathcal{B}_{j+1}^{(p-1)}$$

is an oscillation matrix (cf. the proof of Lemma 4.4) which contains $\mathcal{K}_j^{(p-1)}$ as its leading principal submatrix.

Analogously, the eigenvalues of the oscillation matrices $\mathcal{C}_j^{(k)}$ and $\mathcal{H}_j^{(k)}$ are the same, and $\mathcal{H}_j^{(k+1)}$ is obtained from $\mathcal{C}_j^{(k)}$ by adding 1 to the (j, j) entry (for

$k \in \{0, 1, \ldots, p-2\}$). Also, $\mathcal{H}_j^{(p-1)}$ (which has a Toeplitz structure) is a principal submatrix of $\mathcal{H}_{j+1}^{(0)}$. Theorem E directly leads us now to

OBSERVATION 4.7. *The zeros* $\left\{\lambda_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $K_j^{(k)}$ *are related by*

$$0 < \lambda_{j,1}^{(k)} < \lambda_{j,1}^{(k+1)} < \lambda_{j,2}^{(k)} < \lambda_{j,2}^{(k+1)} < \cdots < \lambda_{j,j}^{(k)} < \lambda_{j,j}^{(k+1)} \ \ (k = 0, 1, \ldots, p-2) \ \ \textit{and}$$
$$0 < \lambda_{j+1,1}^{(0)} < \lambda_{j,1}^{(p-1)} < \lambda_{j+1,2}^{(0)} < \lambda_{j,2}^{(p-1)} < \cdots < \lambda_{j,j}^{(p-1)} < \lambda_{j+1,j+1}^{(0)}.$$

Similarly, the zeros $\left\{\eta_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $H_j^{(k)}$ *are related by*

$$0 < \eta_{j,1}^{(k)} < \eta_{j,1}^{(k+1)} < \eta_{j,2}^{(k)} < \eta_{j,2}^{(k+1)} < \cdots < \eta_{j,j}^{(k)} < \eta_{j,j}^{(k+1)} \ \ (k = 0, 1, \ldots, p-2) \ \ \textit{and}$$
$$0 < \eta_{j+1,1}^{(0)} < \eta_{j,1}^{(p-1)} < \eta_{j+1,2}^{(0)} < \eta_{j,2}^{(p-1)} < \cdots < \eta_{j,j}^{(p-1)} < \eta_{j+1,j+1}^{(0)}.$$

Note that Observation 4.7 establishes Theorem 3.4.

As a fifth step, we seek relations between the eigenvalues of $\mathcal{H}_j^{(k)}$ and the eigenvalues of $\mathcal{K}_j^{(k)}$.

LEMMA 4.8. *The matrices* $\mathcal{K}_j^{(k)}$ *and* $\mathcal{H}_j^{(k)}$ *are connected through*

$$\begin{aligned} \mathcal{K}_j^{(p-1)} &= \mathcal{H}_j^{(p-1)} + (p-1)\mathbf{u}_1\mathbf{u}_1^T \ \ \textit{and} \\ \mathcal{E}_j\left(\mathcal{K}_j^{(k)} - \mathcal{H}_j^{(k)}\right)\mathcal{E}_j^{-1} &= \left(\mathcal{K}_j^{(k+1)} - \mathcal{H}_j^{(k+1)}\right) \ \ (\textit{for } k = 0, 1, \ldots, p-2). \end{aligned}$$

Proof. The first relation is a direct consequence of (4.14). The second relation is established (cf. (4.16)) from

$$\begin{aligned} \mathcal{E}_j\left(\mathcal{K}_j^{(k)} - \mathcal{H}_j^{(k)}\right)\mathcal{E}_j^{-1} &= \mathcal{E}_j\left(\mathcal{B}_j^{(k)} - \mathcal{A}_j^{(k)}\right)\mathcal{E}_j^{-1} \\ &= \mathcal{B}_j^{(k+1)} - \mathcal{A}_j^{(k+1)} = \mathcal{K}_j^{(k+1)} - \mathcal{H}_j^{(k+1)}. \end{aligned}$$

