Electronic Transactions on Numerical Analysis.
Volume 13, pp.1-11, 2002.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

A UNIFORMLY ACCURATE FINITE VOLUME DISCRETIZATION FOR A CONVECTION-DIFFUSION PROBLEM *

DIRK WOLLSTEIN , TORSTEN LINSS†, AND HANS-GÖRG ROOS†

Abstract. A singularly perturbed convection-diffusion problem is considered. The problem is discretized using an inverse-monotone finite volume method on Shishkin meshes. We establish first-order convergence in a global energy norm and a mesh-dependent discrete energy norm, no matter how small the perturbation parameter. Numerical experiments support the theoretical results.

Key words. convection-diffusion problems, finite volume methods, singular perturbation, Shishkin mesh.

AMS subject classifications. 65N30.

1. Introduction.

Let us consider the model convection-diffusion problem

$$-\varepsilon\Delta u + \boldsymbol{a}\cdot \operatorname{grad} u + bu = f \ \text{ in } \ \Omega = (0,1)^2, \quad u = 0 \ \text{ on } \ \Gamma = \partial\Omega, \tag{1.1}$$

with $0 < \varepsilon \ll 1$, $\boldsymbol{a} = \big(a_1(\boldsymbol{x}), a_2(\boldsymbol{x})\big) \geq \big(\alpha_1, \alpha_2\big) > (0,0)$ and $b(\boldsymbol{x}) - \frac{1}{2}\operatorname{div} \boldsymbol{a}(\boldsymbol{x}) \geq \beta > 0$ for $\boldsymbol{x} = (x, y) \in \Omega$. We assume that $\boldsymbol{a}$, b and f are smooth. The solution u of (1.1) has exponential boundary layers at the sides $x = 1$ and $y = 1$ of Ω.

There is a vast literature dealing with numerical methods for convection-diffusion and associated problems; see [13, 15] for a survey. We shall consider an inverse-monotone finite volume discretization on layer-adapted meshes. This scheme was introduced by Baba and Tabata [3] and later generalized by Angermann [1, 2] who also realised that Samarski's scheme [16] fits into this framework. Although we restrict ourselves to piecewise uniform meshes—the so-called Shishkin meshes [12, 18]—our results can be extended to more general meshes, e.g., the Shishkin-type meshes of [14]; see [21, Chapter 3].

A number of numerical methods on Shishkin meshes have been investigated including finite difference schemes [9, 12, 18], Galerkin FEM [7, 19], the streamline diffusion FEM [11, 20] and upwinded FEM with artificial viscosity stabilization [17]. None of these FEM's is inverse-monotone on highly anisotropic meshes. In contrast, we shall study an inverse-monotone finite volume method for (1.1) in this paper. Typically FVM's are interpreted as FEM's with inexact integration and therefore most frequently analysed in a finite element context with convergence established in the L_2 norm or in weighted H^1 norms [2, 4, 6, 21]. Here we shall pursue a similar approach, but we study convergence in a *discrete* mesh-dependent norm. This norm is stronger than the standard ε-weighted energy norm.

An outline of the paper is as follows. In Section 2 we define the upwind FVM, study its stability properties and quote some convergence results. The asymptotic behaviour of the solution of (1.1) is investigated in Section 3. We introduce special piecewise uniform layer-adapted meshes and state our main convergence result. The main ideas of the analysis from [21] are presented in Section 4 for the one-dimensional version of (1.1). Finally, we present results of numerical experiments in Section 5.

Notation: C denotes a generic positive constant that is independent of ε and of the mesh. Also, we set $g_i = g(x_i)$ for any function $g \in C[0,1]$, while u_i^h denotes the ith component of the numerical solution u^h. Similarly, we shall set $g_i = g(\boldsymbol{x}_i)$ and $g_{ij} = g(x_i, y_j)$ for $g \in C(\bar\Omega)$.

*Received May 14, 2001. Accepted for publication March 1, 2002. Recommended by R. S. Varga.

†Institut für Numerische Mathematik, Technische Universität Dresden, D-01062 Dresden, Germany. E-mail: <torsten,roos>@math.tu-dresden.de.

2. The upwind finite volume method and its stability. In this section, let $\Omega \subset \mathbb{R}^2$ be an arbitrary domain with polygonal boundary. We consider the problem

$$-\varepsilon\Delta u + \boldsymbol{a}\cdot \operatorname{grad} u + bu = f \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \Gamma = \partial\Omega, \tag{2.1}$$

with $0 < \varepsilon \ll 1$ and $b - \frac{1}{2}\operatorname{div}\boldsymbol{a} \geq \beta > 0$, but no restriction on the sign of $\boldsymbol{a}$.

