Electronic Transactions on Numerical Analysis.
Volume 14, pp. 142-151, 2002.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

POLYNOMIAL INEQUALITIES, FUNCTIONAL SPACES AND BEST APPROXIMATION ON THE REAL SEMIAXIS WITH LAGUERRE WEIGHTS.*

G. MASTROIANNI †

Abstract. This is a short survey on polynomial approximation with Laguerre weights. Some new polynomial inequalities are presented.

Key words. Orthogonal polynomials, polynomial inequalities, Fourier series, Lagrange interpolation.

AMS subject classifications. 41A10, 26D05.

1. Introduction. The paper is devoted to polynomial approximation, with Laguerre weights, of functions defined on the real semiaxis. Within the literature on polynomial approximation, until recently the case of the real semiaxis was not as complete as, for example, the finite interval case (see, for instance, [9], [11], [2], [19]). This gap was essentially closed in [3]. In that paper, the authors introduced some new functional spaces, suitable K-functionals, and related moduli of smoothness, and they proved the Jackson theorem as well as the associated Stechkin inequality. For this reason, it seems useful to summarize, by means of this short survey, the results in this field which are presently known in the literature. Moreover, some polynomial inequalities are proved.

2. Basic facts on Orthogonal Polynomials and Polynomial Inequalities.

2.1. Orthogonal Polynomials. Let $w_\alpha(x) = x^\alpha e^{-x}$ be the Laguerre weight. Denote by $p_m(w_\alpha)$ the associated system of orthonormal polynomials with leading coefficients $\gamma_m(w_\alpha) > 0$ and by $x_i = x_{m,i}(w_\alpha)$, $i = 1, 2, \ldots, m$, the zeros of $p_m(w_\alpha)$, which are in increasing order. The following bounds hold,

$$\frac{C}{m} < x_1 < \ldots < x_m < 4m - C(4m)^{\frac{1}{3}}, \tag{2.1}$$

and

$$x_{m,k}(w_\alpha) = C_{m,k}\frac{(1+k)^2}{m}, \tag{2.2}$$

where C is a positive constant independent of m and k, and $\left(\frac{3\pi}{16}\right)^2 < C_{m,k} < 4$ holds uniformly with respect to m and k.

The interested reader can find (2.1) and (2.2), with more precise informations about the constants, in ([22]. p.129).

Now set

$$\varphi_m(x) = \sqrt{\frac{x}{|4m - x| + (4m)^{\frac{1}{3}}}}, \quad x \geq 0.$$

The function φ_m is connected with the distance between two consecutive zeros by means of the following equivalences (see for instance [7], [17], [10]),

$$\Delta x_k = x_k - x_{k-1} \sim \varphi_m(x_k) \sim \sqrt{\frac{x_k}{4m - x_k}}, \qquad k = 1, \ldots, m, \quad x_0 = 0, \tag{2.3}$$

*Received October 31, 2000. Accepted for publication February 2001. Communicated by Sven Ehrich.

†Dipartimento di Matematica, Università della Basilicata, C/da Macchia Romana, 85100 Potenza, Italy. This work is supported by a contribution GNIM–INDAM.

where here and in the sequel, if a and b are positive functions of certain parameters, $a \sim b$ means that $\left(\frac{a}{b}\right)^{\pm 1} \le C$ with C being a positive constant independent of the parameters of a and b.

The Christoffel numbers $\lambda_k(w_\alpha) = \lambda_{m,k}(w_\alpha)$ are defined by $\lambda_{m,k}(w_\alpha) = \lambda_m(w_\alpha, x_k)$, $k = 1, 2, \ldots, m$, where $\lambda_m(w_\alpha, x) = [\sum_{k=0}^{m-1} p_k^2(w_\alpha, x)]^{-1}$ are the Christoffel functions. Sharp estimates of $\lambda_m(w_\alpha, x)$ are useful in different contexts. Here we recall the following equivalence, which holds for $\alpha > -1$ and $x \in [0, 4m + A(4m)^{1/3}]$ ($A > 0$ fixed),

$$\frac{1}{C}\varphi_m(x) \le \frac{\lambda_m(w_\alpha, x)}{\left(x + \frac{1}{m}\right)^\alpha e^{-x}} \le C\varphi_m(x), \tag{2.4}$$

where C is independent of m and x. The equivalence (2.4) appeared for the first time in [7], while its complete proof can be found in [13].

