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NEUMANN–NEUMANN METHODS FOR VECTOR FIELD PROBLEMS*

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Abstract. In this paper, we study some Schwarz methods of Neumann–Neumann type for some vector field problems, discretized with the lowest order Raviart–Thomas and Nédélec finite elements. We consider a hybrid Schwarz preconditioner consisting of a coarse component, which involves the solution of the original problem on a coarse mesh, and local ones, which involve the solution of Neumann problems on the elements of the coarse triangulation, also called substructures. We show that the condition number of the corresponding method is independent of the number of substructures and grows logarithmically with the number of unknowns associated with an individual substructure. It is also independent of the jumps of both the coefficients of the original problem. The numerical results presented validate our theoretical bound.

Key words. edge elements, Raviart–Thomas elements, domain decomposition, iterative substructuring, preconditioners, heterogeneous coefficients

AMS subject classifications. 65F10, 65N22, 65N30, 65N55.

1. Introduction.

In this paper, we consider the two boundary value problems

$$
\begin{array}{rcll} Lu := -\mathbf{grad}\,(a\,\mathrm{div}\,\mathbf{u}) + B\,\mathbf{u} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} & = & 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}
$$

with Ω a bounded polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, and

$$
\begin{array}{rcll} Lu := \mathbf{curl}\,(a\,\mathrm{curl}\,\mathbf{u}) + B\,\mathbf{u} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}\times\mathbf{n} & = & 0 & \text{on } \partial\Omega, \end{array} \tag{1.2}
$$

with Ω a bounded polygonal domain in $\mathbb{R}^2$. The domain Ω has unit diameter and $\mathbf{n}$ is its outward normal. The coefficient matrix B is a symmetric uniformly positive definite matrix–valued function with $b_{i,j} \in L^\infty(\Omega)$, $1 \le i, j \le n$, and $a \in L^\infty(\Omega)$ is a positive function bounded away from zero. We refer to [3, 32, 19, 33], for some some significant applications of (1.1) and (1.2) to flow and electromagnetic problems.

The weak formulation of problems (1.1) and (1.2) requires the introduction of the Hilbert spaces $H(\mathrm{div}\,;\Omega)$ and $H(\mathrm{curl}\,;\Omega)$, defined by

$$
\begin{array}{rcl} H(\mathrm{div}\,;\Omega) & := & \left\{\mathbf{v}\in (L^2(\Omega))^n |\quad \mathrm{div}\,\mathbf{v}\in L^2(\Omega)\right\}, \quad n=2,3, \\ H(\mathrm{curl}\,;\Omega) & := & \left\{\mathbf{v}\in (L^2(\Omega))^2 |\quad \mathrm{curl}\,\mathbf{v}\in L^2(\Omega)\right\}. \end{array}
$$

We first consider problem (1.1). The space $H(\mathrm{div}\,;\Omega)$ is equipped with the following inner product and graph norm,

$$
(\mathbf{u},\mathbf{v})_{\mathrm{div}} := \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\,dx, \qquad \|\mathbf{u}\|^2_{\mathrm{div}} := (\mathbf{u},\mathbf{u})_{\mathrm{div}},
$$

and the normal component $\mathbf{u}\cdot\mathbf{n}$, of a vector $\mathbf{u}\in H(\mathrm{div}\,;\Omega)$ on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)$; see [15, 7]. The subspace of vectors in $H(\mathrm{div}\,;\Omega)$ with vanishing normal component on $\partial\Omega$ is denoted by $H_0(\mathrm{div}\,;\Omega)$.

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If, for any $\mathcal{D} \subset \Omega$, we define the bilinear form

$$a_{\mathcal{D}}(\mathbf{u},\mathbf{v}) := \int_{\mathcal{D}} (a \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} + B\, \mathbf{u}\cdot\mathbf{v})\, dx, \quad \mathbf{u}, \mathbf{v} \in H(\operatorname{div};\Omega), \tag{1.3}$$

with

$$a(\cdot,\cdot) := a_{\Omega}(\cdot,\cdot),$$

in case $\mathcal{D} = \Omega$, the variational formulation of Equation (1.1) is then:
Find $\mathbf{u} \in H_0(\operatorname{div};\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega} \mathbf{f}\cdot\mathbf{v}\, dx, \quad \mathbf{v} \in H_0(\operatorname{div};\Omega).$$

We associate an energy norm, defined by $||\cdot||_a^2 := a(\cdot,\cdot)$, to this bilinear form; our assumptions on the coefficients guarantee that this norm is equivalent to the graph norm.

In this paper, we introduce and analyze some Neumann–Neumann preconditioners for the solution of systems arising from the finite element approximation of (1.1) and (1.2). An iterative substructuring method usually consists of local components, which involve the solution of local problems, related to a partition of the original domain Ω into non–overlapping subdomains, called *substructures*. In order to obtain a condition number that is independent of the number of substructures, a coarse component can be added to the method, which typically involves the solution of a problem defined on a coarse space related to the partition into substructures. Depending on the particular method, the substructures can be the elements of a coarse mesh or arbitrary connected subsets of Ω, consisting of unions of elements of the fine mesh. The lack of generous overlap results in bounds for the condition number that are not optimal, but which can be made independent of the jumps of the coefficients. In a Neumann–Neumann method, the degrees of freedom of the local spaces are related to the entire boundaries of the substructures; see [5, 9, 20, 21, 8, 12, 28, 10, 22, 31]. For an introduction to iterative substructuring methods, we refer to [31] and to the references therein.

The study and analysis of preconditioners for Nédélec and Raviart–Thomas approximations is quite new. Extensive work has started only in the past three years, in order to extend classical Schwarz preconditioners to these vector problems. Two–level overlapping Schwarz preconditioners for $H(\operatorname{div};\Omega)$ and $H(\operatorname{curl};\Omega)$ were initially developed for two dimensions, in [3], and then extended to three dimensions, in [32, 19]. Multigrid and multilevel methods were considered in [3, 2, 18, 17, 4, 19], and iterative substructuring methods in [1, 34, 36]. We also mention [16, 13, 6, 23, 24, 8, 29, 30], which report on a study of a class of two- and multi-level methods for mixed approximations of Poisson's equation.

An important element in the definition of a Neumann–Neumann method is a set of scaling functions defined on the boundaries of the substructures, which involve the values of the coefficients of the partial differential equation. The use of these functions can ensure that the condition number of the corresponding preconditioned system be independent of the jumps of the coefficients across the substructures. Here, we propose a set of scaling functions, which involve only the values of one coefficient in (1.1) and (1.2). An important feature of our method is that it is independent of jumps of *both* coefficients. We know of no previous work on a Neumann–Neumann method for the vector problems (1.1) and (1.2), or no previous theoretical study of a Neumann–Neumann method for the case where more than one coefficient has jumps. Our original analysis is contained in [33].

We only consider problem (1.1), in full detail. For problem (1.2) in two dimensions, the associated bilinear form is

$$\int_{\mathcal{D}} (a \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{v} + B\, \mathbf{u} \cdot \mathbf{v})\, dx, \quad \mathbf{u}, \mathbf{v} \in H(\operatorname{curl}; \Omega),$$

and the result follows from that of (1.1) and the observation that the functions in $H(\operatorname{curl}; \Omega)$ are obtained from those in $H(\operatorname{div}; \Omega)$ by a rotation of ninety degrees; see [15, Rem. 2.3, P. 35]. To our knowledge, the case of $H(\operatorname{curl}; \Omega)$ in three dimensions remains to be studied almost completely. We know of only one study of a two–subdomain iterative substructuring method for a problem in $H(\operatorname{curl}; \Omega)$ in three dimensions; see [1]. We also remark that, in this paper, we only consider Dirichlet boundary conditions, in full detail, but that the generalization to the case of Neumann or Robin conditions is straightforward.

The outline of the paper is as follows: In Section 2, we state some additional properties of the space $H(\operatorname{div}; \Omega)$ and describe the Raviart–Thomas finite element spaces together with some trace spaces. In Section 3, we consider the discrete problem and its Schur complement formulation, while in Section 4, we introduce our algorithm. In Section 5, we recall some technical results, which we need in Section 6, for the proof of our main result. After discussing some implementation issues, in Section 7, we present some numerical results in Section 8, in order to validate our analysis.

2. Continuous and discrete spaces. In addition to $H(\operatorname{div}; \Omega)$, we also use some standard Sobolev spaces. Given a bounded open Lipschitz domain $\mathcal{D} \subset \mathbb{R}^n$, with boundary $\partial\mathcal{D}$, let $\|\cdot\|_{0;\mathcal{D}}$ denote the L_2–norm and $|\cdot|_{s;\mathcal{D}}$ the semi–norm of the Sobolev space $H^s(\mathcal{D})$, $s > 0$. In case that $\mathcal{D} = \Omega$, we drop the reference to the region. Throughout, we work with scaled norms for the spaces $H^s(\mathcal{D})$, $s > 0$. We define

$$\|u\|_{1;\mathcal{D}}^2 := |u|_{1;\mathcal{D}}^2 + \frac{1}{H_{\mathcal{D}}{}^2} \|u\|_{0;\mathcal{D}}^2,$$

and

$$\|u\|_{\frac{1}{2};\partial\mathcal{D}}^2 := |u|_{\frac{1}{2};\partial\mathcal{D}}^2 + \frac{1}{H_{\mathcal{D}}} \|u\|_{0;\partial\mathcal{D}}^2.$$

Here and in the following, given a generic subset $\mathcal{D}$ of $\mathbb{R}^n$, we denote its diameter by $H_{\mathcal{D}}$. These scaled norms are obtained from the standard definition of the Sobolev norms on a region with diameter one and a dilation, and dropping, possibly, common multiplicative factors; see, e.g., [31].

