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UNCERTAINTY PRINCIPLES REVISITED*

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Abstract. The Heisenberg uncertainty principle and the uncertainty principle for self-adjoint operators have been known and applied for decades. Both in quantum mechanics and in time-frequency analysis they play an important role. In this paper, the uncertainty principle is extended to symmetric operators and to normal operators. Further, different function spaces are studied in which we obtain a number of uncertainty principles of same type using various operators.

Key words. uncertainty principle, self-adjoint operators, symmetric operators, normal operators, periodic functions, ultraspherical polynomials, sphere.

AMS subject classifications. 26D10, 42C25, 47A05, 47A30, 47B47.

1. Introduction.

The classical uncertainty principle (UP) was established by Heisenberg in [8] bringing a fundamental problem in quantum mechanics to the point: The position and the momentum of particles cannot be both determined explicitly but only in a probabilistic sense with a certain ”uncertainty”. The mathematical equivalent is that a vector in a Hilbert space and its Fourier transform cannot both be arbitrarily localized. This is the fundamental problem in time-frequency analysis, where one would like to have bases of vectors well-localized in both time and frequency.

Studying this problem on different domains and, hence, in different function spaces, the same question leads to a variety of answers and peculiarities puzzling at first sight. Recently, a number of papers has been published including UP’s for periodic functions [9, 11, 15], functions on the interval [14] and on the sphere [7, 10]. These results provide qualitative and quantitative tools in order to determine the time-frequency localization of basis functions, e.g., wavelets, in different function spaces. On the search for the joint root and for the justification of the term ”uncertainty”, the author has been inspired by [1] to take the operator theoretical approach.

The present paper sheds some light behind the scene by extending the classical UP for self-adjoint operators to a wider class of operators, namely to symmetric operators and to normal operators. These issues may be known and, in fact, are not difficult to prove but were not found in the literature available to us. From these, almost all the UP’s cited above follow (except for the sphere because of vector-valued operators). Further, this paper tries to widen the view by proposing the application of this UP to various pairs of operators. It is not conclusive, by far. Questions like the “natural” choice for the equivalent of the position and the momentum operators have not been considered here and will be subject of further investigations. Also, other concepts of UP’s as, e.g., these in [4] could be studied from a more general point of view.

The paper is organized as follows. First, some notation is introduced. In Section 3, UP’s are derived for symmetric and normal operators. Section 4 is devoted to higher moments and higher derivatives for both the real line and the circle. Section 5 deals with ultraspherical expansions on the interval and UP’s for the related Dunkl and Laplace operators. Finally in Section 6, UP’s on the sphere are discussed for the surface curl gradient and, for the first time, for the surface gradient which seems to be more appropriate.

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2. Notation. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and with norm $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$. Further, let A, B be two (possibly unbounded) linear operators with domains $\mathcal{D}(A),\ \mathcal{D}(B) \subseteq \mathcal{H}$ and ranges in $\mathcal{H}$. Let

$$\tau_A(f) \;:=\; \frac{\langle Af, f\rangle}{\langle f, f\rangle}$$

denote the (normalized) *expectation value* of the operator A with respect to $f \in \mathcal{D}(A)$, and

$$\sigma_A(f) \;:=\; \|(A - \tau_A(f))f\|$$

be the *uncertainty* or *standard deviation* of the operator A with respect to $f \in \mathcal{D}(A)$. Taking the operator as the lower index and f as variable shall suggest that the operator is usually fixed in our considerations whereas f varies over the whole domain of the operator. The *variance* is, as usual, the square of the uncertainty and can be written as

$$\begin{aligned}\sigma_A^2(f) \;&=\; \langle Af - \tau_A(f)f, Af - \tau_A(f)f\rangle \\ &=\; \|Af\|^2 - \frac{|\langle Af, f\rangle|^2}{\|f\|^2} \;=\; \|Af\|^2 - |\tau_A(f)|^2\|f\|^2 \,. \end{aligned} \tag{2.1}$$

We will consider non-commuting operators A, B, i.e., those for which $AB \neq BA$. We define the *commutator* and the *anticommutator*, respectively, as

$$[A, B] \;:=\; AB - BA\,, \qquad [A, B]_+ \;:=\; AB + BA\,,$$

both with domain $\mathcal{D}(AB) \cap \mathcal{D}(BA)$. For not-necessarily commuting operators, their *covariance* with respect to a function f is given by

$$\begin{aligned}\mathrm{cov}_{AB}(f) &:= \tfrac{1}{2}\langle[A - \tau_A(f)I, B - \tau_B(f)I]_+ f, f\rangle \\ &= \tfrac{1}{2}\langle[A, B]_+ f, f\rangle - \tau_A(f)\tau_B(f)\langle f, f\rangle \;=\; \tfrac{1}{2}\langle[A, B]_+ f, f\rangle - \frac{\langle Af, f\rangle\langle Bf, f\rangle}{\langle f, f\rangle}\,,\end{aligned}$$

I being the identity operator.

