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A MULTIGRID METHOD FOR SADDLE POINT PROBLEMS ARISING FROM MORTAR FINITE ELEMENT DISCRETIZATIONS*

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Abstract. A multigrid algorithm for saddle point problems arising from mortar finite element discretizations is analyzed. Here, we do not require that the constraints at the interface are satisfied in each smoothing step, but we work on the squared system. Using mesh dependent norms for the Lagrange multipliers, suitable approximation and smoothing properties are established. A convergence rate independent of the meshsize is obtained for the $\mathcal{W}$–cycle.

Key words. mortar finite elements, saddle point problems, multigrid methods.

AMS subject classifications. 65N22, 65N30, 65N55.

1. Introduction. Domain decomposition techniques provide powerful tools for the numerical solution of partial differential equations. Within the framework of mortar methods [6, 7], the flexible coupling of different discretization schemes or of nonmatching triangulations can be realized. Adequate weak continuity conditions replace the pointwise continuity at the interfaces. The arising variational problems are either positive definite nonconforming problems or saddle point problems.

Efficient iterative solvers for the mortar formulation have been considered. The first approaches were based on iterative substructuring methods and preconditioners for the Schur complement system; see [15, 16, 1, 2, 3, 17, 18, 22, 23]. More recently multigrid methods for the discrete nonconforming [12, 19] as well as the discrete saddle point problem have been established; see [10, 11, 28]. Working with the nonconforming variational problem has the drawback that the corresponding nodal basis functions have, in general, non–local support. Thus it might be advantageous to work with the unconstraint product space for the numerical realization of the mortar method. Even if the starting point is the positive definite variational problem, the non–local nodal basis function of the constrained space are,in general, not explicitly used [19]. Instead one works with the local nodal basis functions of the unconstrained product space and the global mortar projection. An alternative approach is to use the equivalent saddle point problem as starting point [4]. In this case, an indefinite problem has to be solved and standard multigrid methods cannot be applied.

Recently, special multigrid techniques for the arising indefinite problems were developed [10, 11, 28]. They are based on the concepts in [13] for the Stokes problem. The characteristic feature of these multigrid methods is the choice of the symmetric smoother for the saddle point problem. It is defined so that the constraints are satisfied in each smoothing step. As a consequence, each smoothing step requires the exact solution of a modified Schur complement system. Using this type of smoother, one works in the positive definite subspace of the saddle point problem, and the multigrid analysis of the standard positive definite case carries over. The drawback is that the solution of the modified Schur complement system in each smoothing step might be too expensive. In [33], a generalization of this type of smoother is investigated defining an inner and outer iteration cycle.

A different approach for the construction of an efficient iterative solver for the saddle point problem is given in [2, 17, 18, 22, 23]. The saddle point problem is solved by a multilevel preconditioned Lanczos iteration. The preconditioner is a block diagonal matrix involving a good preconditioner for the exact Schur complement.

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These observations motivate our new approach. In particular, we do not require that the iterates satisfy the weak continuity conditions at the interfaces exactly. As a consequence, we neither need a good preconditioner for the exact Schur complement nor an exact solver for a modified Schur complement. Following the ideas of [27], we introduce a smoother for the squared positive definite system.

The rest of the paper is organized as follows: In Section 2, we give a brief overview of the mortar method in case of P_1–Lagrangian finite elements. We review the definition of the discrete spaces and the saddle point formulation as well as the a priori estimates. The two basic tools, approximation and smoothing properties, for the convergence analysis of a multigrid method are analyzed in Sections 3 and 4. In Section 3, the approximation property is formulated. Since we are not working in the positive definite subspace, we cannot avoid the use of norms for the Lagrange multiplier. Our analysis is based on mesh dependent norms for the Lagrange multiplier. In Section 4, we establish the smoothing property, where we follow the lines of [27], and introduce a block diagonal smoother. Finally, the convergence rate for the $\mathcal{W}$–cycle is given.

2. Mortar finite elements. Mortar methods are based on domain decomposition techniques. Within this approach, different discretization schemes as well as geometrical nonconforming triangulations can be coupled. The resulting nonconforming method is optimal in the sense that the discretization error is of the same order as the sum of the best approximation errors on the different subdomains. Here, we will only briefly review the definition of a special mortar method. For an overview of more general mortar techniques, we refer the reader to [4, 5, 6, 7, 11].

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider a geometrical conforming non–overlapping decomposition into K polygonal subdomains Ω_k such that

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k .$$

The intersection between the boundaries of two adjacent subdomains Ω_l and Ω_k, $1 \le l \ne k \le K$ is called interface $\Gamma_{lk} := \partial\Omega_l \cap \partial\Omega_k$. Furthermore, the union of subdomain boundaries $\mathcal{S} := \cup_{k=1}^{K} (\partial\Omega_k \setminus \partial\Omega)$ can be decomposed into M disjoint open subsets

$$\mathcal{S} = \bigcup_{m=1}^{M} \overline{\gamma}_m$$

where for each non–mortar γ_m, $1 \le m \le M$, there exist a $l(m)$ and $k(m)$, $1 \le l(m) < k(m) \le K$ such that $\gamma_m = \Gamma_{l(m)k(m)}$.

We consider the following elliptic second order boundary problem

$$\begin{array}{rcll} -\mathrm{div}(a\nabla u) + bu & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \tag{2.1}$$

where a is an uniformly positive function $a \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $0 \le b \in L^\infty(\Omega)$. Each subdomain Ω_k, $1 \le k \le K$ is associated with a quasi–uniform simplicial triangulation $\mathcal{T}_{k;h}$ of meshsize h. For the discretization on the different subdomains Ω_k, we use P_1–conforming finite elements $X(\Omega_k; \mathcal{T}_{k;h})$, $k = 1, ..., K$, satisfying homogeneous boundary conditions on $\partial\Omega_k \cap \partial\Omega$. Then, the unconstraint product space

$$X_h := \prod_{k=1}^{K} X(\Omega_k; \mathcal{T}_{k;h})$$

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satisfies no weak continuity condition at the interface and is not a suitable space for the discretization of (2.1). In particular, the consistency error would be not bounded by the meshsize h. To define an appropriate global discrete space, we have to impose weak continuity conditions at the interfaces of the decomposition. Each non–mortar γ_m will be associated with a 1-D triangulation $\mathcal{S}_{m;h}$, inherited either from $\mathcal{T}_{l(m);h}$ or $\mathcal{T}_{k(m);h}$. Then, the local Lagrange multiplier space $M_h(\gamma_m;\mathcal{S}_{m;h})$ associated with the non–mortar γ_m is given by

$$\begin{aligned} M_h(\gamma_m;\mathcal{S}_{m;h}) \quad := \quad & \{\mu \in C(\gamma_m) \mid \mu_{|_e} \in P_1(e), e \in \mathcal{S}_{m;h}, \\ & \mu_{|_e} \in P_0(e), e \text{ contains an endpoint of } \gamma_m\}. \end{aligned}$$