□

The argument leading from Lemma 4.8 to Observation 4.7 now implies

OBSERVATION 4.9. *The zeros* $\left\{\lambda_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $K_j^{(k)}$ *and the zeros* $\left\{\eta_{j,l}^{(k)}\right\}_{l=1}^{j}$ *of* $H_j^{(k)}$ *are related by*

$$0 < \eta_{j,1}^{(k)} < \lambda_{j,1}^{(k)} < \eta_{j,2}^{(k)} < \lambda_{j,2}^{(k)} < \cdots < \eta_{j,j}^{(k)} < \lambda_{j,j}^{(k)}.$$

Note that Observation 4.9 completes the proof of Theorem 3.3.

As a sixth step, we derive a sharp upper bound for the eigenvalues of $\mathcal{H}_j^{(k)}$ and the eigenvalues of $\mathcal{K}_j^{(k)}$. We first deduce an upper bound for the spectral radius $\rho(\mathcal{K}_j^{(p-1)})$ of $\mathcal{K}_j^{(p-1)}$, which is a nonnegative irreducible matrix. To this end, we define the positive vector $\mathbf{x}$ in $\mathbb{R}^j$, for $j \geq 1$, by

$$\mathbf{x} = [x_1, x_2, \ldots, x_j]^T := \left[\tau^0, \tau^1, \ldots, \tau^{j-1}\right]^T \in \mathbb{R}^j, \ \ \text{where } \tau := 1/(p-1),$$

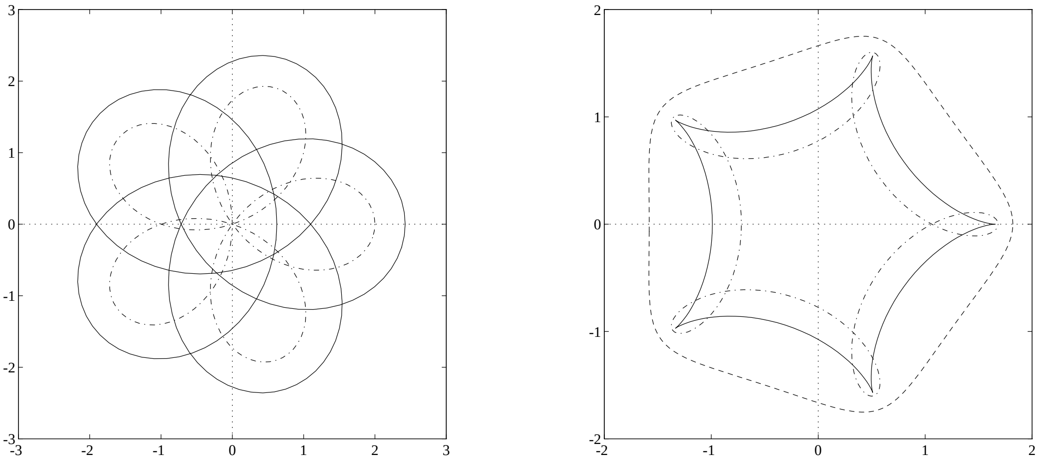

FIG. 4.1. *Shape of* Γ_ρ *for* $p = 5$.

and we compute the components of $\mathbf{y} = [y_1, y_2, \ldots, y_j]^T := \mathcal{K}_j^{(p-1)}\mathbf{x}$. It is easy to verify from (4.14) that $y_k/x_k = \kappa_p := p^p/(p-1)^{p-1}$ for $k = 1, 2, \ldots, j-p+1$ and that $y_k/x_k < \kappa_p$ for $k = j-p+2, \ldots, p$. By a refined form of the "Quotient Theorem" due to Collatz (cf. [21, Theorem 2.2]), we have that $\rho(\mathcal{K}_j^{(p-1)}) < \kappa_p$, for all $j \geq 1$. We already know (cf. Observations 4.7 and 4.9) that $\rho(\mathcal{K}_j^{(k)}) < \rho(\mathcal{K}_j^{(p-1)})$ for $k = 0, 1, \ldots, p-2$, $j \geq 1$, and that $\rho(\mathcal{H}_j^{(k)}) < \rho(\mathcal{K}_j^{(p-1)})$ for $k = 0, 1, \ldots, p-1$, $j \geq 1$. These inequalities could also be derived from the Perron-Frobenius theory on nonnegative matrices.