As already mentioned, the normal component of any vector field $\mathbf{u} \in H(\operatorname{div}; \mathcal{D})$ belongs to $H^{-\frac{1}{2}}(\partial\mathcal{D})$, and the corresponding trace operator is continuous and surjective; see [15, 7]. Here, $H^{-\frac{1}{2}}(\partial\mathcal{D})$ is equipped with the norm

$$||\mathbf{u} \cdot \mathbf{n}||_{-\frac{1}{2};\partial\mathcal{D}} := \sup_{\substack{\phi \in H^{\frac{1}{2}}(\partial\mathcal{D}) \\ \phi \neq 0}} \frac{\langle \mathbf{u} \cdot \mathbf{n}, \phi \rangle}{||\phi||_{\frac{1}{2};\partial\mathcal{D}}}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\partial\mathcal{D})$ and $H^{\frac{1}{2}}(\partial\mathcal{D})$. The following inequality holds

$$\|\mathbf{u} \cdot \mathbf{n}\|_{-\frac{1}{2};\partial\mathcal{D}}^2 \leq C \left(||\mathbf{u}||_{0;\mathcal{D}}^2 + H_{\mathcal{D}}^2 ||\operatorname{div} \mathbf{u}||_{0;\mathcal{D}}^2 \right), \tag{2.2}$$

with a constant C that is independent of $H_{\mathcal{D}}$. The scaling factor is obtained by dilation from a region of unit diameter. From now on, we denote by C a positive generic constant, uniformly bounded from above, and by c a positive generic constant uniformly bounded away from zero.

We now introduce some finite element spaces. Let $\mathcal{T}_H$ be a shape–regular triangulation of Ω, of maximum diameter H, consisting of triangles or rectangles, for $n = 2$, and of tetrahedra or parallelepipeds, for $n = 3$. Let also $\mathcal{T}_h$ be a *shape–regular* and *quasi–uniform* triangulation, with characteristic diameter h, obtained by refining the elements of $\mathcal{T}_H$, in such a way that $\mathcal{T}_h$ is conforming in Ω. A generic element of $\mathcal{T}_h$ and $\mathcal{T}_H$ is denoted by t and T, respectively. For $n = 3$, the sets of faces of the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, are denoted by $\mathcal{F}_h$ and $\mathcal{F}_H$, respectively. For $n = 2$, the sets of edges of the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, are also denoted by $\mathcal{F}_h$ and $\mathcal{F}_H$, respectively. A generic face (or edge, for $n = 2$) is denoted by f or F. In the following, we refer to the elements in $\mathcal{F}_h$ and $\mathcal{F}_H$ as *faces*, both for $n = 2$ and $n = 3$. The elements of the coarse triangulation are called substructures. We define the interface Γ as the union of the parts of the boundaries of the substructures that do not belong to $\partial\Omega$,

$$\Gamma := \bigcup_{T \in \mathcal{T}_H} \partial T \setminus \partial\Omega. \tag{2.3}$$

We also suppose that the coefficients a and B are constant on each substructure T and equal to a_T and B_T, respectively. They may have arbitrary jumps across the interface. In addition, the matrices $\{B_T\}$ satisfy

$$\beta_T\, \eta^T \eta \leq \eta^T B_T \eta \leq \gamma_T\, \eta^T \eta, \quad \forall \eta \in \mathbb{R}^n, \tag{2.4}$$

where β_T and γ_T are positive constants, which can depend on the substructure T.

In the following, we only consider, in full detail, triangulations based on rectangles and parallelepipeds, but our results are equally valid for finite element spaces built on triangles and tetrahedra. Much of the analysis is carried out on a square or cubic substructure divided into square or cubic elements, but the results remain equally valid if the elements and substructures are images of a reference square or cube under sufficiently benign mappings, which effectively means that their aspect ratios have to remain uniformly bounded. We remark that our analysis is carried out locally for one substructure at a time. We can therefore interpret the factor H/h, which appears in our estimates, as

$$\max_{T \in \mathcal{T}_H} \max_{\substack{t \in \mathcal{T}_h \\ t \subset T}} \frac{H_T}{H_t}.$$

We first consider the lowest order Raviart–Thomas elements defined on the fine mesh,

$$X_h = X_h(\Omega) := \left\{ \mathbf{u} \in H(\mathrm{div}\,; \Omega) |\ \mathbf{u}_{|t} \in \mathcal{RT}(t),\, t \in \mathcal{T}_h \right\},$$

where the local space, for a rectangle or a cube with sides parallel to the coordinate axes, is given by

$$\mathcal{RT}(t) := \{\mathbf{u} |\ u_i = \alpha_i + \beta_i\, x_i,\ i = 1, \cdots, n\}\,;$$

see [25, 7]. Here, the i–th component of the vector $\mathbf{u}$ is denoted by u_i, and $\mathbf{x}$ is the position vector. The normal components over the faces (four, if $n = 2$, six, if $n = 3$) of t are constant and the local degrees of freedom in $\mathcal{RT}(t)$ can be taken as the averages of the normal components over the faces. The global degrees of freedom are the averages of the normal component over the faces in $\mathcal{F}_h$:

$$\lambda_f(\mathbf{u}) \;:=\; \frac{1}{|f|}\int_f \mathbf{u}\cdot\mathbf{n}\,ds, \quad f\in\mathcal{F}_h. \tag{2.5}$$

Here $|f|$ is the area of the face f and the direction of the normal can be fixed arbitrarily for each face.

As in the case of nodal elements, the L^2–norm of a vector $\mathbf{u}\in\mathcal{RT}(t)$ can be bounded from above and below by means of its degrees of freedom

$$c\sum_{f\subset\partial t}\left(H_f^{n/2}\,\lambda_f(\mathbf{u})\right)^2 \le \|\mathbf{u}\|_{0;t}^2 \;\le\; C\sum_{f\subset\partial t}\left(H_f^{n/2}\,\lambda_f(\mathbf{u})\right)^2, \tag{2.6}$$

where the constants c and C only depend on the aspect ratio of the element t. The proof given for nodal elements in [26, Prop. 6.3.1] can easily be adapted to the present case.

We also need the finite element space conforming in $H_0(\mathrm{div}\,;\Omega)$,

$$X_{0;h}=X_{0;h}(\Omega):=X_h(\Omega)\cap H_0(\mathrm{div}\,;\Omega),$$

the corresponding space defined on the coarse triangulation $\mathcal{T}_H$,

$$X_{0;H}=X_{0;H}(\Omega):=\left\{\mathbf{u}\in H_0(\mathrm{div}\,;\Omega)|\ \mathbf{u}_{|_T}\in\mathcal{RT}(T),\ T\in\mathcal{T}_H\right\},$$

and the local spaces defined on a generic substructure T,

$$\begin{aligned} X_h(T) &:= \left\{\mathbf{u}\in H(\mathrm{div}\,;T)|\ \mathbf{u}_{|_t}\in\mathcal{RT}(t),\ t\in\mathcal{T}_h,\ t\subset T\right\},\\ X_{0;h}(T) &:= X_h(T)\cap H_0(\mathrm{div}\,;T). \end{aligned}$$

For $n=2$, the Nédélec finite element spaces, conforming in $H(\mathrm{curl}\,;\Omega)$, are obtained from the Raviart–Thomas spaces by a rotation of ninety degrees; see [25, 7].

We now define some finite element spaces on the boundaries of the substructures. Given a substructure T, define $S_H(\partial T)$ as the space of functions which are constant on each coarse face $F\subset\partial T$

$$S_H(\partial T):=\{\psi:\partial T\to\mathbb{R}\mid \psi_{|_F}\ \text{constant},\ F\in\mathcal{F}_H,\ F\subset\partial T\};$$

its dimension is four, for $n=2$, and six, for $n=3$. We also define $S_h(\partial T)$ as the space of functions that are constant on each fine face $f\in\mathcal{F}_h$, $f\subset\partial T$

$$S_h(\partial T):=\{\psi:\partial T\to\mathbb{R}\mid \psi_{|_f}\ \text{constant},\ f\in\mathcal{F}_h,\ f\subset\partial T\},$$

and its subspace $S_{0;h}(\partial T)$, of functions that have mean value zero on ∂T

$$S_{0;h}(\partial T):=\left\{\psi\in S_h(\partial T)\mid \int_{\partial T}\psi\,ds=0\right\}.$$

It is immediate to check that $S_h(\partial T)$ is the space of normal traces on ∂T of vectors in $X_h(T)$. Finally let $S_h(\Gamma)$ be the global space of piecewise constant functions on Γ, such that their restriction to the boundary of a substructure T belongs to $S_h(\partial T)$

$$S_h(\Gamma):=\{\psi:\Gamma\to\mathbb{R}\mid \psi_{|\partial T}\in S_h(\partial T),\ T\subset\mathcal{T}_H\}.$$

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We now define some extension and interpolation operators. For each substructure T and each function $\psi \in S_h(\partial T)$, the discrete "harmonic" extension operator

$$\mathcal{H}_T : S_h(\partial T) \longrightarrow X_h(T),$$

is the vector $\mathbf{u} := \mathcal{H}_T\psi$ which satisfies

$$a_T(\mathbf{u}, \mathbf{v}) = 0, \quad \mathbf{v} \in X_{0;h}(T), \tag{2.7}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial T} = \psi. \tag{2.8}$$

The corresponding global operator

$$\mathcal{H} : S_h(\Gamma) \longrightarrow X_{0;h},$$

satisfies

$$a(\mathcal{H}\psi, \mathbf{v}) = 0, \quad \mathbf{v} \in X_{0;h}(T), \quad T \in \mathcal{T}_H, \tag{2.9}$$

$$\mathcal{H}\psi \cdot \mathbf{n}|_\Gamma = \psi. \tag{2.10}$$

or, equivalently,

$$(\mathcal{H}\psi)_{|_T} = \mathcal{H}_T\left(\psi_{|\partial T}\right).$$

The space of discrete harmonic extensions on the substructures is denoted by $\widetilde{X}_{0;h} \subset X_{0;h}$, and $\mathbf{u} \in \widetilde{X}_{0;h}$ if and only if $\mathbf{u}$ verifies (2.7).