The global Lagrange multiplier space is defined as the product space

$$M_h := \prod_{m=1}^{M} M_h(\gamma_m;\mathcal{S}_{m;h}),$$

see [5, 6, 7]. We refer the reader to [26, 31, 32] for possible modifications of the Lagrange multiplier space. In terms of M_h, a suitable global space V_h for the discretization of (2.1) can be defined by

$$V_h := \{v \in X_h \mid b(v,\mu) = 0, \mu \in M_h\},$$

where the bilinear form $b(\cdot,\cdot)$ is given by the following duality pairing

$$b(v,\mu) := \sum_{m=1}^{M} \langle [v],\mu\rangle_{\gamma_m}, \quad v \in \prod_{k=1}^{K} H^1(\Omega_k),\ \mu \in \prod_{m=1}^{M} \left(H^{\frac{1}{2}}(\gamma_m)\right)'.$$

Here, $[\cdot]$ denotes the jump, $[v] := v_{|_{\Omega_{k(m)}}} - v_{|_{\Omega_{l(m)}}}$. The bilinear form $b(\cdot,\cdot)$ restricted on $X_h \times M_h$ can be written as $b(v,\mu) = \sum_{m=1}^{M} \int_{\gamma_m} [v]\mu\, ds$. With this notation for V_h, the nonconforming formulation of the mortar method can be given as : find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h), \quad v_h \in V_h; \tag{2.2}$$

see [6, 7]. Here, the bilinear form $a(\cdot,\cdot)$ is defined as $a(v,w) := \sum_{k=1}^{K} \int_{\Omega_k} a\nabla v \cdot \nabla w + bv\,w\,dx$, $v,w \in \prod_{k=1}^{K} H^1(\Omega_k)$ and $f(v) := \int_\Omega fv\,dx$, $v \in L^2(\Omega)$. The following saddle point problem is equivalent to (2.2): find $(u_h,\lambda_h) \in X_h \times M_h$ such that

$$\begin{array}{lcll} a(u_h,v_h) + b(v_h,\lambda_h) & = & f(v_h), & v_h \in X_h \\ b(u_h,\mu_h) & = & 0, & \mu_h \in M_h. \end{array} \tag{2.3}$$

The Lagrange multiplier λ_h is an approximation of the flux at the interfaces $\lambda_{\gamma_m} := a\frac{\partial u}{\partial \mathbf{n}_{kl}}$, where $\mathbf{n}_{kl}$ is the unit outer normal on $\Omega_{k(m)}$ restricted to $\partial\Omega_{k(m)} \cap \partial\Omega_{l(m)}$; see [5, 30].

In the rest of this section, we state technical tools which will be necessary for the analysis of the multigrid method. In particular, the coercivity of the bilinear form $a(\cdot,\cdot)$, a discrete inf–sup condition and well–known a priori estimates are given. In the following all constants $0 < c \le C < \infty$ are generic constants depending on the coefficients of (2.1) and on the shape regularity of the triangulation but not on the meshsize.

Under the assumption of H^2–regularity, we have the following $\mathcal{O}(h)$ estimate for the discretization error in the broken H^1–norm and an $\mathcal{O}(h^2)$ estimate for the discretization error in the L^2–norm

$$\begin{aligned} ||u-u_h||_1 &\le C\,h\,||f||_{0;\Omega}, \\ ||u-u_h||_{0;\Omega} &\le C\,h^2\,||f||_{0;\Omega} \end{aligned} \tag{2.4}$$

where $||v||_1^2 := \sum_{k=1}^K ||v||^2_{1;\Omega_k}$, $v \in \prod_{k=1}^K H^1(\Omega_k)$, see [6, 7, 10].

Furthermore, we have ellipticity of the bilinear form $a(\cdot,\cdot)$ on $Y \times Y$, i.e.,

$$a(v,v) \geq c\, ||v||_1^2, \quad v \in Y, \tag{2.5}$$

and V_h is a subspace of Y

$$Y := \{v \in \prod_{k=1}^K H^1(\Omega_k) \mid v_{|\partial\Omega} = 0, \int\limits_{\gamma_m} [v]\, ds = 0,\ 1 \leq m \leq M\};$$

see [8].

Our multigrid approach will be based on the saddle point formulation (2.3). It is introduced and analyzed in [5] where a discrete inf–sup condition is given in the $H_{00}^{1/2}$–norm of the Lagrange multiplier. Here, we work with mesh dependent norms for which a discrete inf–sup condition is established in [30]

$$||\mu||^2_{h^{-s};\mathcal{S}} := \sum_{\gamma_m} \sum_{e \in \mathcal{S}_{m;h}} h_e^{2s} ||\mu||^2_{0;e}, \quad \mu \in M_h,\ s \geq 0;$$

see also [10, 11]. The notation $||\cdot||_{h^{-s};\mathcal{S}}$ is chosen because the mesh dependent norm represents a kind of discrete H^s–dual norm. The inf–sup condition

$$c\, ||\mu_h||_{h^{-\frac{1}{2}};\mathcal{S}} \leq \sup_{\substack{v_h \in X_h \\ v_h \neq 0}} \frac{b(v_h,\mu_h)}{||v_h||_1}, \quad \mu \in M_h \tag{2.6}$$

can be found in [30] where also the a priori estimate for the Lagrange multiplier in the mesh dependent norm,

$$||\lambda - \lambda_h||_{h^{-\frac{1}{2}};\mathcal{S}} \leq C\, h ||f||_{0;\Omega}, \tag{2.7}$$

is established. We remark that the constants in the inf–sup condition (2.6) and in the a priori estimate (2.7) are independent of the meshsize. The proof of (2.7) requires a careful analysis since the bilinear form $b(v_h,\mu_h)$ is not uniformly bounded in h. Thus the general saddle point theory; see [14], cannot be applied directly. Combining the two a priori estimates (2.4) and (2.7), we get an order h^2 a priori estimate for the mesh dependent norm $||(u - u_h, \lambda - \lambda_h)||_{h;\Omega\times\mathcal{S}}$, where

$$||(v,\mu)||^2_{h;\Omega\times\mathcal{S}} := ||v||^2_{0;\Omega} + ||\mu||^2_{h^{-\frac{3}{2}};\mathcal{S}}, \quad (v,\mu) \in L^2(\Omega) \times L^2(\mathcal{S}).$$