Next, we wish to show that, for each $k \in \{0, 1, \ldots, p-1\}$, the eigenvalues of $\mathcal{H}_j^{(k)}$, as well as the eigenvalues of $\mathcal{K}_j^{(k)}$, are *dense* in the interval $\mathbb{I} := [0, \kappa_p]$ (for $j \to \infty$). In view of the interlacing properties described in Observations 4.5, 4.7 and 4.9, it is certainly sufficient to show that the eigenvalues of the finite sections of the Toeplitz matrix $\mathcal{H}$ of (4.2) are dense on the star $\mathcal{S}(p, 1, 1)$ (cf. (3.1)). In other words, we must show that $\mathcal{S}(p, 1, 1) = \Lambda_\infty(\mathcal{H})$ (cf. (2.13)). Since $\Lambda(\mathcal{H}_j) \subseteq \mathcal{S}(p, 1, 1)$ for every $j = 1, 2, \ldots$ (which obviously implies $\Lambda_\infty(\mathcal{H}) \subseteq \mathcal{S}(p, 1, 1)$) has already been shown, the symmetry properties of $\mathcal{S}(p, 1, 1)$ imply that we merely must prove that $[0, \kappa_p^{1/p}] \subseteq \Lambda_\infty(\mathcal{H})$ is valid.

Here, we make use of Theorem F. With $\psi(w) = w + w^{1-p}$ for $p > 2$, we monitor the shape of the curve $\Gamma_\rho := \{z \in \mathbb{C} : z = \psi(w) \text{ with } w = \rho\}$, as ρ increases from 0 to ∞ (cf. Fig. 4.1), and show that $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for every $\rho > 0$:

For $0 < \rho < 1, \Gamma_\rho$ has its largest intersection point with the real axis at $\tau_\rho := \rho + \rho^{1-p} > \kappa_p^{1/p}$. Then, because the winding number $n(\Gamma_\rho, r)$ is nonzero for each r in the interval $[0, \tau_\rho]$, it follows by definition that $[0, \tau_\rho] \subseteq \Lambda_\rho(\mathcal{H})$. Consequently, $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for $0 < \rho < 1$.

For $\rho = 1$, Γ_ρ is a looped hypocycloid centered at the origin, which intersects the positive real axis in exactly one point, namely $\tau_1 = 2 > \kappa_p^{1/p}$. Consequently, $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for $\rho = 1$.

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For $1 < \rho < (p-1)^{1/p}$, Γ_ρ is a looped hypocycloid centered at the origin, which intersects the positive real axis in exactly two points, the larger of which equals $\tau_\rho = \rho + \rho^{1-p} > \kappa_p^{1/p}$. Consequently, $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for $1 < \rho < (\rho - 1)^{1/p}$.

For $\rho = (p-1)^{1/p}$, Γ_ρ is a cusped hypocycloid centered at the origin, which intersects the positive real axis in exactly one point, namely in $\tau_\rho = \kappa_p^{1/p}$. Consequently, $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for $\rho = (p-1)^{1/p}$.

Finally, for $\rho > (p-1)^{1/p}$, Γ_ρ is a blunted hypocycloid centered at the origin, which intersects the positive real axis in exactly one point, namely in $\tau_\rho = \rho + \rho^{1-p} > \kappa_p^{1/p}$. Consequently, $[0, \kappa_p^{1/p}] \subseteq \Lambda_\rho(\mathcal{H})$ for $\rho > (p-1)^{1/p}$.