One of the crucial facts in this context is the pointwise behaviour of $P_m(w_\alpha, x)$. Estimates for $|p_m(w_\alpha, x)|$, $\alpha > -1$, can be found in [1] and [16]. Here we prefer to recall a sharp equivalence which holds in a subinterval of $[0, \infty)$ which contains all the zeros of $p_m(w_\alpha)$. More precisely, if $x_d = x_{m,d}(w_\alpha)$ is a zero of $p_m(w_\alpha)$ closest to x, i.e. $|x - x_d| = \min_k |x - x_k|$, then for any $x \in [0, 4m + A(4m)^{1/3}]$, with an arbitrary fixed $A > 0$, we get

$$\begin{aligned} \frac{1}{C}\left(\frac{x - x_d}{x_d - x_{d\pm1}}\right)^2 &\le P_m^2(w_\alpha, x)e^{-x}\left(x + \frac{1}{m}\right)^{\alpha+\frac{1}{2}} \sqrt{|4m - x| + (4m)^{1/3}} \\ &\le \left(\frac{x - x_d}{x_d - x_{d\pm1}}\right)^2, \end{aligned} \tag{2.5}$$

where $\alpha > -1$, and C is a positive constant independent of x, x_d, m, and $p_n(w_\alpha)$. The relation (2.5) was proved in ([13], Lemma 3.2). From (2.5), we get

$$\sqrt{w_\alpha(x)}|p_n(w_\alpha, x)| \sqrt[4]{x(4m - x + (4m)^{1/3})} \le C, \quad \frac{a}{m} \le x \le 4m, \tag{2.6}$$

with C independent of m and x, and $a < m$ fixed.

2.2. Polynomial inequalities. The main idea in this context, due to G. Freud and P. Nevai, is to prove polynomial inequalities with exponential weight on unbounded intervals, using well–known polynomial inequalities (eventually weighted) on bounded intervals. To this end, the main tool is the so–called "infinite–finite range inequality" (expression coined by D. Lubinsky). First G. Freud proved similar inequalities in different norms. In the case of the real semiaxis and of the Laguerre weights, we can use the same strategy. To this end the following identity of Mhaskar–Saff [15] is crucial (see also [21], p.207),

$$\max_{x \ge 0} |(P_m w_\alpha)(x)| = \max_{a_m \le x \le b_m} |(P_m w_\alpha)(x)|, \tag{2.7}$$

which holds true for any polynomial of degree $m = 1, 2, \ldots$ ($P_m \in \mathbb{P}_m$) with $\alpha > 0$, $a_m = \alpha + m - \sqrt{m^2 + 2\alpha m}$ and $b_m = \alpha + m + \sqrt{m^2 + 2\alpha m}$. The L^p–version of (2.7) is

$$\left(\int_0^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p} \le C\left(\int_0^{2m} |(P_m w_\alpha)(x)|^p dx\right)^{1/p}, \tag{2.8}$$

which holds for $p \in (0, \infty)$, and some C independent of m and P_m. Furthermore, we observe that (2.7) and (2.8) are still true if we replace x^α with $\left(x + \frac{1}{m}\right)^\alpha$, for arbitrary real α.

A second useful tool is to replace e^{-x} with a polynomial on each interval $[0, \sigma m]$, $\sigma > 0$ fixed. Indeed in [3] it was proved that for any fixed σ, there exists a polynomial $Q_{lm} \in \mathbb{P}_{lm}$, with l integer independent of m, such that for $x \in [0, \sigma m]$ we have

$$\frac{e^{-x}}{2} \leq |Q_{lm}(x)| \leq \frac{3}{2}e^{-x} \quad \text{and} \tag{2.9}$$
$$|\sqrt{x}Q'_{lm}(x)| \leq 2\sqrt{\sigma}\sqrt{m}e^{-x}.$$

Hence by (2.7) – (2.9), for any polynomial $P_m \in \mathbb{P}_m$, we get

$$\max_{x\geq 0} |(P_m w_\alpha)(x)| \leq 2 \max_{a_m \leq x \leq b_m} |(P_m Q_{lm})(x)x^\alpha|, \quad \alpha \geq 0, \tag{2.10}$$

and

$$\left(\int_0^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p} \leq 2C \left(\int_0^{2m} |(P_m Q_{lm})(x)x^\alpha|^p dx\right)^{1/p}, \quad \alpha > -1/p, \tag{2.11}$$

where $0 < p < \infty$, and C is the same constant in (2.8). Consequently, inequalities with Laguerre weights of Remez, Schur, Nikolskii and Bernstein types can be deduced from (2.10) and (2.11) by the use of well known polynomial inequalities for the Jacobi weight x^α (see for instance [12], [18]). Here we recall some inequalities since they are useful in different contexts.

PROPOSITION 2.1. *(Remez–type inequalities) Let $P_m \in \mathbb{P}_m$. For any fixed $a > 0$ there exists a constant C, depending on a and independent of m and P_m, such that:*

$$\left(\int_0^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p} \leq C \left(\int_{\frac{a}{m}}^{2m} |(P_m w_\alpha)(x)|^p dx\right)^{1/p}, \tag{2.12}$$

$$\alpha > -1/p, \quad 0 < p < \infty,$$

and

$$\max_{x\geq 0} |(P_m w_\alpha)(x)| \leq C \max_{\left[\frac{a}{m}, 2m\right]} |(P_m w_\alpha)(x)|, \quad \alpha \geq 0. \tag{2.13}$$