The corresponding mesh dependent scalar product on $L^2(\Omega) \times L^2(\mathcal{S})$ is denoted by $(\cdot,\cdot)_{h;\Omega\times\mathcal{S}}$

$$((v,\mu),(w,\nu))_{h;\Omega\times\mathcal{S}} := (v,w)_{0;\Omega} + (\mu,\nu)_{h^{-\frac{3}{2}};\mathcal{S}} := (v,w)_{0;\Omega} + \sum_{\gamma_m} \sum_{e \in \mathcal{S}_{m;h}} h_e^3 (\mu,\nu)_{0;e}.$$

Finally, the following lemma gives a relation between the weighted L^2–norm of the jump and the nonconformity of an element. A similar result can be found in [30] where finite element spaces of different order are used. Here, we use the same order for the finite element spaces but two different triangulations $\mathcal{T}_h$ and $\mathcal{T}_{2h}$. We assume that the triangulation $\mathcal{T}_h$ is obtained by uniform refinement from $\mathcal{T}_{2h}$.

LEMMA 2.1. *Let $v_h \in X_h$ satisfy*

$$b(v_h, \mu_{2h}) = 0, \quad \mu_{2h} \in M_{2h}.$$

Then, the jump of v_h can be bounded by

$$\sum_{\gamma_m} \sum_{e \in \mathcal{S}_{m;h}} \frac{1}{h_e} ||[v_h]||^2_{0;e} \le C \inf_{v \in H^1_0(\Omega)} ||v_h - v||^2_1.$$

Proof. Here, we will sketch the proof only for the quasi–uniform case. For more details, we refer the reader to [30]. Using the approximation property of M_{2h} and a trace Theorem, we obtain for each $v \in H^1_0(\Omega)$ and $\mu_{2h} \in M_{2h}$

$$\begin{aligned} ||[v_h]||^2_{0;\mathcal{S}} &= ([v_h], [v_h - v] - \mu_{2h})_{0;\mathcal{S}} \le C\, h^{\frac{1}{2}} ||[v_h]||_{0;\mathcal{S}} ||[v_h - v]||_{\frac{1}{2};\mathcal{S}} \\ &\le C\, h^{\frac{1}{2}} ||[v_h]||_{0;\mathcal{S}} ||v_h - v||_1. \end{aligned}$$

Here, the norm $|| \cdot ||_{1/2;\mathcal{S}}$ is defined by

$$||\mu||^2_{\frac{1}{2};\mathcal{S}} := \sum_{m=1}^{M} ||\mu||^2_{\frac{1}{2};\gamma_m}, \quad \mu \in \prod_{m=1}^{M} H^{\frac{1}{2}}(\gamma_m),$$

where $|| \cdot ||_{1/2;\gamma_m}$ is the standard $H^{1/2}$–norm on γ_m. □

3. Approximation property for the saddle point problem. A suitable approximation property is one of the basic tools needed to establish a level independent convergence rate for the $\mathcal{W}$–cycle. Here, we will establish an approximation result not only for the first solution component but also for the Lagrange multiplier. An approximation property for the first solution component is given in [10]. Although the bilinear form $b(\cdot, \cdot)$ is not bounded independently of the meshsize, optimal approximation properties can be established.

A family of finite element spaces associated with a nested sequence of triangulations $\mathcal{T}_l$ of meshsize h_l is given by

$$X_0 \times M_0 \subset X_1 \times M_1 \subset \cdots \subset X_l \times M_l \subset \cdots \subset X_n \times M_n.$$

Here, we assume that the triangulations are quasi–uniform and that $h_{l-1} = 2h_l$. The spaces X_l, M_l and $X_l \times M_l$ equipped with the norms $||\cdot||_{0;\Omega}$, $||\cdot||_{h_l^{-3/2};\mathcal{S}}$ and $||\cdot||_{h_l;\Omega\times\mathcal{S}}$, respectively, are Hilbert spaces. Let T be a linear continuous operator $T : H_1 \longrightarrow H_2$. Here, H_1 and H_2 associated with the norms $|| \cdot ||_{H_1}$ and $|| \cdot ||_{H_2}$, respectively, are Hilbert spaces. We use the standard operator norm

$$||T|| := \sup_{\substack{x \in H_1 \\ x \neq 0}} \frac{||Tx||_{H_2}}{||x||_{H_1}}.$$

The following lemma is an adaption of Lemma 4.2 in [27] to mortar finite elements. Based on this lemma a suitable approximation property will be formulated at the end of this section. In contrast to [27], the $b(\cdot, \cdot)$ is not continuous. However, Lemma 2.1 guarantees the continuity of $b(\cdot, \cdot)$ on a suitable subspace of $X_l \times L^2(\mathcal{S})$

$$|b(w_l, \mu)| \le C\|w_l\|_1 \, \|\mu\|_{h^{-\frac{1}{2}};\mathcal{S}}, \quad (w_l, \mu) \in \hat{X}_l \times L^2(\mathcal{S}),$$

where $\hat{X}_l := \{w_l \in X_l \,|\, b(w_l, \mu_{l-1}) = 0\}$.