This completes the proof of $[0, \kappa_p^{1/p}] \subseteq \cap_{\rho>0}\Lambda_\rho(\mathcal{H}) = \Lambda_\infty(\mathcal{H})$ (cf. Theorem F) and also the proof of Theorem 3.2.

5. Concluding Remarks. We finally note that we also implicitly determined the zeros and local extreme points of the Faber polynomials associated with a another class of compact sets, which are defined by mappings of the form

$$\psi(w) = \alpha w \left(1 + \frac{\beta}{w}\right)^p \qquad (p \in \mathbb{N},\ \alpha > 0,\ \beta \in \mathbb{C},\ \beta \neq 0)\,, \tag{5.1}$$

where ψ is conformal in $|w| > 1$ if and only if $|\beta| \leq 1/(p-1)$. (For $p = 1$, this condition is vacuous.) In Fig. 5.1, we present examples of the sets

$$\Omega = \Upsilon(p, \alpha, \beta) := \mathbb{C}_\infty \setminus \{z \in \mathbb{C} : z = \psi(w) \text{ with } |w| > 1\}\,, \tag{5.2}$$

where, e.g., for the figure on the left, $\Upsilon(3, 1, 1/2)$ (which has a cusp on its boundary) and $\Upsilon(3, 3/2, 1/3)$ are shown, along with the zeros of the associated Faber polynomials $\{F_m\}_{m=1}^{50}$ (cf. the second footnote in Section 1). Note that for $p = 1$, $\Upsilon(1, \alpha, \beta)$ represents the closed disk with center $\alpha\beta$ and radius α. It is well known in this case that F_m has then only one zero, namely $\xi = \alpha\beta$ (with multiplicity m). In the sequel, we shall therefore concentrate on the case of $p > 1$[4].

From (5.1), we know that the infinite upper Hessenberg matrix $\mathcal{F}$ associated with these Faber polynomials $\{F_m\}_{m\geq 0}$ has the form (cf. (2.4))

$$\mathcal{F} = \mathcal{F}_\Upsilon := \alpha \begin{bmatrix} \binom{p}{1}\beta & 2\binom{p}{2}\beta^2 & \cdots & p\binom{p}{p}\beta^p & & & \\ \binom{p}{0} & \binom{p}{1}\beta & \binom{p}{2}\beta^2 & & \binom{p}{p}\beta^p & & \\ & \binom{p}{0} & \binom{p}{1}\beta & \ddots & & \binom{p}{p}\beta^p & \\ & & \ddots & \ddots & & & \ddots \\ & & & & & & \end{bmatrix},$$

whereas the the infinite upper Hessenberg Toeplitz matrix $\mathcal{G}$ associated with the Faber polynomials of the second kind $\{G_m\}_{m\geq 0}$ for $\Upsilon(p, \alpha, \beta)$ has the form (cf. (2.9))

$$\mathcal{G} = \mathcal{G}_\Upsilon := \alpha \begin{bmatrix} \binom{p}{1}\beta & \binom{p}{2}\beta^2 & \cdots & \binom{p}{p}\beta^p & & & \\ \binom{p}{0} & \binom{p}{1}\beta & \binom{p}{2}\beta^2 & & \binom{p}{p}\beta^p & & \\ & \binom{p}{0} & \binom{p}{1}\beta & \ddots & & \binom{p}{p}\beta^p & \\ & & \ddots & \ddots & & & \ddots \\ & & & & & & \end{bmatrix}.$$

[4] Notice also that for $p = 2$, we obtain another well-known case: $\Upsilon(2, \alpha, \beta)$ is either an interval (if $|\beta| = 1/2$) or an ellipse together with its interior (if $|\beta| < 1/2$).