We also need the space of the coarse discrete harmonic extensions, $\widetilde{X}_{0;H} \subset \widetilde{X}_{0;h}$: $\widetilde{X}_{0;H}$ is the space of discrete harmonic functions, the normal component of which is constant on each coarse face $F \in \mathcal{F}_H$, i.e., $\mathbf{u} \in \widetilde{X}_{0;H}$ if and only if

$$\mathbf{u} \cdot \mathbf{n}|_{\partial T} \in S_H(\partial T), \quad T \in \mathcal{T}_H.$$

We remark that the functions in the spaces $X_{0;H}$ and $\widetilde{X}_{0;H}$ have the same normal traces on the boundaries of the substructures.

The interpolation operator ρ_H onto the coarse space $X_{0;H}$,

$$\rho_H : \; X_{0;h} \longrightarrow X_{0;H},$$

is defined in terms of the degrees of freedom of the coarse space

$$\lambda_F(\rho_H\mathbf{u}) := \frac{1}{|F|}\int\limits_F \mathbf{n} \cdot \mathbf{u}\, ds, \quad F \in \mathcal{F}_H. \tag{2.11}$$

In a similar fashion, the degrees of freedom (2.11) define a unique interpolation operator Π_H

$$\Pi_H : \; X_{0;h} \longrightarrow \widetilde{X}_{0;H}.$$

Here, for any vector $\mathbf{u} \in X_{0;h}$, $\Pi_H\mathbf{u}$ is the unique function in $\widetilde{X}_{0;h}$, such that

$$(\Pi_H\mathbf{u}) \cdot \mathbf{n}|_{\partial T} = (\rho_H\mathbf{u}) \cdot \mathbf{n}|_{\partial T}, \quad T \in \mathcal{T}_H.$$

3. Discrete problem and Schur complement system. We consider the approximate problem: Find $\mathbf{u} \in X_{0;h}$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in X_{0;h}, \tag{3.1}$$

where $\mathbf{f} \in L^2(\Omega)^n$. The generalization to the case of the $X_h(\Omega)$ (Neumann boundary conditions) does not present any particular difficulty. In particular, we remark that if Neumann conditions are considered on some part of the boundary $\partial\Omega_N \subset \partial\Omega$, $\partial\Omega_N$ has to be added to Γ; see definition (2.3) and, e.g., [12].

We now introduce a Schur complement formulation of problem (3.1). We refer to [31, Ch. 4] for a general discussion of Schur complement methods, and to Section 7 for some implementation issues.

Let T_i be a substructure. Let A and $A^{(i)}$ be the matrices of the bilinear forms $a(\cdot,\cdot)$ and $a_{T_i}(\cdot,\cdot)$. The variational problem (3.1) can then be written as a linear system

$$AU = F,$$

where U is a column vector containing the degrees of freedom of $\mathbf{u}$. The local matrices $A^{(i)}$ can be represented as

$$\begin{bmatrix} A_{II}^{(i)} & A_{IB}^{(i)} \\ A_{BI}^{(i)} & A_{BB}^{(i)} \end{bmatrix},$$

where we divide the local vectors $U^{(i)}$ into two subvectors, $U_I^{(i)}$ and $U_B^{(i)}$, of degrees of freedom corresponding to faces inside T_i and on ∂T_i, respectively.

Since the variables interior to the substructures are associated with only one substructure, they can be eliminated in parallel across the substructures, using a direct method, and the reduced system

$$SU_B = G, \tag{3.2}$$

only involves the variables corresponding to the degrees of freedom on the interface Γ. Once the solution U_B of (3.2) is found, the local values $U_I^{(i)}$ of the solution can be obtained by solving one local problem for each subdomain.

The local Schur complements are

$$S^{(i)} := A_{BB}^{(i)} - A_{BI}^{(i)} \left(A_{II}^{(i)}\right)^{-1} A_{IB}^{(i)}. \tag{3.3}$$

Let $\tilde{R}_i$ be the restriction matrix, such that $\tilde{R}_i U_B$ contains the coefficients relative to the degrees of freedom on ∂T_i. The global Schur complement and the vector G can then be obtained by subassembling local contributions:

$$S = \sum_{T_i \in \mathcal{T}_H} \tilde{R}_i^T S^{(i)} \tilde{R}_i, \tag{3.4}$$

$$G = F_B - \sum_{T_i \in \mathcal{T}_H} \tilde{R}_i^T A_{BI}^{(i)} \left(A_{II}^{(i)}\right)^{-1} F_I^{(i)}. \tag{3.5}$$

For a substructure T_i and a vector $U_B^{(i)}$, it immediately follows that

$${U_B^{(i)}}^T S^{(i)} U_B^{(i)} = \min_{V_B^{(i)} = U_B^{(i)}} {V^{(i)}}^T A^{(i)} V^{(i)}, \tag{3.6}$$

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and that the minimum is obtained for the vector $U^{(i)}$ that satisfies

$$A_{II}^{(i)} U_{II}^{(i)} + A_{IB}^{(i)} U_B^{(i)} = 0. \tag{3.7}$$

We recall that, if $U^{(i)}$ are the degrees of freedom of a local vector $\mathbf{u}$, (3.7) is equivalent to (2.7), and, consequently, (3.7) defines the discrete harmonic functions.

Since the degrees of freedom $U_B^{(i)}$ of a local function $\mathbf{u} \in X_h(T_i)$ are the normal components $\mathbf{u} \cdot \mathbf{n}_{|f}$, on the fine faces f contained in the boundary ∂T_i, and $\mathbf{u} \cdot \mathbf{n}$ belongs to $S_h(\partial T_i)$, the column vector $\Psi^{(i)} := U_B^{(i)}$ determines a unique function $\psi \in S_h(\partial T_i)$. For ψ and ϕ in $S_h(\partial T_i)$, the following local bilinear form is then well–defined

$$s_{T_i}(\psi, \phi) := {\Psi^{(i)}}^T S^{(i)} \Phi^{(i)}.$$

In the same way, a vector $\Psi = U_B$, of degrees of freedom on Γ, uniquely determines a function $\psi \in S_h(\Gamma)$, and, for ψ and ϕ in $S_h(\Gamma)$, we can define the global bilinear form

$$s(\psi, \phi) := \sum_{T_i \in \mathcal{T}_H} {\Psi^{(i)}}^T S^{(i)} \Phi^{(i)},$$

where $\Psi^{(i)}$ and $\Phi^{(i)}$ are the subvectors of Ψ and Φ, respectively, containing the entries relative to ∂T_i.

With these definitions, (3.6) and (3.7) give, for $\psi \in S_h(\partial T_i)$,

$$s_{T_i}(\psi, \psi) = \min_{\substack{\mathbf{u} \in X_h(T_i) \\ \mathbf{u} \cdot \mathbf{n} = \psi}} a_{T_i}(\mathbf{u}, \mathbf{u}) = a_{T_i}(\mathcal{H}_{T_i}\psi, \mathcal{H}_{T_i}\psi), \tag{3.8}$$

and, for $\psi \in S_h(\Gamma)$,

$$s(\psi, \psi) = \min_{\substack{\mathbf{u} \in X_{0;h} \\ \mathbf{u} \cdot \mathbf{n} = \psi}} a(\mathbf{u}, \mathbf{u}) = a(\mathcal{H}\psi, \mathcal{H}\psi) = \sum_{T_i \in \mathcal{T}_H} a_{T_i}(\mathcal{H}_{T_i}\psi^{(i)}, \mathcal{H}_{T_i}\psi^{(i)}), \tag{3.9}$$

where $\psi^{(i)} = \psi_{|\partial T_i}$. In (3.8) and (3.9), the normal component $\mathbf{u} \cdot \mathbf{n}$ is taken on ∂T_i and Γ, respectively.

Finally, the variational formulation of (3.2) can be given as: Find $\psi \in S_h(\Gamma)$, such that,

$$s(\psi, \phi) = \int_\Gamma g\, \psi \, ds, \quad \psi \in S_h(\Gamma), \tag{3.10}$$

where $g \in S_h(\Gamma)$ is the normal trace determined by the degrees of freedom G.

4. Description of the algorithms. In this section, we build a Schwarz preconditioner for the Schur complement system (3.2), corresponding to the variational problem (3.10). Because of (3.9), instead of working with functions in $S_h(\Gamma)$ and the bilinear form $s(\cdot, \cdot)$, we can work with the space of discrete harmonic extensions $\widetilde{X}_{0;h}$ and the original bilinear form $a(\cdot, \cdot)$. We refer to [11, 31, 35, 27] and to the references therein, for an introduction and discussion of Schwarz methods.

An important element in the definition of a Neumann–Neumann method is a set of scaling functions defined on the boundaries of the substructures, which involve the values of the coefficients of the partial differential equation. The use of these functions can ensure that the condition number of the corresponding preconditioned system be independent of the jumps of the coefficients across Γ; see [5, 9, 20, 21, 8, 12, 28, 10, 22, 31]. Here, we propose a set

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of scaling functions, which involve only the values of one coefficient of (1.3). An important feature of our method is that it is independent of jumps of both coefficients a and B in (1.3). This is due to the particular divergence–free extension employed in the proof of Lemma 6.2; see Lemma 5.2. Because of the nature of the degrees of freedom in $X_{0;h}$, our scaling functions are particularly simple, compared to those for problems in $H^1(\Omega)$.