LEMMA 3.1. *Let $(d_l, \delta_l) \in X_l \times M_l$ be orthogonal to $X_{l-1} \times M_{l-1}$, i.e.*

$$\big((d_l, \delta_l), (v_{l-1}, \mu_{l-1})\big)_{h_l, \Omega\times\mathcal{S}} = 0, \qquad (v_{l-1}, \mu_{l-1}) \in X_{l-1} \times M_{l-1},$$

and $(w_l, \nu_l) \in X_l \times M_l$ be the solution of the problem: find $(w_l, \nu_l) \in X_l \times M_l$ such that

$$\begin{array}{lcllll} a(w_l, v_l) & + & b(v_l, \nu_l) & = & (d_l, v_l)_{0;\Omega}, & v_l \in X_l \\ b(w_l, \mu_l) & & & = & (\delta_l, \mu_l)_{h_l^{-\frac{3}{2}};\mathcal{S}}, & \mu_l \in M_l. \end{array} \tag{3.1}$$

Then, there exists a constant satisfying

$$||(w_l, \nu_l)||_{h_l;\Omega\times\mathcal{S}} \le C\, h_l^2\, ||(d_l, \delta_l)||_{h_l;\Omega\times\mathcal{S}}.$$

Proof. The quasi–uniformity of the triangulations yields

$$c\,(||w_l||_{0;\Omega} + h_l^{\frac{3}{2}}||\nu_l||_{0;\mathcal{S}}) \le ||(w_l, \nu_l)||_{h_l;\Omega\times\mathcal{S}} \le C\,(||w_l||_{0;\Omega} + h_l^{\frac{3}{2}}||\nu_l||_{0;\mathcal{S}}).$$

The estimate for the term in the Lagrange multiplier will be based on the discrete inf–sup condition (2.6) whereas the bound for $||w_l||_{0;\Omega}$ is based on duality techniques. We start with an estimate for the upper bound of $h_l^{3/2}||\nu_l||_{0;\mathcal{S}}$. The continuity of $a(\cdot,\cdot)$, the discrete inf–sup condition (2.6) and the orthogonality of d_l, $(d_l, v_{l-1})_{0;\Omega} = 0$, $v_{l-1} \in X_{l-1}$, yield an upper bound for $||\nu_l||_{h_l^{-1/2};\mathcal{S}}$ in terms of $||w_l||_1$

$$\begin{array}{rcl} ||\nu_l||_{h_l^{-\frac{1}{2}};\mathcal{S}} & \le & C \displaystyle\sup_{\substack{v_l\in X_l\\ v_l\ne 0}} \frac{b(v_l, \nu_l)}{||v_l||_1} \\ & = & C \displaystyle\inf_{v_{l-1}\in X_{l-1}} \sup_{\substack{v_l\in X_l\\ v_l\ne 0}} \frac{(d_l, v_l - v_{l-1})_{0;\Omega} - a(w_l, v_l)}{||v_l||_1} \\ & \le & C\ \left(||w_l||_1 + h_l||d_l||_{0;\Omega}\right). \end{array} \tag{3.2}$$

Observing that $w_l \in Y$, applying (2.5) and using (3.2), we find for $w_{l-1} \in X_{l-1}$

$$\begin{array}{rcl} c\,||w_l||_1^2 & \le & a(w_l, w_l) = (d_l, w_l)_{0;\Omega} - b(w_l, \nu_l) \\ & = & (d_l, w_l - w_{l-1})_{0;\Omega} - (\delta_l, \nu_l)_{h_l^{-\frac{3}{2}};\mathcal{S}} \\ & \le & C\,h_l\left(||d_l||_{0;\Omega} + h_l^{\frac{3}{2}}||\delta_l||_{0;\mathcal{S}}\right)||w_l||_1 + C\,h_l^{\frac{7}{2}}||d_l||_{0;\Omega}||\delta_l||_{0;\mathcal{S}}. \end{array}$$

To obtain an upper bound for $||w_l||_1$ in terms of $h_l^{\frac{3}{2}}||\delta_l||_{0;\mathcal{S}}$ and $||d_l||_{0;\Omega}$, it is sufficient to apply Young's inequality and to observe that

$$(h_l^{\frac{7}{2}}||d_l||_{0;\Omega}||\delta_l||_{0;\mathcal{S}})^{1/2} \le \frac{1}{2}(h_l||d_l||_{0;\Omega} + h_l^{\frac{5}{2}}||\delta_l||_{0;\mathcal{S}}).$$

Thus, we have

$$||w_l||_1 \le C\,h_l\left(||d_l||_{0;\Omega} + h_l^{\frac{3}{2}}||\delta_l||_{0;\mathcal{S}}\right). \tag{3.3}$$

Combining (3.3) with the upper bound for $||\nu_l||_{h_l^{-1/2};\mathcal{S}}$, we find

$$h_l^{\frac{3}{2}}\ ||\nu_l||_{0;\mathcal{S}} \le C\,h_l^2\left(||d_l||_{0;\Omega} + h_l^{\frac{3}{2}}||\delta_l||_{0;\mathcal{S}}\right). \tag{3.4}$$

In the next step, we will focus on an estimate for $||w_l||_{0;\Omega}$. Let $\hat{w} \in H_0^1(\Omega)$ be the solution of the continuous variational problem: find $\hat{w} \in H_0^1(\Omega)$ such that

$$a(\hat{w}, v) = (w_l, v)_{0;\Omega} \quad v \in H_0^1(\Omega).$$

Taking into account that w_l, in general, is not contained in $H_0^1(\Omega)$, we get

$$||w_l||_{0;\Omega}^2 = a(\hat{w}, w_l) + b(w_l, \hat{\nu}),$$

where $\hat{\nu} := a\nabla\hat{w}\mathbf{n}$ is the flux of $\hat{w}$ at the interfaces $\mathcal{S}$. Using the orthogonalities $a(w_l, v_{l-1}) + b(v_{l-1}, \nu_l) = 0$, $v_{l-1} \in X_{l-1}$, $b(w_l, \mu_{l-1}) = 0$, $\mu_{l-1} \in M_{l-1}$ and observing $b(\hat{w}, \nu_l) = 0$, $\nu_l \in M_l$, we find for $v_{l-1} \in X_{l-1}$ and $\mu_{l-1} \in M_{l-1}$

$$\begin{aligned} ||w_l||_{0;\Omega}^2 &= a(\hat{w} - v_{l-1}, w_l) + b(w_l, \hat{\nu}) + b(\hat{w} - v_{l-1}, \nu_l) \\ &\le C\left(||\hat{w} - v_{l-1}||_1 ||w_l||_1 + ||[w_l]||_{0;\mathcal{S}} ||\hat{\nu} - \mu_{l-1}||_{0;\mathcal{S}} + ||[\hat{w} - v_{l-1}]||_{0;\mathcal{S}} ||\nu_l||_{0;\mathcal{S}} \right). \end{aligned}$$