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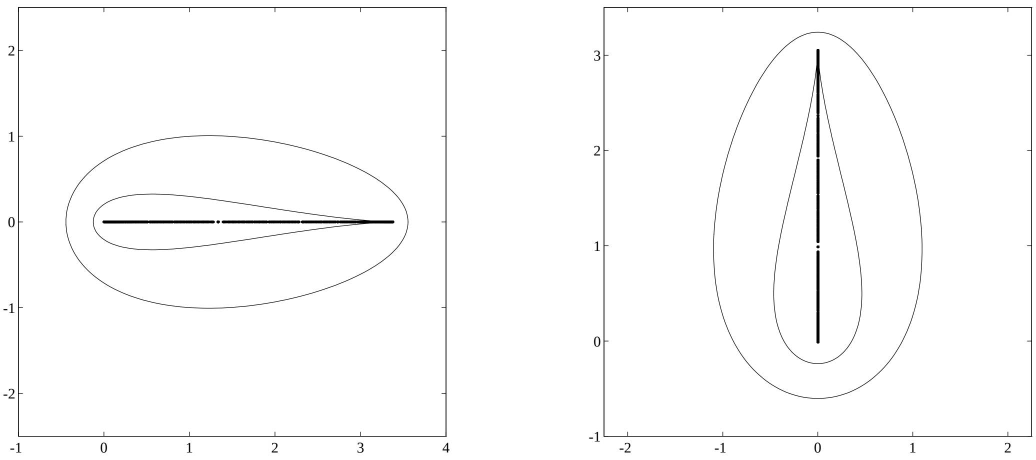

FIG. 5.1. *$\Upsilon(p,\alpha,\beta)$ for $p=3$ and $p=5$, together with the zeros of the associated Faber polynomials $\{F_m\}_{m=1}^{50}$.*

As in the case of the hypocycloidal domains, we apply a diagonal similarity transformation to the above matrices. With the infinite diagonal matrix

$$\mathcal{D} := \operatorname{diag}\left(\beta^0, \beta^1, \beta^2, \ldots\right),$$

there holds

$$\mathcal{D}\mathcal{F}_{\Upsilon}\mathcal{D}^{-1} = \tilde{\mathcal{K}} := \alpha\beta \begin{bmatrix} \binom{p}{1} & 2\binom{p}{2} & \cdots & & p\binom{p}{p} & & & \\ \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & & & \binom{p}{p} & & \\ & \binom{p}{0} & \binom{p}{1} & \ddots & & & \binom{p}{p} & \\ & & \ddots & \ddots & & & & \ddots \\ & & & & & & & \end{bmatrix},$$

and, similarly,

$$\mathcal{D}\mathcal{G}_{\Upsilon}\mathcal{D}^{-1} = \tilde{\mathcal{H}} := \alpha\beta \begin{bmatrix} \binom{p}{1} & \binom{p}{2} & \cdots & & \binom{p}{p} & & & \\ \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & & & \binom{p}{p} & & \\ & \binom{p}{0} & \binom{p}{1} & \ddots & & & \binom{p}{p} & \\ & & \ddots & \ddots & & & & \ddots \\ & & & & & & & \end{bmatrix}.$$

Since $k\binom{p}{k} = p\binom{p-1}{k-1}$ (for $p = 1, 2, \ldots$ and $k = 1, 2, \ldots, p$), the first row of $\tilde{\mathcal{K}}$ equals

$$\alpha\beta\, p \left[\binom{p-1}{0}, \binom{p-1}{1}, \ldots, \binom{p-1}{p-1}, 0, \ldots \quad\right] = \alpha\beta\, p\, \mathbf{b}_{p-1}^T,$$

where the vector $\mathbf{b}_{p-1}$ is defined in (4.11). Thus, we conclude, from (4.11) and the last line of Lemma 4.3, that the above matrices can be expressed simply as

$$\tilde{\mathcal{K}} = \alpha\beta\, \mathcal{K}^{(0)} \quad \text{and} \quad \tilde{\mathcal{H}} = \alpha\beta\, \mathcal{H}^{(p-1)}.$$