Following [28, 29], our family of scaling functions depend on a parameter

$$\delta \geq 1/2. \tag{4.1}$$

Let T be a substructure. We define a piecewise constant function $\mu_T \in S_h(\partial T)$ by

$$\mu_{T|_f} \equiv \frac{\sum_D \gamma_D^\delta}{\gamma_T^\delta}, \quad f \in \mathcal{F}_h, \quad f \subset \partial T, \tag{4.2}$$

where γ_D and γ_T are the largest eigenvalues of the coefficient matrices B_D and B_T, respectively, as in (2.4), and the sum is taken over the substructures that share the face f. We remark that if the coefficient matrix B is constant, $\mu_{T|_f}$ is the total number of subdomains to which f belongs. In addition, due to the nature of the degrees of freedom in $X_{0;h}$, the sum in (4.2) always has just two terms, and the function μ_T belongs to the subspace $S_H(\partial T)$, of piecewise constant functions on the coarse faces $F \subset \partial T$. We also need the corresponding function in $S_h(\Gamma)$, still denoted by μ_T, obtained from μ_T by extending it by zero to all of Γ. Given two substructures T and D that have a common face $\overline{F} = \overline{T} \cap \overline{D}$, we also use the notation

$$\mu_{T;D} := \mu_{T|_F}.$$

We remark that if Neumann conditions are considered on some part of the boundary $\partial\Omega_N \subset \partial\Omega$, the functions μ_T are also defined on $\partial\Omega_N$. In this case, the sum in (4.2) has only one term, if $f \subset \partial\Omega_N$, and $\mu_{T|_f} = 1$.

For $\delta = 1$, the same scaling functions are considered in [8], where a Neumann–Neumann method for the mixed approximation of the Laplace equation is studied.

We now define the pseudoinverses $\{\mu_T^\dagger\}$ of the functions $\{\mu_T\}$ on Γ, by

$$\mu^\dagger_{T|_f} \equiv \mu^{-1}_{T\ |_f}, \ \ f \subset \partial T; \qquad \mu^\dagger_{T|_f} \equiv 0, \ \ f \subset \Gamma \setminus \partial T. \tag{4.3}$$

The functions $\{\mu_T^\dagger\}$ are also constant on each coarse face $F \subset \mathcal{F}_H$.

It is immediate that $\{\mu_T^\dagger\}$ is a partition of unity on Γ:

$$0 \leq \mu_T^\dagger \leq 1, \quad \sum_{T \in \mathcal{T}_H} \mu_T^\dagger \equiv 1, \text{ a.e. on } \Gamma.$$

In order to define a Schwarz algorithm, we need a family of subspaces and a bilinear form for each of them. Given a substructure T, we define $\widetilde{X}_T \subset \widetilde{X}_{0;h}$, as the space of discrete harmonic extensions, that vanish on $\Gamma \setminus \partial T$. We remark that the support of a function $\mathbf{u} \in \widetilde{X}_T$ is contained in the closure of the union of T and the substructures with a common face with T. Since the degrees of freedom of the Raviart–Thomas spaces are defined on the faces of the triangulation, the discrete harmonic extension $\mathbf{u}$ vanishes on the substructures that share only a vertex with T and, if $n = 3$, also on the substructures that share only an edge with T.

We define the following decomposition of the space of discrete harmonic extensions:

$$\widetilde{X}_{0;h} = \widetilde{X}_{0;H} + \sum_{T \in \mathcal{T}_H} \widetilde{X}_T. \tag{4.4}$$

Our next step is to define suitable bilinear forms on the subspaces. On the coarse space, we employ the original bilinear form $a(\cdot,\cdot)$.

On the space $\widetilde{X}_T$, we define an approximate bilinear form. The corresponding local problem only involves the solution of a Neumann problem on the substructure T. It is defined by

$$\tilde{a}_T(\mathbf{u},\mathbf{v}) := a_T\left(\mathcal{H}_T(\mu_T\mathbf{u}\cdot\mathbf{n}), \mathcal{H}_T(\mu_T\mathbf{v}\cdot\mathbf{n})\right), \quad \mathbf{u},\mathbf{v}\in\widetilde{X}_T; \tag{4.5}$$

cf. [28]. We note that the local bilinear form is built with the original one, defined in the substructure T, and with the discrete harmonic extensions of the traces on ∂T, scaled by μ_T. In addition, $\tilde{a}_T(\cdot,\cdot)$ is always positive definite.

We then consider a hybrid Schwarz preconditioner, where one solves first the global problem and then the local ones, in parallel. The error propagation operator is given by

$$E := (I - \sum_{T\in\mathcal{T}_H} Q_T)(I-P_0),$$

where the orthogonal projection P_0 and the operators $\{Q_T\}$ are defined by

$$P_0 : \widetilde{X}_{0;h} \to \widetilde{X}_{0;H}; \quad a(P_0\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}), \quad \mathbf{v}\in\widetilde{X}_{0;H}, \tag{4.6}$$

$$Q_T : \widetilde{X}_{0;h} \to \widetilde{X}_T; \quad \tilde{a}_T(Q_T\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}), \quad \mathbf{v}\in\widetilde{X}_T. \tag{4.7}$$

The corresponding Schwarz operator is

$$I - E = P_0 + \sum_{T\in\mathcal{T}_H} Q_T\,(I-P_0).$$

In the following, we employ the symmetrized operator, obtained by adding an additional coarse solve,

$$\begin{aligned} Q_{hyb} &:= I - (I-P_0)\left(I - \sum_{T\in\mathcal{T}_H} Q_T\right)(I-P_0) \\ &= P_0 + (I-P_0)\left(\sum_{T\in\mathcal{T}_H} Q_T\right)(I-P_0) = P_0 + \sum_{T\in\mathcal{T}_H}\widetilde{Q}_T, \end{aligned} \tag{4.8}$$

where

$$\widetilde{Q}_T := (I-P_0)\,Q_T\,(I-P_0), \quad T\in\mathcal{T}_H.$$

Since P_0 is a projection, this can be done at no extra cost; see Section 7.

Our Schwarz method provides a new equation

$$Q_{hyb}\,U_B = \tilde{G}, \tag{4.9}$$

which can be much better conditioned than the original problem given by (3.2); it can be solved effectively with the conjugate gradient method, without any further preconditioner, employing $s(\cdot,\cdot)$ as the inner product. The right hand side $\tilde{G}$ can be chosen so that the new problem has the same solution as the original one. We refer to Section 7, for some implementation issues.

We choose this hybrid algorithm since the norms of the local operators Q_T are not, in general, uniformly bounded with respect to H, but grow quadratically with $1/H$, thus, giving

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a condition number that depends on the number of substructures. On the other hand, the norms of the projected operators $\widetilde{Q}_T = (I - P_0)Q_T(I - P_0)$ can be bounded by a quantity that only grows logarithmically with H/h, which is only related to the number of degrees of freedom in each substructure. We note that the largest eigenvalue of the Schwarz operator Q_{hyb} is related to the norm of the local operators $\widetilde{Q}_T$; see Lemma 6.3.

Similar hybrid algorithms have been employed successfully in some Neumann–Neumann methods for scalar or vector equations in $H^1(\Omega)$; see [21, 22, 8, 31, 35]. For these methods, the local operators Q_T are not defined on the whole space but only on $Ran(I - P_0)$ and the local bilinear forms are positive definite only on local spaces contained in $Ran(I - P_0)$. For our method, the situation is somewhat different, since the local problems are always solvable, but a good stability constant for the local operators is obtained only on local spaces contained in $Ran(I - P_0)$.

5. Technical tools. In this section, we recall some technical results. For the corresponding proofs, we refer to [34, 36, 33].

We first consider the interpolation operator onto the coarse space ρ_H, defined in Section 2. We have the following stability estimate. For a proof, we refer to [34, 33], for the case $n = 2$, and to [36, 33], for $n = 3$.

LEMMA 5.1. *Let $n = 2, 3$ and let T be a substructure. Then, there exists a constant $C > 0$, which depends only on the aspect ratios of T and the elements in $\mathcal{T}_h$, such that for all $\mathbf{u} \in X_h$,*

$$\|\mathrm{div}\,(\rho_H \mathbf{u})\|^2_{0;T} \le \|\mathrm{div}\,\mathbf{u}\|^2_{0;T}, \tag{5.1}$$

$$\|\rho_H \mathbf{u}\|^2_{0;T} \le C\left(\left(1 + \log\left(\frac{H}{h}\right)\right)\|\mathbf{u}\|^2_{0;T} + H_T^2\|\mathrm{div}\,\mathbf{u}\|^2_{0;T}\right). \tag{5.2}$$

The following lemma ensures the existence of a divergence–free extension of the normal traces in $S_{0;h}(\partial T)$. For a proof, we refer to [36, 33].

LEMMA 5.2. *Let $n = 2, 3$ and let T be a substructure. Then, there exists an extension operator $\widetilde{\mathcal{H}}_T : S_{0;h}(\partial T) \to X_h(T)$, such that, for any $\psi \in S_{0;h}(\partial T)$,*

$$\mathrm{div}\,\widetilde{\mathcal{H}}_T\psi = 0,$$

and

$$||\widetilde{\mathcal{H}}_T\psi||_{0;T} \le C\|\psi\|_{-\frac{1}{2};\partial T}. \tag{5.3}$$

Here C is independent of h, H, and ψ.

We then need a decomposition lemma for the trace of a function on the boundary of a substructure. For a proof, we refer to [33, Sect. 5.5], for the case $n = 2$, and to [36, 33], for $n = 3$.

LEMMA 5.3. *Let $n = 2, 3$ and let T be in $\mathcal{T}_H$. Let $\{\psi_F,\ F \subset \partial T\}$ be functions in $S_{0;h}(\partial T)$, which vanish on $\partial T \setminus F$ and let $\psi := \sum_{F\subset\partial T}\psi_F$. Then there exists a constant C, independent of h and ψ_H, such that, $\forall \psi_H \in S_H(\partial T)$,*

$$||\psi_F||^2_{-\frac{1}{2};\partial T} \le C(1 + \log H/h)\Big((1 + \log H/h)||\psi + \psi_H||^2_{-\frac{1}{2};\partial T} + ||\psi||^2_{-\frac{1}{2};\partial T}\Big).$$

The proof of the following lemma can be found in [33, Sect. 5.5].