We choose $v_{l-1} \in V_{l-1}$ as a local quasi–projection of $\hat{w}$ such that

$$\begin{aligned} ||\hat{w} - v_{l-1}||_1 &\le C\, h_{l-1} ||\hat{w}||_{2;\Omega} \\ ||[\hat{w} - v_{l-1}]||_{0;\mathcal{S}} &\le C\, h_{l-1}^{\frac{3}{2}} ||\hat{w}||_{2;\Omega}; \end{aligned}$$

see [25], and $\mu_{l-1} \in M_{l-1}$ such that

$$||\hat{\nu} - \mu_{l-1}||_{0;\mathcal{S}} \le C\, h_{l-1}^{\frac{1}{2}} ||\hat{\nu}||_{\frac{1}{2};\mathcal{S}}.$$

Finally, we use the H^2–regularity, a trace Theorem and Lemma 2.1 to obtain

$$||w_l||_{0;\Omega}^2 \le C\, h_l \left(||w_l||_1 + ||\nu_l||_{h_l^{-\frac{1}{2}};\mathcal{S}} \right) ||w_l||_{0;\Omega}$$

which proves together with (3.3) and (3.4) the assertion. □

The approximation property of Lemma 3.1 can be also reformulated in an operator setting. Let $A_l : X_l \longrightarrow X_l$, $B_l : M_l \longrightarrow X_l$, $B_l^* : X_l \longrightarrow M_l$ be the operators defined by

$$(A_l v_l, w_l)_{0;\Omega} := a(v_l, w_l),\ (B_l \mu_l, w_l)_{0;\Omega} := b(w_l, \mu_l),\ (B_l^* w_l, \mu_l)_{h_l^{-\frac{3}{2}};\mathcal{S}} := b(w_l, \mu_l).$$

Then, the self–adjoint non–singular operator $K_l : X_l \times M_l \longrightarrow X_l \times M_l$ associated with the saddle point problem (2.3) is given by

$$K_l(v_l, \mu_l) := (A_l v_l + B_l \mu_l, B_l^* v_l), \quad (v_l, \mu_l) \in X_l \times M_l. \tag{3.5}$$

The solution (w_l, ν_l) of the saddle point problem (3.1) satisfies

$$K_l(w_l, \nu_l) = (d_l, \delta_l),$$

and thus $||K_l^{-1}(d_l, \delta_l)||_{h_l;\Omega\times\mathcal{S}} \le C\, h_l^2 ||(d_l, \delta_l)||_{h_l;\Omega\times\mathcal{S}}$ for $(d_l, \delta_l) \in X_l \times M_l$ being orthogonal on $X_{l-1} \times M_{l-1}$ with respect to $(\cdot,\cdot)_{h_l;\Omega\times\mathcal{S}}$. Equivalently, we find

$$||(w_l, \nu_l)||_{h_l;\Omega\times\mathcal{S}} \le C\, h_l^2 ||K_l(w_l, \nu_l)||_{h_l;\Omega\times\mathcal{S}} \tag{3.6}$$

for $(w_l, \nu_l) \in X_l \times M_l$ satisfying $(K_l(w_l, \nu_l), (v_{l-1}, \mu_{l-1}))_{h_l;\Omega\times\mathcal{S}} = 0$, $(v_{l-1}, \mu_{l-1}) \in X_{l-1} \times M_{l-1}$.

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The properties of the operator B_l will be essential for the stability of the iteration process. In the rest of this section, we consider B_l in more detail, whereas in the next section a suitable smoothing property will be established. The following lemma provides upper and lower bounds for $||B_l\mu_l||_{0;\Omega}$. As before in Lemma 3.1, we obtain the qualitative result as if the bilinear form $b(\cdot,\cdot)$ would be uniformly bounded in the meshsize. However, one has to be more careful in the proof.

LEMMA 3.2. *There exist constants such that*

$$c\,\frac{1}{h_l^2}||\mu_l||_{h_l^{-\frac{3}{2}};\mathcal{S}} \le ||B_l\mu_l||_{0;\Omega} \le C\frac{1}{h_l^2}||\mu_l||_{h_l^{-\frac{3}{2}};\mathcal{S}}. \tag{3.7}$$

Proof. The proof of the upper bound is based on the observation that the the bilinear form $b(\cdot,\cdot)$ reflects a duality pairing on the interfaces. Using an inverse estimate and the definition of B_l, we obtain

$$\begin{aligned} ||B_l\mu_l||_{0;\Omega} &= \sup_{\substack{w_l\in X_l\\ w_l\neq 0}} \frac{(B_l\mu_l,w_l)_{0;\Omega}}{||w_l||_{0;\Omega}} = \sup_{\substack{w_l\in X_l\\ w_l\neq 0}} \frac{b(w_l,\mu_l)}{||w_l||_{0;\Omega}} \\ &\le \sup_{\substack{w_l\in X_l\\ w_l\neq 0}} \frac{||[w_l]||_{0;\mathcal{S}}||\mu_l||_{0;\mathcal{S}}}{||w_l||_{0;\Omega}} \le \frac{C}{h_l^2}||\mu_l||_{h_l^{-\frac{3}{2}};\mathcal{S}}. \end{aligned}$$

The proof of the lower bound follows the lines of the proof of the inf–sup condition and is obtained by construction. Our starting point is

$$||B_l\mu_l||_{0;\Omega} \ge \frac{b(w_l,\mu_l)}{||w_l||_{0;\Omega}}, \quad w_l\in X_l,\ w_l\neq 0.$$

Each $w_l \in X_l$ is uniquely defined by its values at the vertices of the triangulation. We define $w_l(p) := \mu_l(p)$ if p is an interior vertex of one of the 1D interface triangulations $\mathcal{S}_{m;l}$, $1\le m\le M$. For all other vertices p of $\mathcal{T}_l$, we set $w_l(p) := 0$. This special choice yields $b(w_l,\mu_l) \ge c\,||[w_l]||_{0;\mathcal{S}}\,||\mu_l||_{0;\mathcal{S}}$, we refer to [30] for details. Now, the lower bound in (3.7) follows from $||[w_l]||_{0;\mathcal{S}} \ge c\,h_l^{-1/2}||w_l||_{0;\Omega}$. ☐