2.1. The upwind finite volume method on arbitrary meshes. Let $\omega = \{\boldsymbol{x}_k\} \subset \bar{\Omega}$ be a set of mesh points. Let Λ and $\partial\Lambda$ be the sets of indices of interior and boundary mesh points, i.e., $\Lambda := \{k : \boldsymbol{x}_k \in \Omega\}$ and $\partial\Lambda := \{k : \boldsymbol{x}_k \in \partial\Omega\}$. Set $\bar{\Lambda} := \Lambda \cup \partial\Lambda$. We partition the domain Ω into subdomains

$$\Omega_k := \big\{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - \boldsymbol{x}_k\| < \|\boldsymbol{x} - \boldsymbol{x}_l\| \ \text{ for all } l \in \bar{\Lambda} \text{ with } k \neq l\big\} \quad \text{for } k \in \bar{\Lambda},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. We define $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ and we say that two mesh nodes $\boldsymbol{x}_k \neq \boldsymbol{x}_l$ are adjacent iff $m_{kl} := \operatorname{meas}_{1D}\Gamma_{kl} \neq 0$. By Λ_k we mean the set of indices of all mesh nodes that are adjacent to $\boldsymbol{x}_k$. Moreover, we define $d_{kl} := \|\boldsymbol{x}_k - \boldsymbol{x}_l\|$, $m_k = \operatorname{meas}_{2D}\Omega_k$, and we denote by n_{kl} the outward normal on the boundary part Γ_{kl} of Ω_k. Let h, the mesh size, be the maximal distance between two adjacent mesh nodes. For a reasonable discretization of the boundary conditions we shall assume that $\Gamma \subset \bigcup_{k\in\partial\Lambda}\Omega_k$.

To simplify the notation we set $N_{kl} = n_{kl}\cdot\boldsymbol{a}\big((\boldsymbol{x}_k + \boldsymbol{x}_l)/2\big)$. Then our discretization of (2.1) is

$$\big[L^h u^h\big]_k = f_k m_k \ \text{ for } \ k \in \Lambda, \ \ u^h_k = 0 \ \text{ for } \ k \in \partial\Lambda, \tag{2.2a}$$

where

$$\big[L^h v\big]_k := \sum_{l\in\Lambda_k} m_{kl}\left(\frac{\varepsilon}{d_{kl}} - N_{kl}\varrho_{kl}\right)(v_k - v_l) + b_k m_k v_k, \tag{2.2b}$$

$\varrho_{kl} = \varrho\,(N_{kl}d_{kl}/\varepsilon)$, and the function $\varrho : \mathbb{R} \to [0,1]$ is assumed to be monotone with

$$\lim_{t\to-\infty}\varrho(t) = 1, \quad \lim_{t\to\infty}\varrho(t) = 0, \tag{2.3a}$$

$$1 + (1-\varrho(t))t \geq 0 \quad \text{for all } t \in \mathbb{R}, \tag{2.3b}$$

$$\big[\varrho(t) + \varrho(-t) - 1\big]t = 0 \quad \text{for all } t \in \mathbb{R}, \tag{2.3c}$$

$$\big[1/2 - \varrho(t)\big]t \geq 0 \quad \text{for all } t \in \mathbb{R}, \tag{2.3d}$$

$$t \to t\varrho(t) \quad \text{is Lipschitz continuous.} \tag{2.3e}$$

For a detailed derivation of the method we refer the reader to [1, 2] or [15, III.3.1.2].

Possible choices for ϱ are

$$\varrho_I(t) = \frac{1}{t}\left(1 - \frac{t}{\exp t - 1}\right), \quad \varrho_S(t) = \begin{cases} 1/(2+t) & \text{for } \ t \geq 0, \\ (1-t)/(2-t) & \text{for } \ t < 0, \end{cases}$$

and

$$\varrho_{U,m}(t) = \begin{cases} 0 & \text{for } \ t > m, \\ \frac{1}{2} & \text{for } \ t \in [-m, m], \\ 1 & \text{for } \ t < -m, \end{cases} \quad \text{with } \ m \in [0,1].$$

The full upwind stabilization $\varrho_{U,0}$ is due to Baba and Tabata [3], while $\varrho_{U,m}$ with $m > 0$ was introduced by Angermann. For ϱ_I and ϱ_S we get the two-dimensional analogs of Il'in's [5] and of Samarski's scheme [16]. Further choices of ϱ are mentioned in [1].

2.2. Stability of the scheme. The construction of the scheme guarantees that the system matrix is an M-matrix if $b > 0$ on $\bar{\Omega}$. Then the discrete problem (2.2) has a unique solution for arbitrary right-hand sides f.

Alternatively, we can derive stability in a special mesh-dependent norm as we shall now show. The FVM can be written in variational form: Find $u^h \in V_0^h = \{v \in \mathbb{R}^{\operatorname{card}\bar{\Lambda}} : v_k = 0 \text{ for } k \in \partial\Lambda\}$ such that

$$a_h(u^h, v^h) = f_h(v^h) \text{ for all } v^h \in V_0^h,$$

where

$$a_h(v, w) := \sum_{k\in\bar{\Lambda}} \left[L^h v\right]_k w_k \quad \text{and} \quad f_h(w) := \sum_{k\in\bar{\Lambda}} f_k m_k w_k.$$

