**Electronic Transactions on Numerical Analysis.**
**Volume 13, pp.12-21, 2002.**

**ISSN 1068-9613.**

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**

# THE ASYMPTOTIC DISTRIBUTION OF GENERAL INTERPOLATION ARRAYS FOR EXPONENTIAL WEIGHTS *

S. B. DAMELIN †

**Abstract.** We study the asymptotic distribution of general interpolation arrays for a large class of even exponential weights on the line and $(-1,1)$. Our proofs rely on deep properties of logarithmic potentials. We conclude with some open problems.

**Key words.** asymptotic distribution, Freud weight, Erdős weight, exponential weight, interpolation, Lebesgue constant, logarithmic potential, Pollaczek weight, sup norm, weighted approximation.

**AMS subject classifications.** 42C15, 42C05, 65D05.

## 1. Introduction.

This article grew out of a recent interesting paper of Szabados [14]. Recently, there has been quite an intense interest in developing the theory of weighted Lebesgue constants on the real line and on $(-1,1)$ for specific and general arrays. The above has been applied, in particular, to the theory of weighted Lagrange and other higher order Hermite-Fejér processes. We refer the reader to [1], [2], [4], [8], [19], [20], [21], [23], [24], and the many references therein for a comprehensive survey of this subject. Our interest in this paper is to study the asymptotic distribution of general interpolation arrays for a large class of even exponential weights, $w$, on the real line and $(-1,1)$. Our main observation will be that provided the Lebesgue constant for the array does not grow geometrically fast for large $n$, points of interpolation cannot distribute themselves asymptotically too far from the scaled endpoints of the equilibrium measure $\mu_w$ for the weight $w$. Moreover, the discrepancy of this distribution may be calculated precisely and simultaneously by the rate of decay of $w$ near $\pm\infty$ or near $\pm 1$. We refer the reader to Remark 1.2 below for a further discussion of this idea and to recent work of [5], [6], [10] and the references cited therein.

One of our main results, Theorem 1.4 below, will cover Freud type weights such as

$$w_\alpha(x) := \exp\left(-|x|^\alpha\right),\ \alpha > 1,\ x \in \mathbb{R}, \tag{1.1}$$

Erdős type weights such as

$$w_{k,\beta}(x) := \exp\left(-\exp_k\left(|x|^\beta\right)\right),\ \beta > 0,\ k \geq 1,\ x \in \mathbb{R}, \tag{1.2}$$

and Pollaczek weights of the form

$$w_{0,\gamma}(x) := \exp\left(-(1-x^2)^{-\gamma}\right),\ \gamma > 0, \tag{1.3}$$

and

$$w_{k,\gamma}(x) := \exp\left(-\exp_k(1-x^2)^{-\gamma}\right),\ \gamma > 0,\ k \geq 1,\ x \in (-1,1). \tag{1.4}$$

Here and throughout, $\exp_k$ denotes the $k$-th iterated exponential. Freud weights are characterised by their smooth polynomial decay at infinity and Erdős weights by their faster than smooth polynomial decay at infinity. Generalised Pollaczek weights decay strongly near $\pm 1$ as exponentials and are of faster decay than classical Jacobi weights. They violate the well known Szegő condition for orthogonal polynomials [9, Chapter 5, p. 208].

*Received October 10, 2001. Accepted for publication March 12, 2002. Recommended by V. V. Andrievskii.
†Department of Mathematics and Computer Science, Georgia Southern University, Post Office Box 8093, Statesboro, GA 30460, U.S.A. E-mail: damelin@gsu.cs.gasou.edu

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etna@mcs.kent.edu



In the opposite direction, for a given exponential weight $w$ and a specific or general array of interpolation points, one may ask for upper and lower bounds for the corresponding Lebesgue constant. These questions are dealt with in [1], [2], [19], [23], and [24].

To set the scene for our investigations, let $I$ be a real interval of positive length and let

$$w : I \longrightarrow (0, \infty)$$

be a continuous weight. If $I$ is unbounded, assume further that

$$\lim_{|x|\to\infty} |x|w(x) = 0, \; x \in I.$$

We set

$$Q := -\log w,$$

and call $w$ *admissible* and $Q$ the *external field* associated with $w$.