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LEMMA 5.4. *Let $n = 2, 3$ and let T_1 and T_2 be two substructures with a common face $F \in \mathcal{F}_H$. Let ψ_F be a function in $L^2(\partial T_1 \cup \partial T_2)$, that vanishes outside F. Then there is a positive constant C, that only depends on the aspect ratios of T_1 and T_2, such that*

$$\|\psi_F\|_{-\frac{1}{2};\partial T_1} \leq C \|\psi_F\|_{-\frac{1}{2};\partial T_2}.$$

We end this section by recalling a well–known result on Schwarz methods. We refer to [11, 31] for a proof.

LEMMA 5.5. *Let the operators $\{Q_T\}$ be defined by (4.7). If a representation*

$$\mathbf{u} = \sum_{T \subset \mathcal{T}_H} \mathbf{u}_T, \quad \mathbf{u}_T \in \widetilde{X}_T,$$

can be found, such that

$$\sum_{T \subset \mathcal{T}_H} \tilde{a}_T(\mathbf{u}_T, \mathbf{u}_T) \leq C_0^2 \; a(\mathbf{u}, \mathbf{u}), \quad \mathbf{u} \in \widetilde{X}_{0;h}, \tag{5.4}$$

then

$$a\left(\sum_{T \subset \mathcal{T}_H} Q_T \mathbf{u}, \mathbf{u}\right) \geq C_0^{-2} \; a(\mathbf{u}, \mathbf{u}), \quad \mathbf{u} \in \widetilde{X}_{0;h}. \tag{5.5}$$

6. Main result.

In this section, we prove a logarithmic bound for the condition number of the hybrid operator Q_{hyb}, introduced in Section 4. We refer to [35], for similar proofs for a hybrid method for the Laplace equation.

Our first lemma gives a uniform bound for the lowest eigenvalue of the hybrid operator Q_{hyb}.

LEMMA 6.1. *We have*

$$a(Q_{hyb}\mathbf{u}, \mathbf{u}) \geq a(\mathbf{u}, \mathbf{u}), \quad \mathbf{u} \in \widetilde{X}_{0;h}.$$

Proof. We first prove a lower bound for the smallest eigenvalue of the operator

$$\sum_{T \in \mathcal{T}_H} Q_T.$$

Given a function $\mathbf{u} \in \widetilde{X}_{0;h}$, let

$$\mathbf{u}_T := \mathcal{H}\left(\mu_T^\dagger \left(\mathbf{u} \cdot \mathbf{n}|_\Gamma\right)\right) \in \widetilde{X}_T, \quad T \in \mathcal{T}_H, \tag{6.1}$$

where the partition of unity $\{\mu_T^\dagger\}$ is defined in (4.3). We remark that $\mathbf{u}_T$ is obtained from $\mathbf{u}$ by taking its normal component on Γ and multiplying it by the cut–off function $\mu_T^\dagger$, obtaining a function that is non–zero only on ∂T. We finally extend it harmonically into T and its neighboring substructures.

Since $\{\mu_T^\dagger\}$ is a partition of unity on Γ and the vectors $\mathbf{u}$ and $\mathbf{u}_T$ are all discrete harmonic extensions, we have

$$\mathbf{u} = \sum_{T \in \mathcal{T}_H} \mathbf{u}_T.$$

In view of Lemma 5.5, we have to bound the sum of the energies of the $\mathbf{u}_T$. Using the definition of $\mathbf{u}_T$ and the fact that the functions $\{\mu_T^\dagger\}$ are the pseudoinverses of the $\{\mu_T\}$, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_H} \tilde{a}_T(\mathbf{u}_T,\mathbf{u}_T) &= \sum_{T\in\mathcal{T}_H} a_T\left(\mathcal{H}_T(\mu_T\,\mathbf{u}_T\cdot\mathbf{n}),\mathcal{H}_T(\mu_T\,\mathbf{u}_T\cdot\mathbf{n})\right)\\ &= \sum_{T\in\mathcal{T}_H} a_T(\mathbf{u},\mathbf{u}) = a(\mathbf{u},\mathbf{u}).\end{aligned}$$

Lemma 5.5, ensures that the smallest eigenvalue of $\sum_{T\in\mathcal{T}_H} Q_T$ is bounded from below by one. The same quantity is also a lower bound for the smallest eigenvalue of Q_{hyb} since P_0 is an orthogonal projection. In fact,

$$\begin{aligned}a(Q_{hyb}\,\mathbf{u},\mathbf{u}) &= a\left(P_0\mathbf{u} + \sum_{T\in\mathcal{T}_H}\widetilde{Q}_T\mathbf{u},\mathbf{u}\right)\\ &= a(P_0\mathbf{u},P_0\mathbf{u}) + a\left((\sum_{T\in\mathcal{T}_H} Q_T)\,(I-P_0)\mathbf{u},(I-P_0)\mathbf{u}\right) \geq a(\mathbf{u},\mathbf{u}).\end{aligned}$$

□

In order to bound the largest eigenvalue, it is enough to find an upper bound for the norm of the local operators $\{\widetilde{Q}_T\}$; see, e.g., [11]. We then use a standard coloring argument.

LEMMA 6.2. *Let T be a substructure. Then there is a constant C, independent of h, H, the coefficients a and B, and the parameter δ, such that*

$$\|\widetilde{Q}_T\|_a \leq C\,\eta\left(1+\log\frac{H}{h}\right)^2,$$

where

$$\|\widetilde{Q}_T\|_a^2 := \sup_{\mathbf{u}\in\widetilde{X}_{0;h}} \frac{a(\widetilde{Q}_T\mathbf{u},\widetilde{Q}_T\mathbf{u})}{a(\mathbf{u},\mathbf{u})},$$

and

$$\eta := \max_{T\in\mathcal{T}_H}\{\eta_T\} := \max_{T\in\mathcal{T}_H}\max\left\{\frac{\gamma_T}{\beta_T},\frac{\gamma_T H_T^2}{a_T}\right\}. \tag{6.2}$$

Proof. Let $\Pi_H:\ X_{0;h}\rightarrow\widetilde{X}_{0;H}$ be the interpolation operator defined in Section 2. Since P_0 is an orthogonal projection and it has the same range as Π_H, we have

$$(I-P_0)\,(I-\Pi_H) = I-P_0,$$

and we can then write

$$\widetilde{Q}_T = (I-P_0)\,Q_T\,(I-P_0) = (I-P_0)\,(I-\Pi_H)\,Q_T\,(I-P_0). \tag{6.3}$$

It is therefore enough to estimate the norm of

$$(I-\Pi_H)\,Q_T,$$

when applied to $Ran(I-P_0)$.

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Let $\mathbf{u} \in Ran(I - P_0)$, and $\mathbf{w} := (I - \Pi_H)\, Q_T \mathbf{u}$. The support of $Q_T \mathbf{u} \in \widetilde{X}_T$ and $\mathbf{w}$ extends to the neighboring substructures of T, while the approximate bilinear form $\tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u})$ only involves the values of $Q_T\mathbf{u}$ in T. We first bound the energy of $\mathbf{w}$ in T and that from the neighboring substructures, in terms of $\tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u})$; see (6.11), (6.14), and (6.15). We remark that we only need to consider the substructures that have a common face with T. We also remark that $\mathbf{w}$ is discrete harmonic. Let D be one of these substructures and let $\overline{F} = \partial T \cap \partial D$. Using then the fact that $\mathbf{w} \cdot \mathbf{n}$ has mean value zero on F and vanishes elsewhere on ∂D, and Lemma 5.2, we have

$$\begin{aligned} a_D(\mathbf{w}, \mathbf{w}) &\leq a_D\left(\widetilde{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n}), \widetilde{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n})\right) \\ &\leq \gamma_D \left\|\widetilde{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n})\right\|^2_{0;D} \leq C\,\gamma_D \|\mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial D}, \end{aligned} \tag{6.4}$$

with a constant C that does not depend on the diameter of D. Using Lemma 5.4 gives

$$\gamma_D\, \|\mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial D} \leq C\gamma_D \|\vartheta_F\, \mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} = C\,\frac{\gamma_D}{\mu^2_{T;D}}\, \|\vartheta_F(\mu_T\, \mathbf{w}\cdot\mathbf{n})\|^2_{-\frac{1}{2};\partial T}, \tag{6.5}$$

where ϑ_F is identically one on F and vanishes on the rest of ∂T. Lemma 5.3 and the fact that $\mu_T\, \mathbf{w}\cdot\mathbf{n}$ has mean value zero on each face of T ensures that the last term in (6.5) can be bounded by

$$\begin{aligned} &C\,\frac{\gamma_D}{\mu^2_{T;D}} \left(1 + \log\frac{H}{h}\right) \times \\ &\quad\left(\|\mu_T\, \mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} + \left(1 + \log\frac{H}{h}\right) \|\mu_T\, \mathbf{w}\cdot\mathbf{n} + \psi_H\|^2_{-\frac{1}{2};\partial T}\right), \end{aligned} \tag{6.6}$$

where ψ_H is an arbitrary function of $S_H(\partial T)$.