We define the norm $\|\cdot\|_{FV}$ associated with the bilinear form a_h:

$$\|v\|_{FV}^2 := \varepsilon|v|_{\omega,1}^2 + |v|_{\omega,\varrho}^2 + \|v\|_{\omega,0}^2,$$

where

$$|v|_{\omega,1}^2 := \frac{1}{2}\sum_{k\in\bar{\Lambda}}\sum_{l\in\Lambda_k} \frac{m_{kl}}{d_{kl}} \left(v_k - v_l\right)^2,$$

$$|v|_{\omega,\varrho}^2 := \frac{1}{2}\sum_{k\in\bar{\Lambda}}\sum_{l\in\Lambda_k} m_{kl}N_{kl}\left(\frac{1}{2} - \varrho_{kl}\right)\left(v_k - v_l\right)^2,$$

and

$$\|v\|_{\omega,0}^2 := \sum_{k\in\bar{\Lambda}}^{N-1} m_k v_k^2.$$

Note that because of (2.3d) this is a well-defined norm and it is stronger than the discrete ε-weighted energy norm

$$\|v\|_{\omega,\varepsilon}^2 := \varepsilon|v|_{\omega,1}^2 + \|v\|_{\omega,0}^2 \le \|v\|_{FV}^2.$$

THEOREM 2.1. *The bilinear form a_h is V_0^h elliptic for h sufficiently small. For any $\kappa \in (0, \beta)$ there exists an $h^* = h^*(\kappa)$ such that*

$$a_h(v, v) \ge \min(1, \kappa)\|v\|_{FV}^2 \text{ for all } v \in V_0^h \text{ and } h \le h^*.$$

Proof. This follows from [2, proof of Lemma 4]. There $|w|_{\omega,\varrho}^2 \ge 0$ is used to prove coercivity with respect to the ε-weighted energy norm, while we have incorporated this term into our mesh-dependent norm $\|\cdot\|_{FV}$. □

In [2] the following convergence results for quasi-uniform meshes in the ε-weighted energy norm are given:

$$\left\|u - u^h\right\| \le C\frac{h}{\sqrt{\varepsilon}}\left[\|u\|_{H^2} + \|f\|_{W_q^1}\right]$$

when $q > 2$, and the stronger bound

$$\left\|u - u^h\right\| \le Ch\left[\|u\|_{H^2} + \|f\|_{W_q^1}\right]$$

if the underlying triangulations have special symmetry properties. Note that neither of these results are uniform, because typically $\|u\|_{H^2} = \mathcal{O}\left(\varepsilon^{-3/2}\right)$.

3. The finite volume method on Shishkin meshes. In this section we shall study convergence of the FVM in the norm $\|\cdot\|_{FV}$ on Shishkin meshes which we shall introduce now. Shishkin meshes [12, 18] are piecewise equidistant meshes, constructed a priori, that partly resolve layers. To construct them correctly, it is crucial to have a precise knowledge of the asymptotic behaviour of the exact solution. Provided $\boldsymbol{a}$, b and f are sufficiently smooth and satisfy certain compatibility conditions, the solution u of (1.1) can be decomposed as $u = S + E_1 + E_2 + E_{12}$, where the regular part S satisfies

$$\left|\frac{\partial^{i+j} S}{\partial x^i y^j}(\boldsymbol{x})\right| \le C,$$

while for the layer terms E_1, E_2 and E_{12} we have

$$\left|\frac{\partial^{i+j} E_1}{\partial x^i y^j}(\boldsymbol{x})\right| \le C\varepsilon^{-i} \exp\bigl(-\alpha_1(1-x)/\varepsilon\bigr),$$
$$\left|\frac{\partial^{i+j} E_2}{\partial x^i y^j}(\boldsymbol{x})\right| \le C\varepsilon^{-j} \exp\bigl(-\alpha_2(1-y)/\varepsilon\bigr),$$

and

$$\left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(\boldsymbol{x})\right| \le C\varepsilon^{-(i+j)} \exp\bigl(-\bigl(\alpha_1(1-x) + \alpha_2(1-y)\bigr)/\varepsilon\bigr),$$

for $\boldsymbol{x} = (x, y) \in \Omega$ and $0 \le i + j \le 2$. Conditions that guarantee the existence of the decomposition are given in [10].

Our construction of the Shishkin mesh is based on this decomposition. Let N be an even positive integer. Let λ_x and λ_y denote two mesh transition parameters defined by

$$\lambda_x = \min\left(\frac{1}{2}, \frac{2\varepsilon}{\alpha_1}\ln N\right) \quad \text{and} \quad \lambda_y = \min\left(\frac{1}{2}, \frac{2\varepsilon}{\alpha_2}\ln N\right).$$

The mesh transition parameters have been chosen so that the boundary layer terms in the asymptotic expansion of u (the terms E_1, E_2 and E_{12} above) are of order N^{-2} on $[0, \lambda_x] \times [0, \lambda_y]$.

We specify the mesh points $\omega = \{(x_i, y_j) \in \Omega : i, j = 0, \dots, N\}$ by

$$x_i = \begin{cases} 2i(1-\lambda_x)/N & \text{for} \quad i = 0, \dots, N/2, \\ 1 - 2(N-i)\lambda_x/N & \text{for} \quad i = N/2+1, \dots, N, \end{cases}$$

with a similar definition for y_j.

THEOREM 3.1. *Let ω be a tensor-product Shishkin mesh. Suppose ϱ satisfies (2.3). Then there exists an $N_0 > 0$ that is independent of ε such that the error of the FVM satisfies*

$$\left\|u^h - u\right\|_{FV} \le CN^{-1}\ln^{3/2} N \quad \textit{for } N \ge N_0.$$

In Section 4 we shall give a proof of this theorem for a one-dimensional version of the FVM. We restrict ourselves to one dimension to keep the presentation as simple as possible. The technique presented there needs only minor modifications to analyse the two-dimensional scheme, although the number of merely technical details increases significantly. A complete analysis of the two-dimensional scheme is given in [21].