Now let

$$\chi_n := \{x_{1,n} < x_{2,n} < ... < x_{n+1,n}, \; n \geq 1\}$$

be a triangular array of $n+1$ points in $I$ and for each $n \geq 1$, we define the *Lebesgue constant* associated with an admissible varying weight $w^n$ and a triangular array $\chi_n$ by

$$\Lambda(w^n, \chi_n) = \Lambda_n := \left\| w^n \sum_{k=1}^{n+1} \frac{|l_{k,n}|}{w^n(x_{k,n})} \right\|_I .$$

Here $\|.\|$ denotes the sup norm and

$$l_{k,n}(x) := \prod_{\substack{i=1\\ i\neq k}}^{n+1} \frac{x - x_{i,n}}{x_{k,n} - x_{i,n}}, \; k = 1, ..., n+1, \; x \in I$$

are the fundamental polynomials in $\Pi_n$, the class of algebraic polynomials of degree at most $n$, satisfying

$$l_{k,n}(x_{j,n}) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

The number $\Lambda_n$ arises in a natural way in the theory of weighted interpolation, see [1], [2], [19], [20], [21], [23], [24], and the references cited therein.

The *equilibrium measure* (see [18] and [22]) in the presence of an admissible external field

$$Q : I \longrightarrow \mathbb{R}$$

is the unique Borel probability measure $\mu_w$ with compact support on $I$ satisfying for a unique constant $F_w$,

$$M_w(x) := U^{\mu_w}(x) + Q(x) - F_w = 0, \; x \in \text{supp}(\mu_w), \tag{1.5}$$

and

$$M_w(x) \geq 0, \; x \in I. \tag{1.6}$$

Here, $U^{\mu_w}$ denotes the logarithmic potential of $\mu_w$, i.e.,

$$U^{\mu_w}(x) := \int_I \log \frac{1}{|x-t|} d\mu_w(t),\ x \in \mathbb{C}.$$

Following is our first result:

THEOREM 1.1. *Let $w$ be an admissible weight and for each $n \geq 1$, let $\chi_n$ be a triangular array of $n+1$ points in $I$ and $\Lambda_n$ the associated Lebesgue constant for the varying weight $w^n$ and the array $\chi_n$. Then uniformly for $i = 1, ..., n+1$,*

$$M_w(x_{i,n}) \leq \frac{\log \Lambda_n}{n}. \tag{1.7}$$

*In particular, if*

$$\limsup_{n\to\infty} \Lambda_n^{1/n} \leq 1, \tag{1.8}$$

*then*

$$\lim_{n\to\infty} M_w(x_{i,n}) = 0. \tag{1.9}$$

REMARK 1.2:

(a) Using (1.5) and the non-negativity of the Lebesgue constant, it is immediate that (1.7) holds for interpolation points in $\mathrm{supp}(\mu_w)$. Thus the essence of formula (1.7) is that it gives us information on the asymptotic location of interpolation points outside $\mathrm{supp}(\mu_w)$. Such information is useful in many aspects of weighted polynomial approximation; see [1], [2], [15], [16], [17], [19], [20], [21], [22], [23], [24], and the references cited therein.

(b) Given an admissible weight $w$ and following an idea of [10, p. 2], we shall say that the pair $(I, w)$ has an *asymptotic interpolation measure* if there exists a compactly supported Borel measure $\mu$ on $I$ such that (1.8) implies

$$\nu(\chi_n) := \frac{1}{n+1}\sum_{k=1}^{n+1} \delta_{x_{k,n+1}} \to \mu,\ n \to \infty \tag{1.10}$$

weak star. The main purpose of this paper is to show that for a class of strongly admissible weights, see (1.1)-(1.4) and Definition 1.3 below, it is possible to estimate the speed of convergence in (1.9), and hence describe the discrepancy in (1.10), for $\mu = \mu_w$, precisely and simultaneously by the rate of decay of $w$ near $\pm\infty$ or near $\pm 1$.

To state our main result, we require some additional notation. To this end, let us agree that henceforth $C$ will denote a positive constant depending on $w$ which may take on different values at different times, $I+$ will denote either $(0, \infty)$ if $I$ is $\mathbb{R}$ and $(0, 1)$ if $I$ is $(-1, 1)$. Moreover, for any two sequences $b_n$ and $c_n$ of non zero real numbers, we shall write $b_n = O(c_n)$ if there exists a positive constant $C$, independent of $n$, such that

$$b_n \leq Cc_n,\ n \to \infty,$$

and $b_n \sim c_n$ if

$$b_n = O(c_n) \text{ and } c_n = O(b_n).$$

Similar notation will be used for functions and sequences of functions.