An easy consequence of (3.7) is

$$\inf_{\substack{v_l\in X_l\\ B_l^*v_l=B_l^*w_l}} ||v_l||_{0;\Omega} \le C\,h_l^2||B_l^*w_l||_{h_l^{-\frac{3}{2}};\mathcal{S}}, \quad w_l\in X_l. \tag{3.8}$$

Remark: *The fact that the bilinear form $b(w_l,\mu_l)$ is not uniformly bounded by $\|w_l\|_1\,\|\mu_l\|_{h^{-1/2};\mathcal{S}}$ has no influence on the estimate (3.7). We have to consider the L^2–norm of the jump $[w_l]$ in more detail. The properties of the operator B_l are strongly connected with the inf–sup condition (2.6), the duality pairing on the interface and an inverse inequality.*

4. Smoothing property for the saddle point problem. The second basic tool required to establish convergence within the multigrid framework is the smoothing property. We follow the lines of [27] and work with the squared system. The advantage of this approach is that, in contrast to [10], modified Schur complements systems do not need to be solved exactly in each smoothing step.

Using the definition (3.5) of the operator K_l, the saddle point problem (2.3) on $X_l\times M_l$ is equivalent to the following operator equation: find $z_l^* := (u_l,\lambda_l)\in X_l\times M_l$ such that

$$K_l z_l^* = f_l,$$

where $f_l := (\tilde f_l, 0)\in X_l\times M_l$ is defined by $(\tilde f_l, v_l)_{0;\Omega} := (f, v_l)_{0;\Omega}$, $v_l\in X_l$.

The smoothing operator $\hat{K}_l : X_l \times M_l \longrightarrow X_l \times M_l$ is defined by means of the symmetric positive definite bilinear forms $\hat{a}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X_l \times X_l$ and $M_l \times M_l$, respectively,

$$(4.1) \qquad (\hat{K}_l(v_l,\mu_l),(w_l,\nu_l))_{h_l;\Omega\times\mathcal{S}} := \hat{a}(v_l,w_l) + d(\nu_l,\mu_l), \quad (w_l,\nu_l) \in X_l \times M_l.$$

It has a block diagonal structure

$$\hat{K}_l(v_l,\mu_l) = (\hat{A}_l v_l, D_l \mu_l),$$

where the operators $\hat{A}_l : X_l \longrightarrow X_l$ and $D_l : M_l \longrightarrow M_l$ are associated with the bilinear forms $\hat{a}(\cdot,\cdot)$ and $d(\cdot,\cdot)$, respectively. One smoothing iteration on level l is given by

$$(4.2) \qquad z_l^m := z_l^{m-1} + \hat{K}_l^{-1} K_l \hat{K}_l^{-1}(r_l - K_l z_l^{m-1}),$$

where r_l stands for the right side of the system $K_l z_l = r_l$ which has to be solved, z_l is the exact solution and z_l^m denotes the iterate in the mth–step. Each smoothing step can be easily performed provided that the application of $\hat{A}_l^{-1}$ and D_l^{-1} is cheap. The following lemma gives the smoothing rate:

LEMMA 4.1. *Let $\hat{K}_l$ be defined as in (4.1), where $\hat{A}_l$, $||\hat{A}_l|| \leq C/h_l^2$, and D_l, $||D_l|| \leq C/h_l^2$, are self–adjoint positive definite operators. Under the assumptions that there exists a α_{w_l}, $0 < \alpha_{w_l} < 1$ for each $w_l \in X_l$ such that*

$$(A_l w_l, w_l)_{0;\Omega} \leq \alpha_{w_l}(\hat{A}_l w_l, w_l)_{0;\Omega}, \quad (B_l D_l^{-1} B_l^* w_l, w_l)_{0;\Omega} \leq (1-\alpha_{w_l})(\hat{A}_l w_l, w_l)_{0;\Omega},$$

the following smoothing property for the iteration (4.2) holds

$$(4.3) \qquad ||K_l e_l^m||_{h_l;\Omega\times\mathcal{S}} \leq \frac{C}{h_l^2\sqrt{m}} ||e_l^0||_{h_l;\Omega\times\mathcal{S}}, \quad m \geq 1,$$

where $e_l^m := z_l^m - z_l$, $0 \leq m$ is the iteration error in the mth–smoothing step.

Furthermore, if $||\hat{A}_l^{-1}|| \leq C\, h_l^2$, we have the following stability estimate for the iteration error

$$(4.4) \qquad ||e_l^m||_{h_l;\Omega\times\mathcal{S}} \leq C\, ||e_l^0||_{h_l;\Omega\times\mathcal{S}}, \quad m \geq 1$$

with a constant independent of m.

Proof. The iteration error e_l^m is given by

$$e_l^m = (\mathrm{Id} - \hat{K}_l^{-1} K_l \hat{K}_l^{-1} K_l)^m e_l^0, \quad m \geq 1.$$

Since $\hat{K}_l$ is a self–adjoint positive definite operator and K_l is self–adjoint, there exists a complete set of eigenfunctions z_l^i satisfying

$$\hat{K}_l^{-\frac{1}{2}} K_l \hat{K}_l^{-\frac{1}{2}} z_l^i = \lambda_i z_l^i.$$

Setting $(w_l^i, \mu_l^i) := \hat{K}_l^{-\frac{1}{2}} z_l^i$, we find for $\lambda_i \neq 0$,

$$A_l w_l^i + \frac{1}{\lambda_i} B_l D_l^{-1} B_l^* w_l^i = \lambda_i \hat{A}_l w_l^i.$$