So far the numerical solution u^h is defined only at the mesh nodes. It can be extended to a function defined on the whole of Ω using linear or bilinear interpolation. Introducing the continuous energy norm $\|v\|_\varepsilon^2 := \int_\Omega \big(\varepsilon \operatorname{grad} v \cdot \operatorname{grad} v + v^2\big)$ for $v \in H_0^1(\Omega)$, we can use Theorem 3.1 and the interpolation error estimates in [14, 19] to derive

COROLLARY 3.2. *Let ω be a tensor-product Shishkin mesh. Let ϱ satisfy (2.3). Then there exists an $N_0 > 0$ that is independent of ε such that the error of the FVM satisfies*

$$\left\|u^h - u\right\|_\varepsilon \le CN^{-1}\ln^{3/2} N \quad \text{for } N \ge N_0.$$

REMARK 1. *In [8] the error of the FVM in the discrete maximum norm $\|\cdot\|_{\omega,\infty}$ on a Shishkin mesh was studied. The error of the scheme satisfies*

$$\left\|u - u^h\right\|_{\omega,\infty} \le CN^{-1}\ln N,$$

and if ϱ is Lipschitz continuous in $(-\delta,\delta)$ with some fixed $\delta > 0$ then the improved bound

$$\left\|u - u^h\right\|_{\omega,\infty} \le CN^{-1}$$

holds for N greater than some threshold value N_δ that depends on δ only.

Our numerical experiments in Section 5 indicate that in the norm $\|\cdot\|_{FV}$ the scheme also has better convergence properties when $\varrho(t)$ is Lipschitz continuous in a neighbourhood of $t = 0$.

REMARK 2. *On $\Omega_c := [0, 1-\lambda_x] \times [0, 1-\lambda_y]$, where the mesh is coarse, we have $h \gg \varepsilon$ and therefore $|v|_\varrho = \mathcal{O}\left(N^{-1/2}\right)|v|_{\omega,1}$. This implies the method gives uniformly convergent approximations of the gradient on Ω_c:*

$$\left|u - u^h\right|_{\omega_c,1} \le CN^{-1/2}\ln^{3/2} N,$$

where

$$|v|^2_{\omega_c,1} := \frac{1}{2}\sum_{\substack{k\in\bar{\Lambda}\\ \boldsymbol{x}_k\in\Omega_c}}\ \sum_{\substack{l\in\Lambda_k\\ \boldsymbol{x}_l\in\Omega_c}} \frac{m_{kl}}{d_{kl}}\left(v_k - v_l\right)^2.$$

4. Analysis of the finite volume method in one dimension. In this section we study the convergence of the finite volume method on a Shishkin mesh for the discretization of the two-point boundary value problem

$$-\varepsilon u'' + au' + bu = f \quad \text{for } x \in (0,1), \quad u(0) = u(1), \tag{4.1}$$

with $0 < \varepsilon \ll 1$, $a \ge \alpha > 0$, and $b - a'/2 \ge \beta > 0$.

The exact solution of (4.1) can be decomposed [12] as $u = S + E$ where, for any fixed order q that depends on the smoothness of the data, the regular part S and the layer term E satisfy

$$\left|S^{(k)}(x)\right| \le C \ \text{ and } \ \left|E^{(k)}(x)\right| \le C\exp\big(-\alpha(1-x)/\varepsilon\big) \ \text{ for } \ x \in [0,1] \text{ and } k = 0,\dots,q. \tag{4.2}$$

The weak formulation of (4.1) is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = f(v) \quad \text{for all } v \in H_0^1(0,1),$$

where

$$a(v,w) = \varepsilon\int_0^1 v'w' + \int_0^1 av'w + \int_0^1 bvw \ \text{ and } \ f(v) = \int_0^1 fv.$$

We shall consider a mesh with mesh points $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$. Let $h_i = x_i - x_{i-1}$ denote the local mesh sizes for $i = 1, \ldots, N$ and $\hbar_i = (h_i + h_{i+1})/2$ the averaged step sizes.

The variational form of the FVM in one dimension is: Find $u^h \in V_0^h = \big\{v \in \mathbb{R}^{N+1} : v_0 = v_N = 0\big\}$ such that

$$a_h(u^h, v^h) = f_h(v^h) \ \text{ for all } v^h \in V_0^h,$$

where

$$a_h(v, w) = \sum_{i=1}^{N-1} \big[L^h v\big]_i w_i, \ \ f_h(w) = \sum_{i=1}^{N-1} \hbar_i f_i v_i,$$

and

$$\big[L^h v\big]_i := -\varepsilon \left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i} \right) + \varrho\Big(\frac{a_{i+1/2} h_{i+1}}{\varepsilon}\Big) a_{i+1/2} \big(v_{i+1} - v_i\big) \\ + \varrho\Big(-\frac{a_{i-1/2} h_i}{\varepsilon}\Big) a_{i-1/2} \big(v_i - v_{i-1}\big) + \hbar_i b_i v_i,$$

where we have set $a_{i+1/2} = a\big((x_i + x_{i+1})/2\big)$.