Combining (6.4), (6.5), and (6.6), we obtain

$$\begin{aligned} &a_D(\mathbf{w}, \mathbf{w}) \leq \frac{\gamma_D}{\mu^2_{T;D}} \left(1 + \log\frac{H}{h}\right) \times \\ &\quad\left(\|\mu_T \mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} + \left(1 + \log\frac{H}{h}\right) \|\mu_T \mathbf{w}\cdot\mathbf{n} + \psi_H\|^2_{-\frac{1}{2};\partial T}\right). \end{aligned} \tag{6.7}$$

The second term in (6.7) can be bounded by taking $\psi_H = -\mu_T\,(\Pi_H\, Q_T\mathbf{u})\cdot\mathbf{n}$, and using the trace estimate (2.2), and the definition (4.5),

$$\begin{aligned} &\|\mu_T\, \mathbf{w}\cdot\mathbf{n} + \psi_H\|^2_{-\frac{1}{2};\partial T} = \|\mu_T\,(Q_T\mathbf{u})\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} \\ &\leq C\left(H_T^2\, \|\mathrm{div}\,(\mathcal{H}_T\,(\mu_T(Q_T\mathbf{u})\cdot\mathbf{n}))\|^2_{0;T} + \|\mathcal{H}_T\,(\mu_T(Q_T\mathbf{u})\cdot\mathbf{n})\|^2_{0;T}\right) \\ &\leq C\,\frac{\eta}{\gamma_T}\, a_T\left(\mathcal{H}_T\,(\mu_T(Q_T\mathbf{u})\cdot\mathbf{n})\,, \mathcal{H}_T\,(\mu_T(Q_T\mathbf{u})\cdot\mathbf{n})\right) \\ &= C\,\frac{\eta}{\gamma_T}\, \tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u}). \end{aligned} \tag{6.8}$$

For the first term in (6.7), we use the definition of $\mathbf{w}$ and obtain

$$\|\mu_T\, \mathbf{w}\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} \leq 2\,\|\mu_T\,(Q_T\mathbf{u})\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T} + 2\,\|\mu_T\,(\Pi_H\, Q_T\mathbf{u})\cdot\mathbf{n}\|^2_{-\frac{1}{2};\partial T}. \tag{6.9}$$

A bound for the second term in (6.9) can be found in the following way: It can easily be checked that the normal component of the vector

$$\mathbf{w}_H := \rho_H \left(\mathcal{H}_T \left(\mu_T \left(Q_T \mathbf{u}\right) \cdot \mathbf{n}\right)\right),$$

on ∂T is equal to $\mu_T \left(\Pi_H\, Q_T \mathbf{u}\right) \cdot \mathbf{n}$. We remark that $\mathbf{w}_H$ is obtained by first extending the normal trace $\left(\mu_T \left(Q_T \mathbf{u}\right) \cdot \mathbf{n}\right)$ harmonically into T and then interpolating into the coarse space $X_{0;H}$. Using the trace estimate (2.2) and the stability estimates for ρ_H in Lemma 5.1, we find

$$\begin{aligned}
\|\mu_T \left(\Pi_H\, Q_T \mathbf{u}\right) \cdot \mathbf{n}\|^2_{-\frac{1}{2};\partial T} &= \|\mathbf{w}_H \cdot \mathbf{n}\|^2_{-\frac{1}{2};\partial T} \leq C \left(H_T^2 \,\|\mathrm{div}\, \mathbf{w}_H\|^2_{0;T} + \|\mathbf{w}_H\|^2_{0;T}\right) \\
&\leq \; C \left(1 + \log \frac{H}{h}\right) \left(H_T^2 \,\|\mathrm{div}\, \left(\mathcal{H}_T \left(\mu_T (Q_T \mathbf{u}) \cdot \mathbf{n}\right)\right)\|^2_{0;T}\right. \\
&\qquad \left. + \|\mathcal{H}_T \left(\mu_T (Q_T \mathbf{u}) \cdot \mathbf{n}\right)\|^2_{0;T}\right) \\
&\leq \; C \frac{\eta}{\gamma_T} \left(1 + \log \frac{H}{h}\right) a_T \left(\mathcal{H}_T \left(\mu_T \left(Q_T \mathbf{u}\right) \cdot \mathbf{n}\right), \mathcal{H}_T \left(\mu_T \left(Q_T \mathbf{u}\right) \cdot \mathbf{n}\right)\right) \\
&= \; C \frac{\eta}{\gamma_T} \left(1 + \log \frac{H}{h}\right) \tilde{a}_T (Q_T \mathbf{u}, Q_T \mathbf{u}).
\end{aligned} \tag{6.10}$$

Finally, combining (6.7), (6.8), and (6.10), we obtain

$$a_D(\mathbf{w}, \mathbf{w}) \leq C\, \eta \left(1 + \log \frac{H}{h}\right)^2 \frac{\gamma_D}{\mu^2_{T;D}\, \gamma_T}\, \tilde{a}_T (Q_T \mathbf{u}, Q_T \mathbf{u}). \tag{6.11}$$

An estimate for the energy $a_T(\mathbf{w}, \mathbf{w})$ can be found in a similar way:
If a substructure D shares a face with T, let $\overline{F}_D := \partial T \cap \partial D$ be the common face. Since $\mathbf{w} \cdot \mathbf{n}$ has mean value zero on each face of ∂T, Lemma 5.2 can still be applied. We have

$$\begin{aligned}
a_T(\mathbf{w}, \mathbf{w}) \;&\leq\; a_T \left(\widetilde{\mathcal{H}}_T (\mathbf{w} \cdot \mathbf{n}), \widetilde{\mathcal{H}}_T (\mathbf{w} \cdot \mathbf{n})\right) \leq \gamma_T \left\|\widetilde{\mathcal{H}}_T (\mathbf{w} \cdot \mathbf{n})\right\|^2_{0;T} \\
&\leq\; C \gamma_T \|\mathbf{w} \cdot \mathbf{n}\|^2_{-\frac{1}{2};\partial T} = \gamma_T \left\| \sum_D \vartheta_{F_D} (\mathbf{w} \cdot \mathbf{n}) \right\|^2_{-\frac{1}{2};\partial T} \\
&\leq\; C \sum_D \frac{\gamma_T}{\mu^2_{T;D}} \|\vartheta_{F_D} (\mu_T\, \mathbf{w} \cdot \mathbf{n})\|^2_{-\frac{1}{2};\partial T}\;,
\end{aligned} \tag{6.12}$$

where the sum is taken over the substructures D that share a face with T, and ϑ_{F_D} is equal to one on F_D and vanishes on the rest of ∂T. Lemma 5.3 thus gives the following bound for the last term in (6.12):

$$\begin{aligned}
&C \sum_D \frac{\gamma_T}{\mu^2_{T;D}} \left(1 + \log \frac{H}{h}\right) \times \\
&\quad \left(\|\mu_T\, \mathbf{w} \cdot \mathbf{n}\|^2_{-\frac{1}{2};\partial T} + \left(1 + \log \frac{H}{h}\right) \|\mu_T\, \mathbf{w} \cdot \mathbf{n} + \psi_H\|^2_{-\frac{1}{2};\partial T} \right),
\end{aligned} \tag{6.13}$$

where ψ_H is an arbitrary function of $S_H(\partial T)$.

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An upper bound for (6.13) can be obtained in the same way as for (6.7), and (6.12) and (6.13) give

$$a_T(\mathbf{w},\mathbf{w}) \le C\,\eta \left(1+\log\frac{H}{h}\right)^2 \left(\sum_D \frac{1}{\mu_{T;D}^2}\right) \tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u}). \tag{6.14}$$

Employing (6.11) and (6.14), and summing over the substructures that share a face with T, we obtain

$$a(\mathbf{w},\mathbf{w}) \le C\,\eta \left(1+\log\frac{H}{h}\right)^2 \left(\sum_D \frac{\gamma_T+\gamma_D}{\mu_{T;D}^2\,\gamma_T}\right) \tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u}).$$

It can easily be checked that the terms

$$\frac{\gamma_T+\gamma_D}{\mu_{T;D}^2\,\gamma_T} = \frac{(\gamma_T+\gamma_D)\,\gamma_T^{2\delta}}{(\gamma_T^\delta+\gamma_D^\delta)^2\,\gamma_T},$$

are homogeneous functions of γ_T and γ_D, that can be bounded by 2, independently of $\gamma_T, \gamma_D > 0$, and $\delta \ge 1/2$. We then obtain

$$a(\mathbf{w},\mathbf{w}) \le C\,\eta \left(1+\log\frac{H}{h}\right)^2 \tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u}). \tag{6.15}$$

Using the definition of Q_T and and the fact that we have chosen $\mathbf{u} \in Ran(I-P_0)$, we can write

$$\begin{aligned} a(\mathbf{w},\mathbf{w}) &\le C\,\eta \left(1+\log\frac{H}{h}\right)^2 a(\mathbf{u}, Q_T\mathbf{u}) \\ &= C\,\eta \left(1+\log\frac{H}{h}\right)^2 a\left(\mathbf{u}, (I-P_0)Q_T(I-P_0)\mathbf{u}\right) \\ &= C\,\eta \left(1+\log\frac{H}{h}\right)^2 a(\mathbf{u},\mathbf{w}). \end{aligned}$$

By applying Schwarz inequality, we finally obtain

$$a\left((I-\Pi_H)\,Q_T\,\mathbf{u}\,,\,(I-\Pi_H)\,Q_T\,\mathbf{u}\right) = a(\mathbf{w},\mathbf{w}) \le C\,\eta^2 \left(1+\log\frac{H}{h}\right)^4 a(\mathbf{u},\mathbf{u}),$$

and the proof is completed by noting that $(I-P_0)$ is an orthogonal projection. ◻

LEMMA 6.3. *There is a constant C, independent of h, H, $\mathbf{u}$, the coefficients a and B, and the parameter δ, such that*

$$a(Q_{hyb}\mathbf{u},\mathbf{u}) \le C\,\eta \left(1+\log\frac{H}{h}\right)^2 a(\mathbf{u},\mathbf{u}), \quad \mathbf{u} \in \widetilde{X}_{0;h}.$$

Proof. The proof employs the bounds for the local operators $\widetilde{Q}_T$ given by the previous lemma and a standard coloring argument. It can be found in [33, Lem. 5.6.3]; see also [31, P. 165] and [3, Th. 4.1]. ◻

Lemmas 6.1 and 6.3 combine to give the following theorem.

THEOREM 6.1. *There is a constant C, independent of h, H, the coefficients a and B, and the parameter δ, such that*

$$\kappa(Q_{hyb}) \leq C\eta \left(1 + \log \frac{H}{h}\right)^2,$$

where η is the constant in Lemma 6.2.

We remark that the same constant η also appears in the estimates for some edge and face–space methods for the same problems considered in this paper; see [34, 36, 33].