A (not necessarily bounded) densely defined linear operator A in a Hilbert space $\mathcal{H}$ is said to be *symmetric* if $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $Af = A^*f$ for $f \in \mathcal{D}(A)$. Equivalently, A is symmetric if and only if $\langle Af, g\rangle = \langle f, Ag\rangle$ for all $f, g \in \mathcal{D}(A)$. If $\mathcal{D}(A) = \mathcal{D}(A^*)$, then A is *self-adjoint*. We call A *normal* if A is closed and densely defined, and if $A^*A = AA^*$. Then we have $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $\|Af\| = \|A^*f\|$ for every $f \in \mathcal{D}(A)$. Hence, an operator is self-adjoint if and only if it is both symmetric and normal.

EXAMPLE 1 (see [4, Corollary 1.35]) The standard operators in the context of uncertainty principles are the position operator A defined by $Af(x) = xf(x)$ and the impulse operator $B = -id/dx$. Their commutator is $[A, B] = iI$, the "imaginary" identity operator. One can show that, with $\mathcal{D}(A) = \{f \in L^2(\mathbb{R}) :\ \int x^2|f(x)|^2\,dx < \infty\}$ and $\mathcal{D}(B) = \{f \in AC_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}) :\ \int |f'(x)|^2\,dx < \infty\}$, both operators are self-adjoint.

EXAMPLE 2 (see [9, Section 5]) On the torus, the choice $Af(x) = xf(x)$ for periodic functions $f \in L^2_{2\pi}$ would yield an unexceptable dependence of $\tau_A(f)$ and $\sigma_A(f)$ on the choice of the integration bounds. Here, we have to take the position operator $Af(x) = e^{ix}f(x)$ instead. Then we have $A^*f(x) = e^{-ix}f(x)$. Hence, A is a unitary operator, with $D(A) = L^2_{2\pi}$. The commutator with $B = -id/dx$ being self-adjoint on $D(B) = \{f \in AC_{2\pi} :\ f' \in L^2_{2\pi}\}$ is $[A, B] = -A$.

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3. UP's for symmetric and normal operators. In the literature we usually find uncertainty principles only for self-adjoint operators (see e.g. [3, Theorem 1.34]). This is a restriction that is sometimes painful. On the one hand, unbounded symmetric operators can have different (or no) self-adjoint extensions where $\mathcal{D}(A) = \mathcal{D}(A^*)$, and these domains are often difficult to determine. On the other hand, certain function spaces require a normal but non-symmetric position operator.

Therefore, we start proving a more general theorem where the two operators in question can be either symmetric or normal (or both), also allowing a pair of one symmetric and one normal operator.

THEOREM 3.1. *If $A, B : \mathcal{H} \to \mathcal{H}$ are symmetric or normal then*

$$\|(A-a)f\| \, \|(B-b)f\| \;\geq\; \tfrac{1}{2}|\langle [A,B]f, f\rangle| \,, \tag{3.1}$$

$$\|(A-a)f\| \, \|(B-b)f\| \;\geq\; \tfrac{1}{2}|\langle [A-aI, B-bI]_+ f, f\rangle| \,, \tag{3.2}$$

for all $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ and all $a, b \in \mathbb{C}$. Equality is attained if and only if there exist constants $c_1, c_2, d_1, d_2 \in \mathbb{C}$ with $(|c_1| + |d_1|)(|c_2| + |d_2|) > 0$ such that

$$c_1(A^* - \overline{a})f = d_1(B-b)f \qquad \text{and} \qquad c_2(A-a)f = d_2(B^* - \overline{b})f, \tag{3.3}$$

and, additionally, either at least one of the constants is zero, or $d_1/c_1 = \pm\overline{d_2}/\overline{c_2}$ with the plus in case of (3.2) and the minus in case of (3.1), respectively.