The one-dimensional equivalent of the mesh-dependent norm is

$$\|v\|_{FV}^2 = \varepsilon |v|_{\omega,1}^2 + |v|_{\omega,\varrho}^2 + \|v\|_{\omega,0}^2 \ \text{ with } \ |v|_{\omega,1}^2 = \sum_{i=1}^{N} h_i^{-1} \big(v_i - v_{i-1}\big)^2,$$

$$|v|_{\omega,\varrho}^2 = \sum_{i=1}^{N} a_{i-1/2} \left[\frac{1}{2} - \varrho\Big(\frac{a_{i-1/2} h_i}{\varepsilon} \Big) \right] \big(v_i - v_{i-1}\big)^2 \ \text{ and } \ \|v\|_{\omega,0}^2 = \sum_{i=1}^{N-1} \hbar_i v_i^2.$$

We shall also use the discrete energy norm $\|v\|_{\omega,\varepsilon}^2 := \varepsilon |v|_{\omega,1}^2 + \|v\|_{\omega,0}^2$ and the continuous energy norm

$$\|v\|_{\varepsilon}^2 := \varepsilon \int_0^1 v'(x)^2 \, dx + \int_0^1 v(x)^2 \, dx.$$

We start our analysis from Theorem 2.1 and follow the standard approach of the Strang Lemma [15, III.3.1.2]. For any $v \in \mathbb{R}^{N+1}$ or $v \in C[0,1]$ let v^I denote the piecewise linear interpolant of v on the mesh given. Set $\eta = (u^h)^I - u^I$. Then

$$C\big\|\eta\big\|_{FV}^2 \le a_h(\eta, \eta) \le \big|a(u - u^I, \eta)\big| + \big|a(u^I, \eta) - a_h(u^I, \eta)\big| + \big|f_h(\eta) - f(\eta)\big| \tag{4.3}$$

for $h = \max h_i$ sufficiently small.

The terms on the right-hand side will be bounded separately.

PROPOSITION 4.1. *On a Shishkin mesh we have*

$$\big|a(u - u^I, \eta)\big| \le C N^{-1} \ln N \|\eta\|_{\omega,\varepsilon}.$$

Proof. From [19] we have

$$\big|a(u - u^I, \eta)\big| \le C N^{-1} \ln N \|\eta\|_{\varepsilon}.$$

To complete the proof, we use the fact that on V_0^h the continuous norm $\|\cdot\|_\varepsilon$ and the discrete norm $\|\cdot\|_{\omega,\varepsilon}$ are equivalent. □

PROPOSITION 4.2. *Let ω be an arbitrary mesh with maximal step size h. Then*

$$\bigl|f_h(\eta) - f(\eta)\bigr| \le Ch\|\eta\|_{\omega,0}.$$

Proof. Denoting by φ_i the usual basis functions for linear finite elements, we have

$$\Bigl|\int_{x_{i-1}}^{x_i} \bigl(f\varphi_i\bigr)(x)dx - \frac{h_i}{2}f_i\Bigr| = \Bigl|\int_{x_{i-1}}^{x_i}\Bigl\{f_i + \int_{x_i}^{x} f'(s)ds\Bigr\}\varphi_i(x)dx - \frac{h_i}{2}f_i\Bigr| \le \frac{h_i^2}{2}\bigl\|f'\bigr\|_\infty .$$

Thus

$$\begin{aligned}\bigl|f(\eta) - f_h(\eta)\bigr| &= \Bigl|\sum_{i=1}^{N-1}\eta_i\Bigl\{\int_{x_{i-1}}^{x_{i+1}}\bigl(f\varphi_i\bigr)(x)dx - \hbar_i f_i\Bigr\}\Bigr| \\ &\le \|f'\|_\infty h\sum_{i=1}^{N-1}\hbar_i|\eta_i| \le \|f'\|_\infty h\|\eta\|_{\omega,0}.\end{aligned}$$

□

Finally we bound $\bigl|a(u^I,\eta) - a_h(u^I,\eta)\bigr|$. We have

$$a(u^I,\eta) - a_h(u^I,\eta) = a_r(u^I,\eta) - a_{h,r}(u^I,\eta) + a_c(u^I,\eta) - a_{h,c}(u^I,\eta), \tag{4.4}$$

where

$$a_r(u^I,\eta) = \int_0^1 bu^I\eta, \quad a_{h,r}(u^I,\eta) = \sum_{i=1}^{N-1}\hbar_i b_i u_i\eta_i, \quad a_c(u^I,\eta) = \int_0^1 a(u^I)'\eta,$$

and

$$\begin{aligned}a_{h,c}(u^I,\eta) = \sum_{i=1}^{N-1}\Bigl\{&\varrho\Bigl(\frac{a_{i+1/2}h_{i+1}}{\varepsilon}\Bigr)a_{i+1/2}\bigl(u_{i+1} - u_i\bigr) \\ &+\varrho\Bigl(-\frac{a_{i-1/2}h_i}{\varepsilon}\Bigr)a_{i-1/2}\bigl(u_i - u_{i-1}\bigr)\Bigr\}\eta_i.\end{aligned}$$

PROPOSITION 4.3. *Let ω be a Shishkin mesh. Then*

$$\bigl|a_r(u^I,\eta) - a_{h,r}(u^I,\eta)\bigr| \le CN^{-1}\ln N\|\eta\|_\omega.$$