The estimate given in Theorem 6.1 remains bounded when the coefficient matrix B tends to zero, but becomes unbounded when a becomes small. The following lemma ensures that in the limit case $a = 0$, the condition number of the hybrid operator is bounded independently of H/h and the jumps of the coefficient B.

LEMMA 6.4. *In the limit case $a = 0$, there is a constant C, independent of h, H, the coefficient matrix B, and the parameter δ, such that*

$$\kappa(Q_{hyb}) \leq C\,\xi,$$

where

$$\xi := \max_{T \in \mathcal{T}_H} \left\{ \frac{\gamma_T}{\beta_T} \right\}.$$

Proof. It is immediate to check that Lemma 6.1 still holds. It is then enough to prove a bound for $\|\widetilde{Q}_T\|_a$, as in Lemma 6.2, since the coloring argument in the proof of Lemma 6.3 can also be employed in the limit case $a = 0$.

Let T be a substructure and let $\widetilde{Q}_T = (I - P_0)Q_T(I - P_0)$ be defined in Section 4. Since $a = 0$, the bilinear forms $a(\cdot,\cdot)$ and $a_T(\cdot,\cdot)$ are just weighted L^2–scalar products, and we can employ (2.6).

Let $\mathbf{u} \in \widetilde{X}_{0;h}$, and $\mathbf{w} := Q_T\mathbf{u} \in \widetilde{X}_T$. The proof is similar to the one of Lemma 6.2. We first bound the energy of $\mathbf{w}$ in T, and that from the neighboring substructures, in terms of $\tilde{a}_T(Q_T\mathbf{u}, Q_T\mathbf{u})$. We only need to consider the substructures that have a common face with T. Let D be one of these substructures and $\overline{F} = \partial T \cap \partial D$. We have

$$\begin{aligned} a_D(\mathbf{w},\mathbf{w}) &\leq a_D\left(\hat{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n}), \hat{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n})\right) \\ &\leq \gamma_D \left\|\hat{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n})\right\|_{0;D}^2 \leq C\,\gamma_D \sum_{f\subset F} H_f^n\,\lambda_f(\mathbf{w})^2, \end{aligned} \tag{6.16}$$

where $\hat{\mathcal{H}}_D(\mathbf{w}\cdot\mathbf{n})$ is the extension by zero of $\mathbf{w}\cdot\mathbf{n}$ into D and, for the last inequality, we have used (2.6).

The last term in (6.16) can be bounded using the fact that the function μ_T is constant on F and the degree of freedom $\lambda_f(\mathbf{w})$ is the normal component of $\mathbf{w}$ on $f \subset F$. We can then

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write

$$
\begin{aligned}
\sum_{f\subset F} H_f^n\,\lambda_f(\mathbf{w})^2 &= \frac{1}{\mu_{T;D}^2}\sum_{f\subset F}\mu_{T;D}^2\,H_f^n\,\lambda_f(\mathbf{w})^2\\
&= \frac{1}{\mu_{T;D}^2}\sum_{f\subset F} H_f^n\,\lambda_f\left(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n})\right)^2\\
&\leq \frac{1}{\mu_{T;D}^2}\sum_{f\subset \overline{T}} H_f^n\,\lambda_f\left(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n})\right)^2\\
&\leq C\,\frac{1}{\mu_{T;D}^2}\,\|\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n})\|_{0;T}^2\\
&\leq C\,\xi\,\frac{1}{\mu_{T;D}^2\,\gamma_T}\,a_T\left(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n}),\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n})\right)\\
&= C\,\xi\,\frac{1}{\mu_{T;D}^2\,\gamma_T}\,\widetilde{a}_T\left(\mathbf{w},\mathbf{w}\right),
\end{aligned}
\tag{6.17}
$$

where we have also used (2.6). Combining (6.16) and (6.17), and using the definition of $\mathbf{w}$, we obtain

$$
a_D(\mathbf{w},\mathbf{w}) \leq C\,\xi\,\frac{\gamma_D}{\mu_{T;D}^2\,\gamma_T}\,\widetilde{a}_T\left(Q_T\mathbf{u},Q_T\mathbf{u}\right); \tag{6.18}
$$

see (6.11).

An estimate for the energy $a_T(\mathbf{w},\mathbf{w})$ can be found in a similar way:
If a substructure D shares a face with T, let $\overline{F}_D := \partial T\cap\partial D$ be the common face. We have

$$
\begin{aligned}
a_T(\mathbf{w},\mathbf{w}) &\leq a_T\left(\hat{\mathcal{H}}_T(\mathbf{w}\cdot\mathbf{n}),\hat{\mathcal{H}}_T(\mathbf{w}\cdot\mathbf{n})\right)\\
&\leq \gamma_T\left\|\hat{\mathcal{H}}_T(\mathbf{w}\cdot\mathbf{n})\right\|_{0;T}^2 \leq C\,\gamma_T\sum_{f\subset\partial T} H_f^n\,\lambda_f(\mathbf{w})^2,
\end{aligned}
\tag{6.19}
$$

where $\hat{\mathcal{H}}_T(\mathbf{w}\cdot\mathbf{n})$ is the extension by zero of $\mathbf{w}\cdot\mathbf{n}$ into T and, for the last inequality, we have used (2.6).

The right hand side of (6.19) can be bounded using similar arguments as in (6.17):

$$
\begin{aligned}
\gamma_T\sum_{f\subset\partial T} H_f^n\,\lambda_f(\mathbf{w})^2 &\leq C\,\gamma_T\sum_{F_D\subset\partial T}\frac{1}{\mu_{T;D}^2}\sum_{f\subset F_D} H_f^n\,\lambda_f(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n}))^2\\
&\leq C\sum_{F_D\subset\partial T}\frac{\gamma_T}{\mu_{T;D}^2}\sum_{f\subset\overline{T}} H_f^n\,\lambda_f(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n}))^2\\
&\leq C\,\xi\,a_T\left(\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n}),\mathcal{H}_T(\mu_T\,\mathbf{w}\cdot\mathbf{n})\right)\sum_{F_D\subset\partial T}\frac{1}{\mu_{T;D}^2}\\
&= C\,\xi\,\widetilde{a}_T\left(Q_T\mathbf{u},Q_T\mathbf{u}\right)\sum_{F_D\subset\partial T}\frac{1}{\mu_{T;D}^2},
\end{aligned}
\tag{6.20}
$$

where we have used (2.6) and the definition of $\mathbf{w}$.

Employing (6.18), (6.19), and (6.20), and summing over the substructures that share a face with T, we obtain

$$
a(\mathbf{w},\mathbf{w}) \leq C\,\xi\left(\sum_{D}\frac{\gamma_T+\gamma_D}{\mu_{T;D}^2\,\gamma_T}\right)\widetilde{a}\left(Q_T\mathbf{u},Q_T\mathbf{u}\right),
$$

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and a bound for $\|\widetilde{Q}_T\|_a$ can then be found as in the proof of Lemma 6.2. □

We remark that Lemma 6.4 gives an optimal bound for the limit case $a = 0$. Thus, as the ratio between the coefficients B and a becomes large, we expect an upper bound for the condition number which is independent of H/h; see the numerical results in the Section 8.

7. Implementation of the method. In this section, we describe a particular implementation of our method. We refer to [31, Ch. 4] and to the references therein, for additional comments on the implementation of iterative substructuring methods.

The hybrid operator Q_{hyb} has been defined in (4.8). We first note that the Schur complement S does not need to be calculated explicitly, but its action can be evaluated using Equations (3.4) and (3.3), at the expense of solving one Dirichlet problem for each substructure.

The matrix representation of the projection P_0, still denoted by P_0, is

$$P_0 = B_H\, S := (\widetilde{R}_H^T\, S_H^{-1}\, \widetilde{R}_H)\, S, \tag{7.1}$$

where $\widetilde{R}_H^T$ is the natural extension matrix from the coarse space into the fine space, defined on the interface Γ, and S_H is the matrix representation of the bilinear form $s(\cdot,\cdot)$ on the coarse space, defined by

$$S_H := \widetilde{R}_H\, S\, \widetilde{R}_H^T.$$

We remark that, if n_H is the dimension of the coarse space, n_H applications of the Schur complement are required, in order to calculate S_H. Since the basis functions of the coarse space are supported on the unions of two substructures that have a common face, only the solution of at most two Dirichlet problems on two substructures are required for the application of S to a coarse basis function.

For a generic substructure T_i, the matrix representation of the local operator Q_{T_i}, still denoted by Q_{T_i}, is

$$Q_{T_i} = B_i\, S := \left(\widetilde{R}_i^T\, D_i^{-1}\, S_i^{-1}\, D_i^{-1}\, \widetilde{R}_i\right) S; \tag{7.2}$$

see (4.5). Here the natural extension $\widetilde{R}_i^T$ maps the local degrees of freedom on ∂T_i into the corresponding global ones on Γ, the diagonal matrix D_i represents the multiplication by the scaling function μ_{T_i}, and S_i is the Schur complement of the local bilinear form $a_{T_i}(\cdot,\cdot)$, with respect to the variables on ∂T_i. The matrix S_i does not need to be calculated explicitly, but the action of its inverse on a local vector can be calculated by solving a Neumann problem on the substructure T_i; see [31, Sect. 4.2.1]. We also note that, as is often the case for Neumann–Neumann methods, the local averaging operators $\{D_i^{-1}\}$ satisfy

$$\sum_i \widetilde{R}_i\, D_i^{-1}\, \widetilde{R}_i = I;$$

see, e.g., [10, 31].

If we define the local preconditioner

$$\hat{S} := \sum_{T_i \in \mathcal{T}_H} B_i,$$

for a particular choice of the initial guess, the conjugate gradient method applied to the new equation (4.9) is equivalent to the following preconditioned projected conjugate gradient method.

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TABLE 8.1
Estimated condition number and number of CG iterations necessary for a reduction of 10^{-6} *of the norm of the preconditioned residual (in parentheses), versus* H/h *and* n. *Case of* $a = 1$, $b = 1$.