Then, the assumptions on A_l and D_l yield $|\lambda_i| \leq 0.5(\alpha_{w_l^i} + \sqrt{\alpha_{w_l^i}^2 + 4(1-\alpha_{w_l^i})}) \leq 1$. Without loss of generality, we can assume that the eigenfunctions are normalized $(z_l^i, z_l^j)_{h_l;\Omega\times\mathcal{S}} = \delta_{ij}$. Then, the norm of $K_l e_l^m$ is bounded by

$$\begin{aligned} ||K_l e_l^m||_{h_l;\Omega\times\mathcal{S}} &\leq ||\hat{K}_l^{\frac{1}{2}} \hat{K}_l^{-\frac{1}{2}} K_l \hat{K}_l^{-\frac{1}{2}} (\mathrm{Id} - \hat{K}_l^{-\frac{1}{2}} K_l \hat{K}_l^{-1} K_l \hat{K}_l^{-\frac{1}{2}})^m \hat{K}_l^{\frac{1}{2}} e_l^0||_{h_l;\Omega\times\mathcal{S}} \\ &\leq \sup_{s\in\sigma(\hat{K}_l^{-\frac{1}{2}} K_l \hat{K}_l^{-\frac{1}{2}})} |s(1-s^2)^m|\; ||\hat{K}_l||\; ||e_l^0||_{h_l;\Omega\times\mathcal{S}}. \end{aligned}$$

Using $\sup_{t\in[0;1]}(t(1-t^2)^m) \leq C/\sqrt{m}$ and $||\hat{K}_l|| \leq C/h_l^2$, we obtain (4.3). To obtain the stability estimate (4.4), we use the same type of arguments. The assumption on D_l and (3.8) yield an upper bound for $||D_l^{-1/2}||$,

$$\begin{aligned} ||D_l^{-\frac{1}{2}}|| &= \sup_{\substack{\mu_l\in M_l\\ \mu_l\neq 0}} \frac{||D_l^{-\frac{1}{2}}\mu_l||_{h_l^{-3/2};\mathcal{S}}}{||\mu_l||_{h_l^{-3/2};\mathcal{S}}} = \sup_{\substack{\mu_l\in M_l\\ B_l^* w_l=\mu_l\neq 0}} \frac{||D_l^{-\frac{1}{2}}B_l^* w_l||_{h_l^{-3/2};\mathcal{S}}}{||B_l^* w_l||_{h_l^{-3/2};\mathcal{S}}} \\ &\leq \sup_{\substack{\mu_l\in M_l\\ \mu_l\neq 0}} \inf_{\substack{w_l\in X_l\\ B_l^* w_l=\mu_l}} \frac{||\hat{A}_l^{\frac{1}{2}} w_l||_{0;\Omega}}{||B_l^* w_l||_{h_l^{-3/2};\mathcal{S}}} \leq C\, h_l^{-1} h_l^2 \leq C\, h_l. \end{aligned}$$

The last inequality together with the assumption on $A_l^{-1/2}$ gives $||\hat{K}_l^{-1/2}|| \leq h_l$, and thus

$$\begin{aligned} ||e_l^m||_{h_l;\Omega\times\mathcal{S}} &= ||(\mathrm{Id}-\hat{K}_l^{-1}K_l\hat{K}_l^{-1}K_l)^m||\ ||e_l^0||_{h_l;\Omega\times\mathcal{S}} \\ &\leq \sup_{s\in[-1;1]}(1-s^2)^m ||\hat{K}_l^{\frac{1}{2}}||\ ||\hat{K}_l^{-\frac{1}{2}}||\ ||e_l^0||_{h_l;\Omega\times\mathcal{S}} \leq C\,||e_l^0||_{h_l;\Omega\times\mathcal{S}}. \end{aligned}$$

□

Remark: *Replacing the iteration (4.2) by a conjugate residual algorithm improves the upper bound (4.3). As in the positive definite case with a suitable Jacobi–type smoothing operator, we obtain*

$$||K_l e_l^m||_{h_l;\Omega\times\mathcal{S}} \leq \frac{C}{h_l^2 m}||e_l^0||_{h_l;\Omega\times\mathcal{S}}, \quad m\geq 1.$$

For more details, we refer the reader to [27].

Combining the approximation property (3.6) and the smoothing property (4.3), we obtain a mesh size independent convergence rate for the two–grid algorithm. The stability estimate (4.4) is necessary in case of the $\mathcal{W}$–cycle analysis. Under the assumptions of Lemma 4.1 the convergence rate of the $\mathcal{W}$–cycle in the $||\cdot||_{h_l;\Omega\times\mathcal{S}}$–norm is independent of the number of refinement levels provided that the number of smoothing steps is large enough [21, 27]. For numerical results in the saddle point framework, where the iterates does not satisfy the constraints exactly, we refer the reader to [29]. In case that the constraints are satisfied in each smoothing step a weaker approximation result than that given in Lemma 3.1 can be used. Working in the more general framework has the advantage that no mass system has to be solved in each smoothing step as is required in [10, 11].

Remark: *A suitable smoother in the algebraic formulation of the method is given by the following diagonal matrix*

$$\hat{K} := \begin{pmatrix} \alpha_1\mathrm{Id} & 0 \\ 0 & h_l^2\alpha_2\mathrm{Id} \end{pmatrix}$$

with some constants $\alpha_1, \alpha_2 > 0$. An optimal scaling of the constants has to be obtained by numerical results. Here, nodal basis functions for the finite elements as well as the Lagrange multiplier are used. Then, the level independent convergence rate of the $\mathcal{W}$–cycle is obtained for the energy norm $||\cdot||_{\hat{K}}$ of $\hat{K}$.

REFERENCES

[1] Y. ACHDOU AND Y. KUZNETSOV, *Substructuring preconditioners for finite element methods on nonmatching grids*, East–West J. Numer. Math. 3 (1995), pp. 1-28.

[2] Y. ACHDOU, Y. KUZNETSOV, AND O. PIRONNEAU, *Substructuring preconditioners for the Q_1 mortar element method.* Numer. Math. 71 (1995), pp. 419-449.

[3] Y. ACHDOU, Y. MADAY, AND O.B. WIDLUND, *Iterative substructuring preconditioners for mortar element methods in two dimensions*, SIAM J. Numer. Anal., 36 (1999), pp. 551-580.

[4] F. BEN BELGACEM, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999) pp. 173-197.

[5] F. BEN BELGACEM AND Y. MADAY, *The mortar element method for three dimensional finite elements.* RAIRO Modél. Anal. Numér. 31 (1997), pp. 289-302.