Proof. By the definition of $a_{h,r}$ and a_r, we have

$$\begin{aligned}a_{h,r}(u^I,\eta)_i - a_r(u^I,\eta) &= \sum_{i=1}^{N-1}\Bigl\{\int_{x_{i-1}}^{x_i}\bigl(bu^I\varphi_i\bigr)(x)dx - \frac{h_i}{2}b_iu_i\Bigr\}\eta_i \\ &\quad + \sum_{i=1}^{N-1}\Bigl\{\int_{x_i}^{x_{i+1}}\bigl(bu^I\varphi_i\bigr)(x)dx - \frac{h_{i+1}}{2}b_iu_i\Bigr\}\eta_i.\end{aligned} \tag{4.5}$$

A Taylor expansion with the integral form of the remainder gives

$$d_{r,i}^{-} := \int_{x_{i-1}}^{x_i} \big(bu^I\varphi_i\big)(x)dx - \frac{h_i}{2}b_iu_i = \int_{x_{i-1}}^{x_i}\int_{x_i}^{x}\Big(b'u^I + b\frac{u_i-u_{i-1}}{h_i}\Big)(s)\,ds\,\varphi_i(x)\,dx \tag{4.6}$$

Using the decomposition $u = S + E$, we see that

$$\big|u_i - u_{i-1}\big| \le CN^{-1}\ln N.$$

We apply this bound to (4.6) to get

$$\Big|\sum_{i=1}^{N-1} d_{r,i}^{-}\eta_i\Big| \le CN^{-1}\ln N\sum_{i=1}^{N/2} h_i|\eta_i| \le CN^{-1}\ln N\|\eta\|_{\omega}.$$

We obtain

$$\Big|\sum_{i=1}^{N-1} d_{r,i}^{-}\eta_i\Big| \le CN^{-1}\ln N\|\eta\|_{\omega},$$

with a similar bound for the second sum in (4.5). □

PROPOSITION 4.4. *Let ϱ satisfy (2.3). Suppose ω is a Shishkin mesh. Then*

$$\big|a_c(u^I,\eta) - a_{h,c}(u^I,\eta)\big| \le CN^{-1}\ln^{3/2}N\|\eta\|_{\omega,\varepsilon}.$$

Proof. We have

$$\begin{aligned}
&a_c(u^I,\eta) - a_{h,c}(u^I,\eta)\\
&\quad= \sum_{i=1}^{N}\Big\{\int_{x_{i-1}}^{x_i}\big(a(u^I)'\eta\big)(x)\,dx\\
&\qquad\qquad - \Big[\varrho\Big(\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\eta_{i-1} + \varrho\Big(-\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\eta_i\Big]a_{i-1/2}\big(u_i-u_{i-1}\big)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i}\big(a(u^I)'\eta\big)(x)\,dx\\
&\quad= a_{i-1/2}\big(u_i-u_{i-1}\big)\frac{\eta_i+\eta_{i-1}}{2} + \int_{x_{i-1}}^{x_i}\Big\{\int_{x_{i-1/2}}^{x} a'(s)\,ds\frac{u_i-u_{i-1}}{h_i}\eta(x)\Big\}dx.
\end{aligned}$$

We combine these two equations and use (2.3c). We get

$$\begin{aligned}
a_c(u^I,\eta) - a_{h,c}(u^I,\eta) &= \sum_{i=1}^{N}\Big[\frac{1}{2} - \varrho\Big(\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\Big]\big(\eta_{i-1}-\eta_i\big)\big(u_i-u_{i-1}\big)a_{i-1/2}\\
&\quad + \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\Big\{\int_{x_{i-1/2}}^{x} a'(s)\,ds\frac{u_i-u_{i-1}}{h_i}\eta(x)\Big\}dx.
\end{aligned} \tag{4.7}$$

The second sum can be bounded using the argument from the proof of Proposition 4.3. We get

$$\Big|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\Big\{\int_{x_{i-1/2}}^{x} a'(s)\,ds\frac{u_i-u_{i-1}}{h_i}\eta(x)\Big\}dx\Big| \le CN^{-1}\ln N\|\eta\|_{\omega}. \tag{4.8}$$

Next we bound the first sum in (4.7). For $i > N/2$ we have $\left|u_i - u_{i-1}\right| \le CN^{-1} \ln N$. Thus

$$\begin{aligned}(4.9)\quad &\Big|\sum_{i=N/2+1}^{N} \left[\frac{1}{2} - \varrho\Big(\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\right] \big(\eta_{i-1} - \eta_i\big)\big(u_i - u_{i-1}\big)a_{i-1/2}\Big| \\ &\qquad \le CN^{-1} \ln N \sum_{i=N/2+1}^{N} \left|\eta_i - \eta_{i-1}\right| \le CN^{-1} \ln^{3/2} N \varepsilon^{1/2} |\eta|_{\omega,1}.\end{aligned}$$

For $i \le N/2$ we use the splitting $u = S + E$ of the exact solution. We start with E. We have $E_i \le CN^{-2}$ for $i \le N/2$. Hence

$$\begin{aligned}(4.10)\quad &\Big|\sum_{i=1}^{N/2} \left[\frac{1}{2} - \varrho\Big(\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\right] \big(\eta_{i-1} - \eta_i\big)\big(E_i - E_{i-1}\big)a_{i-1/2}\Big| \\ &\qquad \le CN^{-2} \sum \big(|\eta_i| + |\eta_{i-1}|\big) \le CN^{-1} \|\eta\|_\omega.\end{aligned}$$