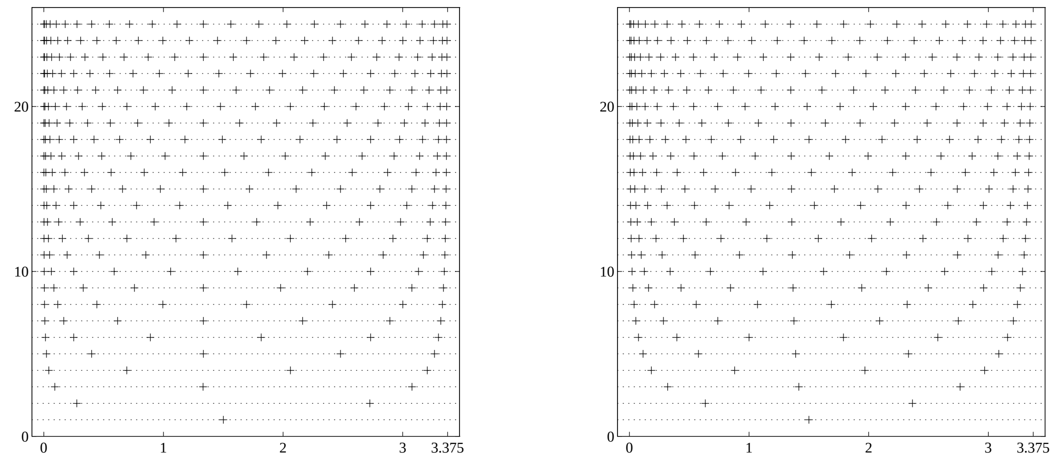

FIG. 5.2. *Zeros of* $\{F_m\}_{m=1}^{25}$ *(on the left) and of* $\{G_m\}_{m=1}^{25}$ *(on the right) for* $\Upsilon(3,1,1/2)$ *plotted versus* m.

Consequently, their associated Faber polynomials are explicitly given by

$$F_m(z) = \alpha^m K_m^{(0)}\left(\frac{z}{\alpha\beta}\right) \quad \text{and} \quad G_m(z) = \alpha^{m+1} H_m^{(p-1)}\left(\frac{z}{\alpha\beta}\right) \qquad (m \in \mathbb{N}_0),$$

where the polynomials $K_m^{(0)}$ and $H_m^{(p-1)}$ are recursively defined in Theorem 3.1.

From Theorems 3.2 and 3.3, we immediately obtain our last new result:

THEOREM 5.1. *Let* F_m *and* G_m *denote, respectively, the* m*th Faber polynomial of the first and second kind associated with the compact set* $\Upsilon(p,\alpha,\beta)$ *of* (5.2)*, where we assume that* $p > 1$*. Then, all zeros* $\{\xi_{m,k}\}_{k=1}^m$ *of* F_m*, as well as all zeros* $\{\zeta_{m,k}\}_{k=1}^m$ *of* G_m*, are simple and are located in the (complex) open interval* $(0, \alpha\beta p^p/(p-1)^{p-1})$*. Moreover, for* $m \to \infty$*, both sets,* $\{\xi_{m,k}\}_{k=1}^m$ *and* $\{\zeta_{m,k}\}_{k=1}^m$*, become dense on the closed interval* $[0, \alpha\beta p^p/(p-1)^{p-1}]$.

With the orderings

$$|\xi_{m,1}| < |\xi_{m,2}| < \cdots < |\xi_{m,m}| \quad \textit{and} \quad |\zeta_{m,1}| < |\zeta_{m,2}| < \cdots < |\zeta_{m,m}|,$$

the following interlacing properties are valid:

$$\begin{aligned} &0 < |\xi_{m+1,1}| < |\xi_{m,1}| < |\xi_{m+1,2}| < |\xi_{m,2}| < \cdots < |\xi_{m,m}| < |\xi_{m+1,m+1}|, \quad \textit{and} \\ &0 < |\zeta_{m+1,1}| < |\zeta_{m,1}| < |\zeta_{m+1,2}| < |\zeta_{m,2}| < \cdots < |\zeta_{m,m}| < |\zeta_{m+1,m+1}|, \end{aligned}$$

for every $m \in \mathbb{N}_0$.

These interlacing properties are illustrated in Fig. 5.2 for $\Upsilon(3,1,1/2)$.

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