[6] C. BERNARDI, Y. MADAY, AND A.T. PATERA, *Domain decomposition by the mortar element method*, in Asymptotic and numerical methods for partial differential equations with critical parameters,H. Kaper et al., eds., pp. 269-286, Reidel, Dordrecht, 1993.

[7] C. BERNARDI, Y. MADAY, AND A.T. PATERA, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear Partial differential Equations and Their Applications.Collège de France Seminar. Vol. XII. Papers from the Seminar on Applied Mathematics held at the Collège de France, Paris, 1991–1993, H.H. Brézis and J.-L. Lions eds., Pitman Research Notes in Mathematics Series, 302. Longman Scientific & Technical, Harlow, pp. 13-51.

[8] C. BERNARDI AND Y. MADAY, *Raffinement de maillage en elements finis par la methode des joints*, C. R. Acad. Sci., Paris, Ser. I Math.320, No.3 (1995), pp. 373-377. This paper appeared also as a preprint, Laboratoire d'Analyse Numérique, Univ. Pierre et Marie Curie, Paris, R94029, including more details.

[9] D. BRAESS, *Finite Elemente*, Springer–Verlag, Berlin, 1992.

[10] D. BRAESS, W. DAHMEN, AND C. WIENERS, *A multigrid algorithm for the mortar finite element method*, SIAM J. Numer. Anal., 37, (1999), pp. 48-69.

[11] D. BRAESS AND W. DAHMEN, *Stability estimates of the mortar finite element method for 3–dimensional problems*, East–West J. Numer. Math., 6 (1998), pp. 249-263.

[12] D. BRAESS, M. DRYJA, AND W. HACKBUSCH, *A multigrid method for nonconforming FE–discretizations with applications to non–matching grids*, Computing, 63 (1999), pp. 1-25.

[13] D. BRAESS AND R. SARAZIN, *An efficient smoother for the Stokes problem*, Appl. Numer. Math., 23 (1997), pp. 3-19.

[14] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, Springer–Verlag, New York, 1991

[15] M. CASARIN AND O.B. WIDLUND, *A hierarchical preconditioner for the mortar finite element method.* Electron. Trans. Numer. Anal., 4 (1996), pp. 75-88. http://etna.mcs.kent.edu/vol.4.1996/pp75-88.dir/pp75-88.pdf.

[16] M. DRYJA, *An iterative substructuring method for elliptic mortar finite element problems with discontinuous coefficients*, in Domain Decomposition Methods 10, J. Mandel, C. Farhat, and X.-C. Cai, eds., Contemp. Math., 218, American Mathematical Society, 1998, pp. 94-103.

[17] B. ENGELMANN, R.H.W. HOPPE, Y. ILIASH, Y. KUZNETSOV, Y. VASSILEVSKI, AND B.I. WOHLMUTH, *Adaptive finite element methods for domain decompositions on nonmatching grids*, to appear in Parallel solution of PDEs, IMA, Springer, Berlin–Heidelber–New York, 120, 2000, pp.57-84.

[18] B. ENGELMANN, R.H.W. HOPPE, Y. ILIASH, Y. KUZNETSOV, Y. VASSILEVSKI, AND B.I. WOHLMUTH, *Adaptive macro–hybrid finite element methods*, in ENUMATH(Heidelberg) 97, H.G. Bock, F. Brezzi, R. Glowinski, G. Kanschat, Y. Kuznetsov,J. Périaux, and R. Rannacher, eds., World Scientific, Singapore, 1998, pp. 294-302.

[19] J. GOPALAKRISHNAN AND J. PASCIAK, *Multigrid for the mortar finite element method*, to appear in SIAM J. Numer. Anal.

[20] W. HACKBUSCH, *Multigrid methods and applications*, Springer Berlin, 1985.

[21] W. HACKBUSCH, *Iterative Solution of Large Sparse Systems of Equations*, Springer-Verlag, 1993.

[22] R.H.W. HOPPE, Y. ILIASH, Y. KUZNETSOV, Y. VASSILEVSKI, AND B.I. WOHLMUTH, *Analysis and parallel implementation of adaptive mortar finite element methods*, East–West J. Numer. Math., 6 (1998), pp.223-248.

[23] Y. KUZNETSOV, *Efficient iterative solvers for elliptic finite element problems on nonmatching grids*, Russian J. Numer. Anal. Math. Modelling, 10 (1995), pp. 127-143.

[24] P. LE TALLEC AND T. SASSI, *Domain decomposition with nonmatching grids: augmented Lagrangian approach*, Math. Comput. 64 (1995), pp. 1367-1396.

[25] L.R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. 54 (1990), pp. 483-493.

[26] P. SESHAIYER AND M. SURI, *Convergence results for non–conformin g hp methods: The mortar finite element method*, Domain Decomposition Methods 10, J. Mandel, C. Farhat, and X.-C. Cai, eds., Contemp. Math., 218, American Mathematical Society, 1998, pp. 453-459.

[27] R. VERFÜRTH, *A multilevel algorithm for mixed problems*, SIAM J. Numer. Anal., 21 (1984), pp. 264-271.

[28] C. WIENERS AND B.I. WOHLMUTH, *The coupling of mixed and conforming finite element discretizations*, in Domain Decomposition Methods 10, J. Mandel, C. Farhat, and X.-C. Cai, eds., Contemp. Math., 218, American Mathematical Society, 1998, pp. 547-554.

[29] C. WIENERS AND B.I. WOHLMUTH, *A general framework for multigrid methods for mortar finite elements*, Submitted to Notes on Numerical Fluid Mechanics, Vieweg

[30] B.I. WOHLMUTH, *Hierarchical a posteriori error estimators for mortar finite element methods with Lagrange multipliers*, SIAM J. Numer. Anal., 36 (1999), pp. 1636-1658.
[31] B.I. WOHLMUTH, *A mortar finite element method using dual spaces for the Lagrange multiplier*, to appear in SIAM J. Numer. Anal.
[32] B.I. WOHLMUTH AND R. KRAUSE, *Multigrid Methods Based on the Unconstrained Product Space Arising from Mortar Finite Element Discretizations*, Preprint A18–99, FU Berlin.
[33] W. ZULEHNER, *A class of smoothers for saddle point problems* Preprint 546, 1998, Inst. of Analysis and Numerics, Johannes Kepler University.