Our class of admissible weights $w$ will then be assumed to be *strongly admissible* in the sense of the following definition which is taken from a combination of [13, Theorem 1.1], [12, Definition 1.1], and [14, Definition 1.1].

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**1.1. Class of strongly admissible weights.** We start with

DEFINITION 1.3. *Let $w$ be admissible and even.*

(a) *Assume that $Q''$ is continuous in $I+$ and $Q''$, $Q' \geq 0$ in $I+$.*

(b) *The function*

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)},\ x \in I+$$

*satisfies for large enough $x$ or $x$ close enough to $\pm 1$*

$$T(x) \sim \frac{xQ'(x)}{Q(x)}.$$

*Moreover $T$ satisfies either*:

(b1) *There exist $A > 1$ and $B > 1$ such that*

$$A \leq T(x) \leq B,\ x \in I+\,.$$

(b2) *$T$ is increasing in $I+$ with $\lim_{x\to 0+} T(x) > 1$. If $I = \mathbb{R}$,*

$$\lim_{|x|\to\infty} T(x) = \infty,$$

*and if $I = (-1, 1)$, for $x$ close enough to $\pm 1$,*

$$T(x) \geq \frac{A}{1-x^2},$$

*for some $A > 2$.*

*Then $w$ shall be called a strongly admissible weight.*

Canonical examples are the weights listed in $(1.1) - (1.4)$.

We shall prove:

THEOREM 1.4. *Let $w$ be a strongly admissible weight and for each $n \geq 1$, let $\chi_n$ be a triangular array of $n+1$ points in $I$ and $\Lambda_n$ the associated Lebesgue constant for the weight $w$ and the array $\chi_n$. Let $\pm a_n$, $n \geq 1$ denote the endpoints of $\mathrm{supp}(\mu_{w^{1/n}})$ given by*

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt.$$

*Suppose in addition that*

$$\limsup_{n\to\infty} \frac{\log \Lambda_n}{n(T(a_n))^{1/2}} < 1. \tag{1.11}$$

*Then there exists $N_0$, such that for $n \geq N_0$,*

$$\max\{|x_{j,n}| : 1 \leq j \leq n+1\} \leq C a_n \left(1 + \frac{\log \Lambda_n}{nT(a_n)}\right)^{2/3}. \tag{1.12}$$

We now show how Theorem 1.4 may be applied with the weights given by (1.1)-(1.4). In doing so, we will describe for these weights, the growth of the sequences $a_n$ and $T(a_n)$. For general results in this direction, we refer the reader to [11], [12], [13], and [14]. We remark that it is possible, using recent results in [11], to weaken our admissiblity assumptions to allow for weights where $Q^{(2)}$ need not exist and rather $Q'$ satisfies a local Lipchitz $1/2$ condition.

COROLLARY 1.5a: *Freud weights. Let $w_\alpha$ be given by (1.1) and for each $n \geq 1$, let $\chi_n$ be a triangular array of points in $\mathbb{R}$ and $\Lambda_n$ the associated Lebesgue constant. Then there exists $N_0$ such that for $n \geq N_0$,*

$$\max\{|x_{j,n}| : 1 \leq j \leq n\} \leq Cn^{1/\alpha}\left(1 + \frac{\log \Lambda_n}{n}\right)^{2/3}. \tag{1.13}$$

COROLLARY 1.5b: *Erdős weights. Let $w_{k,\beta}$ be given by (1.2) and for each $n \geq 1$, let $\chi_n$ be a triangular array of points in $\mathbb{R}$ and $\Lambda_n$ the associated Lebesgue constant. Then there exists $N_0$ such that for $n \geq N_0$,*

$$\begin{aligned} &\max\{|x_{j,n}| : 1 \leq j \leq n\} \\ &\leq C(\log_k n)^{1/\beta}\left(1 + \frac{\log \Lambda_n}{n\prod_{l=1}^{k}\log_l n}\right)^{2/3}. \end{aligned} \tag{1.14}$$