It is easy to deduce Schur type inequalities from (2.12) and (2.13). For instance, from (2.13), and for $\beta \geq 0$, we get

$$\max_{x\geq 0} |(P_m w_\alpha)(x)| \leq C m^\beta \max_{x\geq 0} |(P_m w_{\alpha+\beta})(x)|, \quad \alpha \geq 0. \tag{2.14}$$

From (2.11), changing variables on $[0, 1]$, we derive the Nikolskii inequality

$$\left(\int_0^\infty |(P_m w_\alpha)(x)|^q dx\right)^{1/q} \leq C m^{\frac{1}{p}-\frac{1}{q}} \left(\int_0^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p}, \tag{2.15}$$

which holds true for arbitrary $P_m \in \mathbb{P}_m$, with $0 < p < q \leq \infty$, and C independent of m and P_m. Inequality (2.15), for $1 \leq p < q \leq \infty$, can be found in [14] with a different proof. A general form of the Bernstein inequality is given in the following proposition.

PROPOSITION 2.2. *Let $P_m \in \mathbb{P}_m$, $P'_m(x) = \frac{d}{dx}P_m(x)$, and let γ be an arbitrary real number. The inequality*

$$\left(\int_0^\infty \left|P'_m(x)\left(x+\frac{1}{m}\right)^{\gamma+\frac{1}{2}} e^{-x}\right|^p dx\right)^{1/p} \le C\sqrt{m}\left(\int_0^\infty \left|P_m(x)\left(x+\frac{1}{m}\right)^{\gamma} e^{-x}\right|^p dx\right)^{1/p} \tag{2.16}$$

holds true with $0 < p < \infty$ and a constant C independent of m and P_m. Moreover,

$$\max_{x\ge 0}\left|P'_m(x)\left(x+\frac{1}{m}\right)^{\gamma+\frac{1}{2}} e^{-x}\right| \le C\sqrt{m}\max_{x\ge 0}\left|P_m(x)\left(x+\frac{1}{m}\right)^{\gamma} e^{-x}\right|, \tag{2.17}$$

where C is a positive constant independent of m and P_m. The interested reader can find (2.17) in [6] and (2.16) in [3], with $\gamma > -\frac{1}{p}$.

For the moment, we want to remark that in the case $\gamma > -\frac{1}{p}$, using (2.12), (2.16) can be rewritten as

$$\left(\int_0^\infty |P'_m(x)\sqrt{x}w_\gamma(x)|^p\right)^{1/p} \le C\sqrt{m}\left(\int_0^\infty |P_m(x)w_\gamma(x)|^p dx\right)^{1/p} \tag{2.18}$$

or

$$\left(\int_0^\infty |P'_m(x)w_\gamma(x)|^p\right)^{1/p} \le Cm\left(\int_0^\infty |P_m(x)w_\gamma(x)|^p dx\right)^{1/p}, \tag{2.19}$$

with $0 < p < \infty$ and C independent of m and P_m. Analogously, in the uniform norm we get

$$\max_{x\ge 0}|P'_m(x)\sqrt{x}w_\gamma(x)| \le C\sqrt{m}\max_{x\ge 0}|P_m(x)w_\gamma(x)|, \tag{2.20}$$

$$\max_{x\ge 0}|P'_m(x)w_\gamma(x)| \le Cm\max_{x\ge 0}|P_m(x)w_\gamma(x)|, \tag{2.21}$$

with $\gamma \ge 0$ and C independent of m and P_m.

The inequalities (2.18) – (2.21), for $p \ge 1$ and with different proofs, can be found in [8] (see also [3]).

For completeness, we also remark that concerning the weight $w_\alpha(x) = x^\alpha e^{-x}$, the Mhaskar–Saff interval is essentially given by $[0, 2m]$. In the case of the weight $w_{\alpha,\lambda}(x) = x^\alpha e^{-\lambda x}$, $\lambda > 0$, using a simple dilation, the corresponding interval is $[0, (2/\lambda)m]$. For instance, in the case $\lambda = 1/2$, the interval is $[0, 4m]$, according to (2.6). Finally we want to recall two inequalities which are connected with (2.7) – (2.8). Indeed, for an arbitrary polynomial P_m, and for any fixed $\delta > 0$, we have

$$\left(\int_{2m(1+\delta)}^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p} \le Ce^{-Am}\left(\int_0^\infty |(P_m w_\alpha)(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty, \tag{2.22}$$

$$|(P_m w_\alpha)(x)| \le Ce^{-Am}\max_{[0,\infty)}|(P_m w_\alpha(x)|, \quad x > 2m(1+\delta), \tag{2.23}$$

where C and A depend on δ and are independent of m and P_m. Inequalities (2.22) and (2.23) can be obtained by ([21], p.207) and ([4], p. 112).

Finally, in the inequalities of this section, we can replace the weight $w_\alpha(x)$ by $u_\alpha(x) = \frac{w_\alpha(x)}{(1+x)^\beta}$, $\beta > 0$. In fact we can also associate to the function $\frac{e^{-x}}{(1+x)^\beta}$ a polynomial that satisfies (2.9).