Finally, we consider the regular solution component S. To simplify the notation let

$$\gamma_{i-1/2} := a_{i-1/2} \left[\frac{1}{2} - \varrho\Big(\frac{a_{i-1/2}h_i}{\varepsilon}\Big)\right].$$

Using summation by parts we get

$$\begin{aligned}&\sum_{i=1}^{N/2} \gamma_{i-1/2}\big(S_i - S_{i-1}\big)\big(\eta_{i-1} - \eta_i\big) = \gamma_{N/2-1/2}\big(S_{N/2} - S_{N/2-1}\big)\eta_{N/2} \\ &\qquad - \sum_{i=1}^{N/2-1} \gamma_{i+1/2}\big(S_{i+1} - 2S_i + S_{i-1}\big)\eta_i + \sum_{i=1}^{N/2-1} \big(\gamma_{i-1/2} - \gamma_{i+1/2}\big)\big(S_i - S_{i-1}\big)\eta_i.\end{aligned}$$

Taylor expansions for S give $\left|S_{i+1} - 2S_i + S_{i-1}\right| \le CN^{-2}$ and $\left|S_i - S_{i-1}\right| \le CN^{-1}$, while (2.3e) implies $\left|\gamma_{i-1/2} - \gamma_{i+1/2}\right| \le CN^{-1}$. Thus

$$\begin{aligned}(4.11)\quad &\Big|\sum_{i=1}^{N/2} \gamma_{i-1/2}\big(S_i - S_{i-1}\big)\big(\eta_{i-1} - \eta_i\big)\Big| \\ &\qquad \le CN^{-1}\big(\|\eta\|_\omega + |\eta_{N/2}|\big) \le CN^{-1} \ln^{1/2} N \|\eta\|_{\omega,\varepsilon},\end{aligned}$$

because

$$|\eta_{N/2}| \le \sum_{i=N/2+1}^{N} \left|\eta_i - \eta_{i-1}\right| \le \ln^{1/2} N \varepsilon^{1/2} |\eta|_{\omega,1}.$$

Collecting (4.7)–(4.11), we complete the proof. □

We combine (4.3), (4.4) and propositions 4.1–4.4 to get our main convergence result.

THEOREM 4.5. *Let ϱ satisfy (2.3) and let ω be a Shishkin mesh. Then there exists an $N_0 > 0$ that is independent of ε such that*

$$\left\|u^h - u\right\|_{FV} \le CN^{-1} \ln^{3/2} N \quad \text{for } N \ge N_0.$$

5. Numerical results. We study the performance of the method when applied to the test problem

$$-\varepsilon\Delta u + (3-x)u_x + (4-y)u_y + u = f(x,y) \text{ in } \Omega = (0,1)^2, \quad u = 0 \text{ on } \Gamma = \partial\Omega,$$

where the right-hand side is chosen so that

$$u(x,y) = \sin x\Big(1 - \exp\big(-2(1-x)/\varepsilon\big)\Big)y^2\Big(1 - \exp\big(-3(1-y)/\varepsilon\big)\Big)$$

is the exact solution. This function exhibits typical boundary layer behaviour. For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem. Almost identical results are obtained for smaller values of ε.

Tables 5.1 and 5.2 display the results of our numerical experiments. They contain the errors of the FVM and the corresponding rates of convergence measured in both the mesh-dependent FV norm and the discrete ε-weighted energy norm for various choices of ϱ. The

	$\varrho_{U,0}$				$\varrho_{U,1}$			
	$\|u-u^h\|_{FV}$		$\|u-u^h\|_{\omega,\varepsilon}$		$\|u-u^h\|_{FV}$		$\|u-u^h\|_{\omega,\varepsilon}$	
N	error	rate	error	rate	error	rate	error	rate
16	2.0194e-1	0.69	1.5947e-1	0.57	9.3700e-2	0.92	5.6303e-2	0.89
32	1.2529e-1	0.74	1.0753e-1	0.66	4.9547e-2	0.95	3.0347e-2	0.93
64	7.5247e-2	0.78	6.8198e-2	0.73	2.5565e-2	0.97	1.5947e-2	0.95
128	4.3870e-2	0.81	4.1159e-2	0.78	1.3028e-2	0.98	8.2497e-3	0.97
256	2.4983e-2	0.84	2.3951e-2	0.82	6.5919e-3	0.99	4.2205e-3	0.98
512	1.3978e-2	0.86	1.3581e-2	0.84	3.3206e-3	0.99	2.1422e-3	0.99
1024	7.7178e-3	—	7.5616e-3	—	1.6680e-3	—	1.0814e-3	—

TABLE 5.1
The FVM on Shishkin meshes; $\varrho_{U,m}$.

numbers are clear illustrations of the theoretical results of Theorem 3.1. We also observe better (first-order) convergence when a $\varrho(t)$ is used that is Lipschitz continuous near $t = 0$, i. e., for ϱ_S, $\varrho_{U,1}$ and ϱ_I, while for $\varrho_{U,0}$ we observe convergence that is slightly slower than first order.