COROLLARY 1.5c: *Pollaczek weights. Let $w_{k,\gamma}$ be given by (1.3) and (1.4) and for each $n \geq 1$, let $\chi_n$ be a triangular array of points in $(-1, 1)$ and $\Lambda_n$ the associated Lebesgue constant. Then there exists $N_0$ such that for $n \geq N_0$,*

$$\begin{aligned} &\max\{|x_{j,n}| : 1 \leq j \leq n\} \\ &\leq C\left(1 - n^{\frac{-1}{1/2+\gamma}}\right)\left(1 + \frac{\log \Lambda_n}{n^{\frac{2\gamma+3}{2\gamma+1}}}\right)^{2/3}, \ k = 0, \end{aligned} \tag{1.15}$$

$$\begin{aligned} &\max\{|x_{j,n}| : 1 \leq j \leq n\} \\ &\leq C\left(1 - (\log_k n)^{\frac{-1}{\gamma}}\right)\left(1 + \frac{\log \Lambda_n}{n(\log_k n)^{1+1/\gamma}\prod_{l=1}^{k}\log_l n}\right)^{2/3}, \ k \geq 1. \end{aligned} \tag{1.16}$$

REMARK 1.6:

(a) Given a strongly admissible weight $w$, Theorem 1.4 gives information on the asymptotic location of points $|x_{j,n}|$, $1 \leq j \leq n$ in general interpolation arrays, uniformly for any $j$, assuming (1.11). Notice that (1.11) is stronger than (1.8), which is expected if we want discrepancy estimates and not just (1.10). In particular, (1.11) and (1.12) show that there exists $N_0$ such that for $n \geq N_0$, and uniformly for $1 \leq j \leq n$,

$$|x_{j,n}| \leq a_n\left(1 + T(a_n)^{-1/3}\right). \tag{1.17}$$

Thus, at least in the case when $\mathrm{supp}(\mu_{w^{1/n}})$ consists of one interval and $w$ is strongly admissible, interpolation points whose Lebesgue constants satisfy (1.11) cannot accumulate too far from the endpoints of the support. Moreover, if we know more about the Lebesgue constant in advance, then we are able to improve (1.17) considerably. Indeed, it is the factor $T(a_n)$ in the right hand side of (1.12) that allows the distribution of interpolation points to be described precisely and simultaneously by the rate of decay of $w$ near $\pm\infty$ or near $\pm 1$.

(b) For general arrays, Vértesi in [23] and [24] has shown that $\log \Lambda_n$ admits a lower bound of $\log \log n$ always. On the other hand, in [19], [1], and [2], Szabados and Damelin have shown that for some specific arrays, this lower bound is achieved and for others, we obtain an upper bound of $\log n$ for $\log \Lambda_n$. Albeit in all cases (1.11) is satisfied although the choice of the points in these latter papers admit better estimates than (1.12) because of their special properties. We refer the reader to those papers for a deeper perspective.

(c) Theorem 1.4 for $w$ given by (1.1) appears as [20, Proposition 1] without the important assumption (1.11). In light of (1.8) and [10, Lemma 5.1], the author believes that a condition such as (1.11) is necessary even in this special case.

The remainder of this paper is devoted to the proofs of Theorem 1.1, Theorem 1.4 and Corollaries 1.5(a-c).

**2. Proofs.** *The Proof of Theorem 1.1*: For the proof of Theorem 1.1, we rely on an important idea which first appeared in [19, Lemma 1]. Let us set

$$P_{k,n} = \frac{l_{k,n}}{w^n(x_{k,n})},\ 1 \leq k \leq n+1.$$

Notice that $P_{k,n} \in \Pi_n$ for every $k$. Then we recall, see [18, Theorem 3.5.1 and Corollary 3.5.3], that given any $x \in I$,

$$|P_{k,n}w^n(x)| \leq \exp(-nM_w(x))\,\|P_{k,n}w^n\|_{S_w}\,. \tag{2.1}$$

For notational simplicity, let us write $P_k := P_{n,k}$. The first step in the proof is to choose $Q_n \in \Pi_n$ so that

$$\|Q_nw^n\|_\Sigma = ||Q_nw^n||_{S_w} = \left\|\sum_{k=1}^{n}|P_k|w^n\right\|_\Sigma\,. \tag{2.2}$$

This is done as follows: First pick $x_0 \in I$ for which

$$\left\|\sum_{k=1}^{n}|P_k|w^n\right\|_I = \sum_{k=1}^{n}|P_k|(x_0)w^n(x_0). \tag{2.3}$$

Then set for any $y \in I$,

$$Q_n(y) := \sum_{k=1}^{n}P_k(y)\mathrm{sgn}(P_k(x_0)). \tag{2.4}$$

Notice that using (2.1), (2.3), and (2.4) we have

$$|Q_nw^n(y)| \leq \left\|\sum_{k=1}^{n}|P_k|w^n\right\|_I =$$
$$\sum_{k=1}^{n}|P_k|(x_0)w^n(x_0) = |Q_nw^n(x_0)| \leq ||Q_nw^n||_I = ||Q_nw^n||_{S_w}.$$