3. Functional spaces, Weighted K-Functionals and Moduli of Smoothness. Let $w_\alpha(x) = x^\alpha e^{-x}, x > 0, \alpha > -\frac{1}{p}, 1 \le p < \infty$, be the Laguerre weight and denote by $L^p_{w_\alpha}(a,b), 0 \le a < b \le \infty$, the set of all functions such that

$$\|fw_\alpha\|^p_{L^p_{w_\alpha}(a,b)} := \int_a^b |f(x)w_\alpha(x)|^p dx < \infty.$$

In the sequel we will write $\|fw_\alpha\|_p$ instead of $\|fw_\alpha\|_{L^p_{w_\alpha}(0,\infty)}$ and $L^p_{w_\alpha}$ instead of $L^p_{w_\alpha}(0,\infty)$.

Let us introduce the space

$$C_{w_\alpha} = \left\{ f \in C^0_{loc} : \lim_{x\to 0^+} |(fw_\alpha)(x)| = 0 = \lim_{x\to\infty} |(fw_\alpha)(x)| \right\}, \quad \alpha > 0,$$

where C^0_{loc} denotes the set of all locally continuous functions on $\mathbb{R}^+$ (i.e. the set of all continuous functions on every interval $[a,b]$ such that $0 < a < b < \infty$), equipped with the usual norm

$$\|f\|_{C_{w_\alpha}} := \|fw_\alpha\|_\infty = \max_{x\ge 0} |f(x)w_\alpha(x)|.$$

If $\alpha = 0$ (i.e. $w_0(x) = e^{-x}$), then C_{w_0} is the space of all functions f which are continuous on $[0,\infty)$ and satisfy the single condition

$$\lim_{x\to\infty} |f(x)e^{-x}| = 0.$$

Under the above assumptions, we set $L^\infty_{w_\alpha} = C_{w_\alpha}, \alpha \ge 0$.

For more regular functions and $1 \le p \le \infty$, we introduce the Sobolev space of order $r \ge 1$,

$$W^p_r := W^p_r(w_\alpha) = \left\{ f \in L^p_{w_\alpha} : \|f^{(r)}\varphi^r w_\alpha\|_p < \infty \right\}, \qquad \varphi(x) = \sqrt{x},$$

equipped with the norm

$$\|f\|_{W^p_r} := \|fw_\alpha\|_p + \|f^{(r)}\varphi^r w_\alpha\|_p.$$

The J. Peetre K-functional is given by

$$K_{r,\varphi}(f,t^r)_{w_\alpha,p} := \inf_{g^{(r-1)}\in AC_{loc}} \{\|(f-g)w_\alpha\|_p + t^r\|\varphi^r g^{(r)} w_\alpha\|_p\}, \tag{3.1}$$

where $0 < t \le 1, 1 \le p \le \infty$, and AC_{loc} is the set of the locally absolutely continuous functions.
In the sequel we will also use the so-called main part K-functional defined by

$$\tilde{K}_{r,\varphi}(f,t^r)_{w_\alpha,p} := \sup_{0<h\le t} \inf_{g^{(r-1)}\in AC_{loc}} \{\|(f-g)w_\alpha\|_{L^p(I_{rh})} + h^r\|\varphi^r g^{(r)} w_\alpha\|_{L^p(I_{rh})}\}, \tag{3.2}$$

where $0 < t \le 1, 1 \le p \le \infty, \varphi(x) = \sqrt{x}$ and $I_{rh} = \left[4r^2h^2, \frac{1}{h^2}\right], h > 0$.

In order to characterize the K-functional of f by means of structural properties of the same function, we introduce some weighted moduli of smoothness.

The first one is the analogue of the "main part of the $\varphi-$modulus of smoothness in $[-1,1]$" introduced in [5], and it is defined by

$$\Omega^r_\varphi(f,t)_{w_\alpha,p} := \sup_{0<h\le t} \|w_\alpha \Delta^r_{h\varphi} f\|_{L^p(I_{rh})}, \tag{3.3}$$

where $0 < t \le 1, 1 \le p \le \infty, \varphi(x)$ and I_{rh} are the same as above and

$$\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\Big(x + \frac{h\varphi(x)}{2}\Big(r-2k\Big)\Big).$$

The complete modulus of smoothness $\omega^r_\varphi(f,t)_{w_\alpha,p}$ is given by

$$\begin{aligned} \omega^r_\varphi(f,t)_{w_\alpha,p} &:= \Omega^r_\varphi(f,t)_{w_\alpha,p} + \inf_{P\in\mathbb{P}_{r-1}} \|w_\alpha(f-P)\|_{L^p(0,4r^2t^2)} \\ &\quad + \inf_{Q\in\mathbb{P}_{r-1}} \|w_\alpha(f-Q)\|_{L^p(\frac{1}{t^2},\infty)}, \end{aligned} \tag{3.4}$$

where $0 < t \le 1, 1 \le p \le \infty$, and $\mathbb{P}_m$ is the set of all algebraic polynomials of degree at most m.