REFERENCES

[1] L. ANGERMANN, *Numerical solution of second-order elliptic equations on plane domains*, Math. Model. Numer. Anal., 25 (1991), pp. 169-191.

[2] ———, *Error estimates for the finite-element solution of an elliptic singularly perturbed problem*, IMA J. Numer. Anal., 15 (1995), pp. 161-196.

[3] K. BABA AND M. TABATA, *On a conservative upwind finite element scheme for convective diffusion equations*, RAIRO Anal. Numer., 15 (1981), pp. 3-25.

[4] J. BEY, *Finite-Volumen- und Mehrgitter-Verfahren für elliptische Randwertprobleme*, B. G. Teubner, Stuttgart, Leipzig, 1998.

[5] A. M. IL'IN, *A difference scheme for a differential equation with a small parameter affecting the highest derivative*, Mat. Zametki, 6 (1969), pp. 237-248, in Russian.

[6] P. KNABNER AND L. ANGERMANN, *Numerik partieller Differentialgleichungen. Eine anwendungsorientierte Einführung*, Springer, Berlin 2000.

[7] T. LINSS, *Uniform superconvergence of a Galerkin finite element method on Shishkin-type meshes*, Numer. Methods Partial Differential Equations, 16 (2000), pp. 426-440.

[8] ———, *Uniform pointwise convergence of an upwind finite volume method on layer-adapted meshes*, ZAMM Z. Angew. Math. Mech., 82 (2002), pp. 247-254.

[9] T. LINSS AND M. STYNES, *A hybrid difference scheme on a Shishkin mesh for linear convection-diffusion problems*, Numer. Appl. Math., 31 (1999), pp. 255-270.

	ϱ_S				ϱ_I			
	$\lVert u-u^h\rVert_{FV}$		$\lVert u-u^h\rVert_{\omega,\varepsilon}$		$\lVert u-u^h\rVert_{FV}$		$\lVert u-u^h\rVert_{\omega,\varepsilon}$	
N	error	rate	error	rate	error	rate	error	rate
16	1.1005e-1	0.96	7.6391e-2	0.94	9.8911e-2	0.94	6.2918e-2	0.92
32	5.6488e-2	1.00	3.9821e-2	1.01	5.1571e-2	0.97	3.3236e-2	0.96
64	2.8210e-2	1.02	1.9754e-2	1.04	2.6325e-2	0.99	1.7087e-2	0.98
128	1.3951e-2	1.02	9.6145e-3	1.04	1.3299e-2	0.99	8.6629e-3	0.99
256	6.8947e-3	1.01	4.6752e-3	1.03	6.6837e-3	1.00	4.3612e-3	1.00
512	3.4161e-3	1.01	2.2868e-3	1.02	3.3504e-3	1.00	2.1879e-3	1.00
1024	1.6973e-3	—	1.1260e-3	—	1.6773e-3	—	1.0958e-3	—

TABLE 5.2
The FVM on Shishkin meshes; ϱ_S, ϱ_I.

[10] ———, *Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem*, J. Math. Anal. Appl., 261 (2001), pp. 604-632.

[11] ———, *The SDFEM on Shishkin meshes for linear convection-diffusion problems*, Numer. Math., 87 (2001), pp. 457-484.

[12] J. J. H. MILLER, E. O'RIORDAN AND G. I. SHISHKIN, *Solution of Singularly Perturbed Problems with ε-uniform Numerical Methods*, World Scientific, Singapore, 1996.

[13] K. W. MORTON, *Numerical Solution of Convection-Diffusion Problems*, Applied Mathematics and Mathematical Computation, vol. 12, Chapman & Hall, London, 1996.

[14] H.-G. ROOS AND T. LINSS, *Sufficient conditions for uniform convergence on layer adapted grids*, Computing, 63 (1999), pp. 27-45.

[15] H.-G. ROOS, M. STYNES AND L. TOBISKA, *Numerical Methods for Singularly Perturbed Differential Equations*. Springer Series in Computational Mathematics, vol. 24, Springer, Berlin, 1996.

[16] A. A. SAMARSKI, *Monotone difference schemes for elliptic and parabolic equations in the case of a non-selfconjugate elliptic operator*, Zh. Vychisl. Mat. Mat. Fis., 5 (1965), pp. 548-551, in Russian.

[17] D. SCHNEIDER, H.-G. ROOS AND T. LINSS, *Uniform convergence of an upwind finite element method on layer adapted grids*, Comp. Meth. Appl. Mech. Engrg., 190 (2001), pp. 4519-4530.

[18] G. I. SHISHKIN, *A difference scheme for a singularly perturbed equation of parabolic type with discontinuous initial condition*, Sov. Math., Dokl., 37 (1988), pp. 792-796.

[19] M. STYNES AND E. O'RIORDAN, *A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem*, J. Math. Anal. Appl., 214 (1997), pp. 36-54.

[20] M. STYNES AND L. TOBISKA, *Analysis of the streamline-diffusion finite element method on a Shishkin mesh for a convection-diffusion problem with exponential layers*, East-West J. Numer. Math., 9 (2001), pp. 59-76.

[21] D. WOLLSTEIN, *Untersuchung einer FVM-Diskretisierung von Konvektions-Diffusions-Gleichungen auf grenzschichtangepaßten Gittern*, Diplomarbeit, TU Dresden, 2000.