Thus it follows that (2.2) indeed holds for the polynomial $Q_n$. Now let us apply (2.2) above with $y = x_{j,n}$. Then (2.1) and the definition of the Lebesgue constant easily yields

$$\begin{aligned}1 &= |Q_nw^n(x_{j,n})| \leq \exp(-nM_w(x_{j,n}))\|Q_nw^n\|_I \\ &= \exp(-nM_w(x_{j,n}))\Lambda_n.\end{aligned}$$

Rearranging gives (1.7). □

In order to apply Theorem 1.1 for a given strongly admissible weight $w$, we scale the weight and obtain a sequence of weights $w(a_n;)$, $n \geq 1$. We then set $w_n := w(a_n;;)^{1/n}$, $n \geq 1$. Using a combination of (2.1), [12, Lemma 5.1] and [16, Theorem 6.1.6], it follows that (1.5), (1.6) and (2.1) become:

LEMMA 2.1. *Let $w$ be strongly admissible. Define for $n \geq 1$:*

$$\mu_{w,n}(x) := \frac{2}{\pi^2}\int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1-t^2}}\frac{a_n tQ'(a_n t) - a_n xQ'(a_n x)}{n(t^2-x^2)}dt,\ x \in [-1,1].$$

$$U_{w,n}^{\mu_{w,n}}(x) := \int_{-1}^1 \log\frac{1}{|x-t|}\mu_{w,n}(t)dt,\ x \in \mathbb{C}. \tag{2.5}$$

$$F_{w,n} := \frac{\log 1/2}{n} - \frac{2}{n\pi}\int_0^1 \frac{Q(a_n t)}{\sqrt{1-t^2}}. \tag{2.6}$$

*Then*

$$\mu_{w,n}(x) > 0,\ x \in (-1,1),\ \int_{-1}^1 \mu_{w,n}(t)dt = 1, \tag{2.7}$$

$$M_{w,n}(x) = U_{w,n}^{\mu_{w,n}}(x) + \frac{Q(a_n x)}{n} - F_{w,n} = 0,\ x \in [-1,1], \tag{2.8}$$

*and*

$$M_{w,n}(x) > 0,\ |x| \in J_n, \tag{2.9}$$

*where*

$$J_n := \begin{cases} (1, 1/a_n), & \text{if } I = [-1,1], \\ (1,\infty), & \text{if } I = \mathbb{R}. \end{cases}$$

LEMMA 2.2. *Let $w$ be strongly admissible. Then for every polynomial $P_n \in \Pi_n$, $n \geq 1$,*

$$|P_n w|(x) \leq \exp\left(-nM_{w,n}(x/a_n)\right)\|P_n w\|_{[-a_n,a_n]},\ |x| \in K_n, \tag{2.10}$$

*where*

$$K_n := \begin{cases} (a_n, 1), & \text{if } I = [-1,1], \\ (a_n,\infty), & \text{if } I = \mathbb{R}. \end{cases}$$

Using Lemmas 2.1 and 2.2, we now prove the following sup norm inequality which is of independent interest.

LEMMA 2.3. *Let $w$ be a strongly admissible weight. Then there exists $\alpha > 1$ depending only on $w$ such that for every polynomial $P_n \in \Pi_n$, $n > C$,*

(2.11)

$$|Pw|(x) \leq \begin{cases} \exp\left(-nC(|x|/a_n - 1)^{3/2}T(a_n)\right)\|Pw\|_{[-a_n,a_n]}, & a_n < |x| \leq a_{\alpha n}, \\ \exp\left(-C\frac{n}{T(a_n)^{1/2}}\right)\|Pw\|_{[-a_n,a_n]}, & |x| > a_{\alpha n}. \end{cases}$$

For the range $a_n < |x| \leq a_{\alpha n}$, Lemma 2.3 includes estimates for the scaled difference

$$|U_{w,n}^{\mu_{w,n}}(x) + \frac{Q(a_n x)}{n} - F_{w,n}|$$

for $|x|/a_n$ close to 1 or equivalently for $|x|$ close to the endpoints of the scaled support. Indeed, we have (cf. the method of [13, Lemma 5.2c]) that given any $\lambda > 1$, we have uniformly for $u \in [v/\lambda, \lambda v]$, $v \in I^+$,