The choice of the extreme values of the interval $I_{rh} = \left[4r^2h^2, \frac{1}{h^2}\right]$ is, in short, suggested by the circumstance that “close” to zero, $w_\alpha(x)$ behaves like x^α (see for instance [5], p. 59), and for $x \to \infty$ (after a quadratic transformation), $w_\alpha(x^2)$ behaves like a Hermite weight (see [5], p. 182). Sometimes, in the proofs, $\frac{1}{h^2}$ is replaced by $\frac{A_1}{h^2}$, $A_1 > 1$, and $4r^2h^2$ by $A_2 4r^2h^2$ with A_2 sufficiently large (say $4A_2 > \frac{1}{4}$) (see [5], p. 49). Since, after such a modification, the behaviour of $\tilde{K}_{r,\varphi}(f,t^r)_{w_\alpha,p}$ and $\Omega^r_\varphi(f,t)_{w_\alpha,p}$ does not change for $t \to 0$, we will preserve the same notations. We will use a similar convention for $\omega^r_\varphi(f,t)_{w_\alpha,p}$ if the intervals $(0, 4r^2t^2)$ and $\left(\frac{1}{t^2}, \infty\right)$ are sometimes replaced by $(0, A_1 4r^2t^2)$ and $\left(\frac{A_2}{t^2}, \infty\right)$, respectively.

Moreover, for positive α, we can define the modulus

$$\begin{aligned} \tilde{\omega}^r_\varphi(f,t)_{w_\alpha,p} &:= \Omega^r_\varphi(f,t)_{w_\alpha,p} \\ &\quad + \sup_{0<h\le t^2} \|w_\alpha \vec{\Delta}^r_h f\|_{L^p(0,4r^2t^2)} \\ &\quad + \inf_{Q\in\mathbb{P}_{r-1}} \|w_\alpha(f-Q)\|_{L^p(\frac{1}{t^2},\infty)}, \end{aligned} \tag{3.5}$$

where $\vec{\Delta}^r_h$ is the ordinary $r-$th forward finite difference.

We observe that in the case $p = \infty$ and $r = 1$ from (3.5), we have

$$\tilde{\omega}_\varphi(f,t)_{w_\alpha,\infty} \ge |(fw_\alpha)(0)| + |(fw_\alpha)(+\infty)|.$$

Therefore, in the definition of C_{w_α}, $\alpha > 0$, the condition

$$\lim_{x\to 0^+} |(fw_\alpha)(x)| = 0 = \lim_{x\to\infty} |(fw_\alpha)(x)|$$

is necessary in order to have $\lim_{t\to 0} \omega^r_\varphi(f,t)_{w_\alpha,\infty} = 0$.

THEOREM 3.1. *Let* $f \in L^p_{w_\alpha}$ *and let* $K_{r,\varphi}(f,t^r)_{w_\alpha,p}$, $\tilde{K}_{r,\varphi}(f,t^r)_{w_\alpha,p}$, $\Omega^r_\varphi(f,t)_{w_\alpha,p}$, $\omega^r_\varphi(f,t)_{w_\alpha,p}$ *and* $\tilde{\omega}^r_\varphi(f,t)_{w_\alpha,p}$ *be defined as in (3.1) – (3.5) respectively, with* r *an arbitrary positive integer. Then, for* $\alpha > -\frac{1}{p}$ *and* $1 \le p \le \infty$, *we have*

$$\Omega^r_\varphi(f,t)_{w_\alpha,p} \sim \tilde{K}_{r,\varphi}(f,t^r)_{w_\alpha,p}, \tag{3.6}$$

and

$$\omega^r_\varphi(f,t)_{w_\alpha,p} \sim K_{r,\varphi}(f,t^r)_{w_\alpha,p}. \tag{3.7}$$

Moreover, for $\alpha > 0$ *and* $1 \le p \le \infty$, *we get*

$$\omega^r_\varphi(f,t)_{w_\alpha,p} \sim \tilde{\omega}^r_\varphi(f,t)_{w_\alpha,p}. \tag{3.8}$$

Here $t < t_0$, where t_0 and the constants in " $\sim$ " are independent of f and t.

Theorem 3.1 will be useful in different contexts. For the time being we remark that $\Omega_\varphi^r(f,t)_{w_\alpha,p} \leq \omega_\varphi^r(f,t)_{w_\alpha,p}$ and, in general, the two moduli of smoothness are not equivalent. However, later on we will show that the equivalence holds for important classes of functions. In that case, it is more convenient to use the modulus Ω_φ^r, since, by (3.6), we have

$$\Omega_\varphi^r(f,t)_{w_\alpha,p} \leq \mathcal{C} \sup_{0<h\leq t} h^r \|f^{(r)}\varphi^r w_\alpha\|_{L^p(I_{rh})},$$

assuming the boundedness of the norm at the right-hand side. For example, if $f(x) = |\log x|, x > 0, \alpha > -1$ and $p = 1$, for an arbitrary $r \geq 1$, we have