H/h	32	16	8	4	2
n=32	-	3.075 (4)	2.881 (10)	2.182 (8)	1.505 (5)
n=64	4.004 (4)	3.791 (11)	3.023 (10)	2.113 (7)	1.508 (5)
n=128	4.860 (12)	3.985 (11)	2.935 (8)	2.033 (6)	x
n=192	-	3.978 (10)	2.854 (7)	1.974 (5)	x
n=256	5.112 (12)	4.01 (10)	2.854 (7)	1.974 (5)	x

1. Initialize

$$U_0 = B_H\, G$$
$$r_0 = G - SU_0$$

2. Iterate $k = 1, 2, \cdots$ until convergence

$$\begin{aligned}
\text{Project:}\quad & W_{k-1} = (I - P_0)^T r_{k-1} \\
\text{Precondition:}\quad & Z_{k-1} = \hat{S}\, W_{k-1} \\
\text{Project:}\quad & Y_{k-1} = (I - P_0) Z_{k-1} \\
& \beta_k = Y_{k-1}^t W_{k-1} / Y_{k-2}^t W_{k-2} \quad [\beta_1 = 0] \\
& p_k = Y_{k-1} + \beta_k p_{k-1} \quad [p_1 = Y_0] \\
& \alpha_k = Y_{k-1}^t W_{k-1} / p_k^t S p_k \\
& U_k = U_{k-1} + \alpha_k p_k \\
& r_k = r_{k-1} - \alpha_k S p_k
\end{aligned}$$

We remark that the residuals r_k are perpendicular to the coarse space, since $\widetilde{R}_H r_k = 0$, for every k. In addition, the first projection can be omitted, since $W_k = r_k$ for every k, thanks to the choice of the initial vector U_0. See [14], for a similar algorithm.

8. Numerical results. In this section, we present some numerical results on the performance of the hybrid Neumann–Neumann method described in the previous sections, when varying the diameters of the coarse and fine meshes, and the coefficients a and B. We only consider two–dimensional problems.

For our numerical experiments, we have considered the domain $\Omega = (0,1)^2$ and uniform rectangular triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$. The fine triangulation $\mathcal{T}_h$ consists of n^2 square elements, with $h = 1/n$. The matrix B is given by

$$B = \text{diag}\{b, b\}.$$

In Table 8.1, we show the estimated condition number and the number of iterations in order to obtain a reduction of the norm of the preconditioned residual by a factor 10^{-6}, as a function of the dimensions of the fine and coarse meshes. For a fixed ratio H/h, the condition number and the number of iterations are quite insensitive to the dimension of the fine mesh. Our results compare very well with those for finite element approximations in H^1 of the Laplace equation; see, e.g. [31], and with those for the iterative substructuring method based

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TABLE 8.2
Estimated condition number and number of CG iterations necessary for a reduction of 10^{-6} of the norm of the preconditioned residual (in parentheses), versus H/h and b. Case of $n = 128$ and $a = 1$.

H/h	4	8	16	32
b=1e-5	2.033 (6)	2.935 (8)	3.988 (11)	4.871 (12)
b=1e-4	2.033 (6)	2.936 (8)	3.988 (11)	4.871 (12)
b=1e-3	2.033 (6)	2.936 (8)	3.988 (11)	4.871 (12)
b=1e-2	2.033 (6)	2.936 (8)	3.988 (11)	4.871 (12)
b=1e-1	2.033 (6)	2.935 (8)	3.988 (11)	4.87 (12)
b=1	2.033 (6)	2.935 (8)	3.985 (11)	4.86 (12)
b=1e+1	2.032 (6)	2.931 (8)	3.959 (11)	4.765 (12)
b=1e+2	2.026 (6)	2.881 (8)	3.705 (10)	4.15 (10)
b=1e+3	1.932 (5)	2.507 (7)	2.806 (8)	2.862 (7)
b=1e+4	1.613 (4)	1.717 (5)	1.751 (5)	1.771 (5)
b=1e+5	1.124 (3)	1.134 (3)	1.15 (3)	1.154 (3)

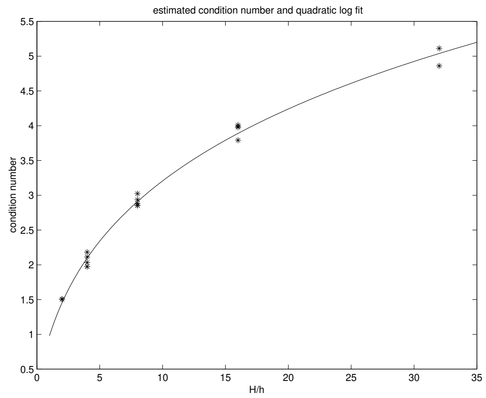

FIG. 8.1. *Estimated condition number from Table 8.1 (asterisk) and least–square second order logarithmic polynomial (solid line), versus H/h; relative fitting error about 2.8 per cent.*

on individual edges, described in [36].

In Figure 8.1, we plot the results of Table 8.1, together with the best second order logarithmic polynomial least–square fit. The relative fitting error is about 2.8 per cent. Our numerical results are therefore in good agreement with the theoretical bound obtained in the previous section and suggest that our bound is sharp.

In Table 8.2, we show some results when the ratio of the coefficients b and a is changed. For a fixed value of $n = 128$ and $a = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and b. In accordance with Theorem 6.1, the condition number is independent of the ratio b/a, when $b/a \leq 1$. Table 8.2 also shows that, in practice, this holds for $b/a \geq 1$, and that, when b/a is very large, the condition number tends to be independent of H/h. Our numerical results then confirm our analysis of the limit case $a = 0$, in Lemma 6.4.

We finally consider some cases where the coefficients have jumps. In Table 8.3, we show some results when the coefficient b has jumps across the substructures. We consider the checkerboard distribution shown in Figure 8.2, where b is equal to b_1 in the shaded area

TABLE 8.3
Checkerboard distribution for b: (b_1, b_2). Estimated condition number and number of CG iterations for a reduction of 10^{-6} of the norm of the preconditioned residual (in parentheses), versus H/h and b_2. Case of $n = 128$, $a = 1$, and $b_1 = 100$.

H/h	4	8	16
$b_2 = 1e-4$	5.344 (14)	7.514 (16)	9.989 (19)
$b_2 = 1e-3$	5.321 (14)	7.481 (16)	9.945 (19)
$b_2 = 1e-2$	5.248 (13)	7.379 (16)	9.81 (19)
$b_2 = 1e-1$	5.031 (12)	7.073 (15)	9.402 (18)
$b_2 = 1$	4.442 (11)	6.239 (14)	8.289 (17)
$b_2 = 1e+1$	3.249 (8)	4.532 (11)	5.987 (14)
$b_2 = 1e+2$	2.026 (6)	2.881 (8)	3.705 (10)
$b_2 = 1e+3$	3.15 (8)	4.138 (11)	4.932 (13)
$b_2 = 1e+4$	3.556 (9)	4.043 (11)	4.384 (13)
$b_2 = 1e+5$	2.638 (8)	3.24 (11)	3.912 (13)
$b_2 = 1e+6$	2.417 (8)	3.176 (11)	3.919 (13)

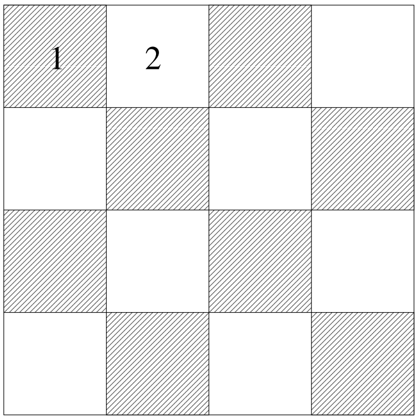

FIG. 8.2. *Checkerboard distribution of the coefficients in the unit square.*

and to b_2 elsewhere. For a fixed value of $n = 128$, $b_1 = 100$, and $a = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and b_2. For $b_2 = 100$, the coefficient b has a uniform distribution, and this corresponds to a minimum for the condition number and the number of iterations. When b_2 decreases or increases, the condition number and the number of iterations also increase, but they can still be bounded independently of b_2.

In Table 8.4, we show some results when the coefficient a has jumps. We consider the checkerboard distribution shown in Figure 8.2, where a is equal to a_1 in the shaded area and to a_2 elsewhere. For a fixed value of $n = 128$, $a_1 = 0.01$, and $b = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and a_2. We remark that for $a_2 = 0.01$, the coefficient a has a uniform distribution. A slight increase in the number of iterations and the condition number is observed, when a_2 is decreased or increased and when H/h is large.

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TABLE 8.4
Checkerboard distribution for a: (a_1, a_2). Estimated condition number and number of CG iterations for a reduction of 10^{-6} of the norm of the preconditioned residual (in parentheses), versus H/h and a_2. Case of $n = 128$, $b = 1$, and $a_1 = 0.01$.

H/h	4	8	16
$a_2 = 1e-7$	2.1 (6)	3.399 (9)	5.909 (13)
$a_2 = 1e-6$	2.1 (6)	3.196 (9)	5.537 (13)
$a_2 = 1e-5$	2.059 (6)	2.882 (8)	4.165 (11)
$a_2 = 1e-4$	2.051 (6)	2.857 (8)	3.403 (10)
$a_2 = 1e-3$	1.944 (5)	2.853 (8)	3.611 (10)
$a_2 = 1e-2$	2.026 (6)	2.881 (8)	3.705 (10)
$a_2 = 1e-1$	2.032 (6)	2.933 (8)	3.948 (11)
$a_2 = 1$	2.033 (6)	2.938 (8)	3.975 (11)
$a_2 = 1e+1$	2.033 (6)	2.939 (8)	3.977 (11)
$a_2 = 1e+2$	2.033 (6)	2.939 (8)	3.978 (11)
$a_2 = 1e+3$	2.033 (6)	2.939 (8)	3.978 (11)

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