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{T(a_u)}. \tag{2.12}$$

*Proof*. Suppose first that $w$ is Freud weight. Then in this case $T \sim 1$. By [13, Lemma 7.1], there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,

$$M_{w,n}(1+\varepsilon) \sim \varepsilon^{3/2}. \tag{2.13}$$

Choose $D > 0$ in the right hand inequality of (2.12) which recall is independent of $u$ and $v$ there and fix it. Now set $\alpha := \varepsilon_0/D+1 > 1$. Then applying (2.12) gives for $a_n < |x| \le a_{\alpha_n}$,

$$|x|/a_n - 1 \le \varepsilon_0.$$

Setting $\varepsilon = |x|/a_n - 1$ in (2.13) which we may and applying (2.10), gives the Lemma in this case. Suppose next that $w$ is a Generalised Pollaczek weight. By [12, Theorem 5.3], there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0/T(a_n)$,

$$M_{w,n}(1+\varepsilon) \sim \varepsilon^{3/2}T(a_n) + \varepsilon^2 T(a_n)^{3/2}. \tag{2.14}$$

Choose $D_1 > 0$ in the right hand inequality of (2.12) as before and set $\alpha := \varepsilon_0/D_1 + 1 > 1$. Then applying (2.12) gives for $a_n < |x| \le a_{\alpha_n}$,

$$|x|/a_n - 1 \le \varepsilon_0/T(a_n).$$

Setting $\varepsilon = |x|/a_n - 1$ in (2.14) which we may, recalling that $a_{\alpha n} < 1$ and applying (2.10) gives the Lemma in this case as well. Finally we consider the case when $w$ is a Erdős weight. This follows almost exactly as the previous case using the results of [14]. For $|x| > a_{\alpha_n}$, Lemma 2.3 follows from [12, The proof of Theorem 1.7], [13, The proof of Theorem 1.8], and [14, The proof of Theorem 1.7]. Thus the lemma is proved. □

We are now ready to provide the remaining details in:

*The Proof of Theorem 1.4*. Suppose first that $|x_{j,n}| \le a_n$. Then clearly

$$|x_{j,n}| \le a_n \left(1 + \frac{\log \Lambda_n}{nT(a_n)}\right)^{2/3}. \tag{2.15}$$

Next let $\alpha > 1$ be as in Lemma 2.3 and suppose that $a_n < |x_{j,n}| \le a_{\alpha n}$. Observe first that (2.12) and the results of [23] and [24] easily imply the crude estimate

$$a_n \le |x_{j,n}| \le a_n \left(1 + \frac{C \log \Lambda_n}{T(a_n) \log\log n}\right).$$

To improve this, let us now apply Lemma 2.3 with $Q_n$ as defined in the proof of Theorem 1.2. This then gives

$$1 \le \exp\left(-n(|x_{j,n}|/a_n - 1)^{3/2}\right) \Lambda_n,$$

and so rearranging we obtain (2.15), which is more natural and in many cases better. Finally suppose that $|x_{j,n}| > a_{\alpha n}$. Then applying Lemma 2.3 and the argument of the previous case, we obtain

$$1 \le \exp\left(-n/T(a_n)^{1/2}\right) \Lambda_n,$$

which contradicts (1.11). So again (2.15) holds. (1.12) then follows. □

*The Proof of Corollaries 1.5(a-c):* These follow using [19, Theorem 1], [1, Theorems 1.2 and 1.4], and [2, Theorems 2.1 and 2.4]. Observe that in each of the eight cases proved, (1.11) holds. □

**3. Conclusions.** We close with some conclusions and possible extensions for future research.

Firstly, as mentioned earlier, Vértesi in [23] and [24] has shown that given a strongly admissible weight and any triangular array

$$\log n = O(\Lambda_n),\ n \to \infty.$$

Corresponding upper bounds for such general triangular schemes is still an open and interesting problem. One immediate application would be to Theorem 1.4. Suppose next that $\mathrm{supp}(\mu_{w^{1/n}})$ consists of more than one interval, such as for example if $w$ is analytic on a finite interval but not necessarily convex or if $\mathrm{supp}(\mu_{w^{1/n}})$ consists of one interval but with non symmetric endpoints, such as for example if $w$ is non-even and convex, see [11] and [6], then hardly anything is known for both lower and upper bounds of $\Lambda_n$ even for specific arrays such as in Corollaries 1.5(a-c). Natural analogues of Theorem 1.4 in these settings would also be of great interest and Theorem 1.1, we believe, is a natural starting point for such investigations.

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**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



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