$$\Omega_\varphi^r(f,t)_{w_\alpha,1} \sim t^{2\alpha+2}.$$

Now using $\Omega_\varphi^r(f,t)_{w_\alpha,p}$, as in the trigonometric case, we can define the Besov spaces $B_{s,q}^p(w_\alpha)$. To this end, set

$$\|f\|_{p,q,s} = \begin{cases} \left(\displaystyle\int_0^1 \left[\frac{\Omega_\varphi^r(f,t)_{w_\alpha,p}}{t^{s+1/q}}\right]^q dt\right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{w_\alpha,p}}{t^s}, & q = \infty, \end{cases}$$

with $1 \leq p \leq \infty$ and $0 < s < r$, and define $B_{s,q}^p(w_\alpha)$ as

$$B_{s,q}^p(w_\alpha) = \{f \in L_{w_\alpha}^p : \|f\|_{p,q,s} < \infty\},$$

equipped with the norm

$$\|f\|_{B_{s,q}^p(w_\alpha)} = \|fw_\alpha\|_p + \|f\|_{p,q,s},$$

$1 \leq p, q \leq \infty$, $s > 0$. These spaces were introduced in [3], and until now they have not been studied much. Later on, using the error of best polynomial approximation, we will give some equivalent expression for the norms.

4. Polynomial approximation. Denote by $E_m(f)_{w_\alpha,p} = \inf_{p\in\mathbb{P}_m} \|(f-p)w_\alpha\|_p$, $1 \leq p \leq \infty$, the error of best approximation of $f \in L_{w_\alpha}^p$ by means of algebraic polynomials. One of the basic ideas in the theory of polynomial approximation is to characterize the smoothness of a function by means of the convergence order of its best approximation. To this end some suitable moduli of smoothness are the main tools. The next theorem, recently proved in [3], is the analogue to the trigonometric case (Jackson theorem and Stechkin inequality).

THEOREM 4.1. *For all $m, r \in \mathbb{N}$ and $f \in L_{w_\alpha}^p$, $\alpha > -1/p$, $1 \leq p \leq \infty$, there exists a positive constant C, independent of m and f, such that*

$$E_m(f)_{w_\alpha,p} \leq C\omega_\varphi^r\left(f, \frac{1}{\sqrt{m}}\right)_{w_\alpha,p}, \quad r < m, \tag{4.1}$$

$$\omega_\varphi^r\left(f, \frac{1}{\sqrt{m}}\right)_{w_\alpha,p} \leq \frac{C}{(\sqrt{m})^r} \sum_{k=0}^m (1+k)^{\frac{r}{2}-1} E_k(f)_{w_\alpha,p}. \tag{4.2}$$

Moreover, if $\alpha > 0$, ω_φ^r can be replaced by $\tilde{\omega}_\varphi^r$ in (4.1) and (4.2). If we want to estimate $E_m(f)_{w_\alpha,p}$ by means of the "main part" modulus of smoothness $\Omega_\varphi^k(f,t)_{w_\alpha,p}$, we obtain a weaker version of the Jackson theorem.

THEOREM 4.2. *Under the assumptions of Theorem 4.1, we have*

$$E_m(f)_{w_\alpha,p} \le C \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f,t)_{w_\alpha,p}}{t} dt, \quad r < m, \tag{4.3}$$

where C is independent of m and f.

Obviously, since $\Omega_\varphi^r(f,t)_{w_\alpha,p} \le \omega_\varphi^r(f,t)_{w_\alpha,p}$, (4.2) is still true with Ω_φ^r instead of ω_φ^r. From this last remark it follows that if r is a positive integer, then for any positive $\lambda < r$, $E_m(f)_{w_\alpha,p} \sim \frac{1}{(\sqrt{m})^\lambda}$ is equivalent to $\omega_\varphi^r(f,t)_{w_\alpha,p} \sim t^\lambda$ and $\Omega_\varphi^r(f,t)_{w_\alpha,p} \sim t^\lambda$. A generalization of this fact is given by the following equivalences which are true for $1 \le p \le \infty$ and $r > 0$,

$$\left(\sum_{k=1}^{\infty}[(1+k)^{\frac{r}{2}-\frac{1}{q}} E_k(f)_{w_\alpha,p}]^q\right)^{1/q} \sim \left(\int_0^1 \left[\frac{\Omega_\varphi^k(f,t)_{w_\alpha,p}}{t^{r+\frac{1}{q}}}\right]^q dt\right)^{1/q}, \tag{4.4}$$

$1 \le q < \infty$, $k > r$,

$$\sup_{k\ge 1}(1+k)^{\frac{r}{2}} E_k(f)_{w_\alpha,p} \sim \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{w_\alpha,p}}{t^r}, \quad k > r. \tag{4.5}$$

Consequently, (4.4) and (4.5) are equivalent expression of the seminorms which characterize the Besov type spaces defined in Section 2. The case $p = 2$ is of special interest, since the error of best approximation can be expressed by means of the Fourier coefficients of the function f. Here the details are omitted, but the interested reader can consult [3]. Finally we like to recall the following equivalence [3],

$$\omega_\varphi^r\left(f, \frac{1}{\sqrt{m}}\right)_{w_\alpha,p} \sim \inf_{P_m \in \mathbb{P}_m} \left\{\|(f-P_m)w_\alpha\|_p + \frac{1}{(\sqrt{m})^r}\|P_m^{(r)}\varphi^r w_\alpha\|_p\right\}, \tag{4.6}$$

which holds true for $1 \le p \le \infty$, $\alpha > -1/p$, $f \in L^p_{w_\alpha}$ and with the constant in "$\sim$" being independent of m and f.

Unfortunately, while the theory on best approximation appears to be complete, there are not many concrete approximation processes in $\mathbb{R}^+$ available in the literature which are based on Laguerre polynomials. This circumstance is due to the behaviour of $p_m(w_\alpha, x)$ in a wide neighbourood of the point $4m$. A negative example is the following. If $L_m(w_\alpha, f)$ denotes the Lagrange polynomial interpolating a function $f \in L^\infty_{\sqrt{w_\alpha}}$ on the zeros of $p_m(w_\alpha)$, then the corresponding sequence of the Lebesgue constants $\{\|L_m(w_\alpha)\|_{\sqrt{w_\alpha}}\}_m$, where $\|L_m(w_\alpha)\|_{\sqrt{w_\alpha}} = \sup_{\|f\sqrt{w_\alpha}\|_\infty=1}\|L_m(w_\alpha,f)\sqrt{w_\alpha}\|_\infty$, diverges with order $\sqrt[6]{m}$ [13]. Moreover, for the time being, the behaviour of $L_m(w_\alpha, f)$ in $L^p_{w_\alpha}$ is not known, and consequently there are still only few and unsatisfactory results on numerical quadrature. Probably, concerning this subject some new idea is needed.

Now we would like to mention two classical results concerning the Fourier and de la Valleé-Poussin sums. The first result is due to R. Askey and S. Weinger [1] and to Muckenhoupt [16]. Denote by $S_m(w_\alpha, f)$ the m–the Fourier sum of the function f in the system of the Laguerre polynomials. For any $f \in L^p_{\sqrt{w_\alpha}}$, we have

$$\|S_m(w_\alpha, f)\sqrt{w_\alpha}\|_p \le C\|f\sqrt{w_\alpha}\|_p \tag{4.7}$$

for $\alpha > -\frac{1}{2p}$ and $\frac{4}{3} < p < 4$. The authors also show that (4.7) does not hold for $p \in (1, 4/3] \cup [4, \infty)$. The small range of p motivated B. Muckenhoupt [16] to obtain inequalities of the type

$$\|S_m(w_\alpha, f)u\|_p \le \|fv\|_p \quad 1 < p < \infty, \tag{4.8}$$

where u and v are suitable weight functions with $u \neq v \neq \sqrt{w_\alpha}$ in general. In [1] the authors also showed that the proved results, in some sense, were not improvable. Some years later in [20], E. L. Poiani, a Ph.D student of Muckenhoupt, proved a theorem on the boundedness of the first Cesaro sum $\sigma_m(f,x) = \frac{1}{m}\sum_{k=0}^{m-1} S_k(w_\alpha, f, x)$. Here we recall a consequence of this result.

THEOREM 4.3. *Let $V_m(w_\alpha, f) = \frac{1}{m}\sum_{k=m}^{2m-1} S_k(w_\alpha, f) \in \mathbb{P}_{2m-1}$ with $\alpha > -1$ and $u(x) = x^{\frac{\alpha}{2}} e^{-x} x^r$. If*

$$-\frac{1}{p} - \min\left(\frac{\alpha}{2}, \frac{1}{4}\right) < r < 1 - \frac{1}{p} + \min\left(\frac{\alpha}{2}, \frac{1}{4}\right)$$

and

$$-\frac{2}{3p} - \frac{1}{2} \leq r \leq -\frac{2}{3p} + \frac{7}{6},$$

then, for any $f \in L^p_u$, $1 \leq p \leq \infty$, we have

$$\|[f - V_m(w_\alpha, f)]u\|_p \leq c E_m(f)_{u,p},$$

where C is independent of m and f, and $E_m(f)_{u,p}$ is the error of best polynomial approximation in L^p_u.

Obviously, Theorem 4.3 is very useful from the theoretical point of view, but the construction of V_m is very hard due to the computation of the Fourier coefficients. Moreover, if these coefficients are approximated by means of a quadrature formula, then the nature of the operator $V_m(w_\alpha) : L^p_u \to L^p_u$ is modified (since the function must be Riemann–integrable), and the behaviour of the new operator is not known. In spaces of weighted continuous functions, and in the numerical applications, the next theorem is more useful. Denote by $w_{2\gamma}(x) = x^{2\gamma} e^{-x}$, $\gamma \geq 0$, a Laguerre weight and let $L^\infty_{\sqrt{w_{2\gamma}}}$ be the corresponding functional space (see Section 3). For any $f \in L^\infty_{\sqrt{w_{2\gamma}}}$, denote by $L_{m+1}(w_\alpha, f)$ the Lagrange interpolation polynomial of the function f on the knots

$$x_1, x_2, \ldots, x_m, 4m,$$

where $x_i = x_{m,i}(w_\alpha)$ are the zeros of $p_n(w_\alpha)$. With these notations, we can state the following theorem

THEOREM 4.4. *For any function $f \in L^\infty_{\sqrt{w_{2\gamma}}}$ and for some positive constant C independent of f and m, we get*

$$\|(f - L_{m+1}(w_\alpha, f))\sqrt{w_{2\gamma}}\|_\infty \leq C E_m(f)_{\sqrt{w_{2\gamma}},\infty} \log m,$$

if and only if the parameters $\alpha > -1$ and $\gamma \geq 0$ satisfy the conditions

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}.$$

Unfortunately, as said before, for the L^p norm there is no analogous theorem in the literature for the time being, but the author will return to this argument elsewhere.

REFERENCES

[1] R. ASKEY AND S. WAINGER, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87 (1965) 695–708
[2] W. CHEN AND Z. DITZIAN, *Best Approximation and $K-$Functionals*, Acta Math. Hungar. 75 (3) (1997), 165-208.
[3] M. C. DE BONIS, G. MASTROIANNI AND M. VIGGIANO, *K-functionals, moduli of smoothness and weighted best approximation on the semiaxis*, to appear in the Proceedings of the Alexits Memorial Conference.

[4] Z. DITZIAN AND D. S. LUBINSKY, *Jackson and Smoothness Theorems for Freud weights in L_p $(0 < p \leq \infty)$*, Constructive approximation, (1997) 13 99–152.

[5] Z. DITZIAN AND TOTIK V., *Moduli of Smoothness*, SCM, Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo, 9 (1987).

[6] V. M. FEDOROV, *Constructive characteristics of functions satisfying a Lipschitz condition on a semiaxis*, Vestnik Moskovskogo Universiteta. Matematika 38 2 (1983) 64–69

[7] G. FREUD, *A certain class of orthogonal polynomials*,(Russian) Matematicseszkije Zametki 9 (1971), 511-520

[8] G. FREUD, *On two polynomial inequalities*, Acta Math. 23 (1972), 137–145.

[9] E. GÖRLICH AND C. MARKETT, *On approximation by Cesáro means of Laguerre expansion and best approximation*, Resultate der Mathematik, 2 (1979), 124-150.

[10] I. JOÓ, *On some problems of M. Horváth*, Annales Univ. Sci. Budapest, Sect. Math. 31 (1988), 243-260.

[11] I. JOÓ N. X. KY, *Answer to a problem of Paul Turán*, Annales Univ. Sci. Budapest, Sect. Math. 31 (1988), 229-241.

[12] B. A. KHALILOVA, *On some estimates for polynomials*(Russian), Izv. Akad. Nauk Azerbaidzhan SSR, 2 (1974), 46-55.

[13] G. MASTROIANNI AND D. OCCORSIO, *Lagrange interpolation at Laguerre zeros in some weighted uniform spaces*, Acta Math. Hung., 91 (1-2) (2001), 27-52.

[14] C. MARKETT, *Nikolskii–type inequalities for Laguerrre and Hermite expansions*, Functions, Series, Operators (Proceedings, Conf. Budapest 1980), (J. Szabados, Ed.), pp. 811–834, North–Holland, Amsterdam, 1984.

[15] H. N. MHASKAR. AND E. B. SAFF, *Extremal Problems for Polynomials with Laguerre Weights*, in Approx. Theory IV, Academic Press eds. (1983).

[16] B. MUCKENHOUPT, *Mean convergence of Hermite and Laguerre series*, Transactions of the American Mathematical Society, vol.147, (1970) 433–460.

[17] P. NEVAI, *A Laguerre-Polinomok gyökein alapuló Lagrange-féle interpolációról*, Matematika Lapok 22 (1971) 149–164

[18] P. G. NEVAI, *Bernstein's Inequality in L_p for* $0 < p < 1$,, J. Approx. Theory, 27 (1979), 239-243.

[19] P. NEVAI AND GEZA–FREUD, *Orthogonal polynomials and Christoffel functions. A case study*, J. Approx. Theory, 48 (1986), 3–167

[20] E. L. POIANI, *Mean Cesaro summability of Laguerre and Hermite series*, Transactions AMS 173 (1972) 1–31.

[21] E. B. SAFF AND V. TOTIK, *Logarithmic Potentials with External Fields*, Springer-Verlag, Berlin, Heidelberg, New York (1997).

[22] G. SZEGÖ, *Orthogonal Polynomials*, 4th edn. American Mathematical Society Colloquium Publications, vol. 23. Providence, RI, American Mathematical Society (1975).