Proof . For any linear operator A and any $a \in \mathbb{C}$, the adjoint of $A - aI$ is $A^* - \overline{a}I$. Under the assumptions of the theorem, we have that

$$\|(A^* - \overline{a})f\| = \|(A-a)f\| \,, \qquad \|(B^* - \overline{b})f\| = \|(B-b)f\| \,. \tag{3.4}$$

Let us denote

$$\begin{aligned} C_- &= \langle [A-a, B-b]f, f\rangle = \langle [A,B]f, f\rangle \,, \\ C_+ &= \langle [A-a, B-b]_+ f, f\rangle \,. \end{aligned}$$

For both of these, it follows that

$$|C_\pm| \;\leq\; |\langle (B-b)f, (A^* - \overline{a})f\rangle| + |\langle (A-a)f, (B^* - \overline{b})f\rangle| \tag{3.5}$$

$$\leq\; \|(A^* - \overline{a})f\|\|(B-b)f\| + \|(A-a)f\|\|(B^* - \overline{b})f\| \tag{3.6}$$

using Cauchy-Schwarz-Bunjakovski inequality. Now, (3.4) yields (3.1) and (3.2).

Equality is attained for the second inequality (3.6) if and only if (3.3) holds for constants $c_1, c_2, d_1, d_2 \in \mathbb{C}$ with $|c_1| + |d_1| > 0$ and $|c_2| + |d_2| > 0$. If all these constants are nonzero then from (3.4) we obtain $\left|\frac{d_1}{c_1}\right| = \left|\frac{d_2}{c_2}\right|$, and equality in (3.5) means that $\left|\frac{\overline{d_1}}{\overline{c_1}} \pm \frac{d_2}{c_2}\right| = \left|\frac{d_1}{c_1}\right| + \left|\frac{d_2}{c_2}\right|$ Hence, we have $\frac{d_1}{c_1} = \pm\frac{\overline{d_2}}{\overline{c_2}}$, where the plus stands in case of C_+ and the minus in case of C_- . If at least one of the constants is zero, then equality in (3.5) follows trivially. ∎

As far as equality is concerned, note that the case that (at least) one of the constants c_1, c_2, d_1, d_2 is zero means that f is in the kernel of at least one of the operators $(A - aI), (A^* - \overline{a}I), (B - bI), (B^* - \overline{b}I)$. If $(A - a)f = 0$ then f is an eigenvector of A and $a = \tau_A(f)$ is the corresponding eigenvalue.

For symmetric operators and real numbers a, b, the inequalities can be united thereby sharpening each other. A special case of this was given in [1, formula (15.87)].

THEOREM 3.2. *If A and B are symmetric operators in a Hilbert space $\mathcal{H}$, then*

$$\|(A-a)f\|\,\|(B-b)f\| \;\geq\; \frac{1}{2}\sqrt{|\langle [A,B]f,f\rangle|^2 + |\langle [A-aI,B-bI]_+ f,f\rangle|^2}\,, \tag{3.7}$$

for all $f \in \mathcal{D}(AB)\cap\mathcal{D}(BA)$ and all $a, b \in \mathbb{R}$. Equality holds if and only if $(A-a)f$ and $(B-b)f$ are scalar multiples of one another.

Proof. In the proof of Theorem 3.1, we replace inequality (3.5) by the exact value

$$|\langle (B-b)f,(A-a)f\rangle| \;=\; \sqrt{(\Im\langle (B-b)f,(A-a)f\rangle)^2 + (\Re\langle (B-b)f,(A-a)f\rangle)^2}$$

and realize that

$$\begin{aligned} 2\Re\langle (B-b)f,(A-a)f\rangle &= \langle (B-b)f,(A-a)f\rangle + \langle (A-a)f,(B-b)f\rangle \\ &= \langle [A-aI,B-bI]_+ f,f\rangle\,, \end{aligned}$$

$$2\Im\langle (B-b)f,(A-a)f\rangle = \langle [A,B]f,f\rangle\,. \qquad \square$$

So far, we have stated inequalities for arbitrary numbers a, b. They can be used if we consider the case of equality and wish to have a variety of solutions. Or, we can ask for which a, b the left-hand side of the functional inequalities (3.1), (3.2) and (3.7) is minimized. In our case, the minimum of $\|(A-a)f\|$ is, for all f, the uncertainty of A, namely when a is the orthogonal projection of Af to f, i.e.,

$$\min_a \|Af - af\| \;=\; \left\| Af - \frac{\langle Af,f\rangle}{\langle f,f\rangle} f \right\| \;=\; \|Af - \tau_A(f)f\| \;=\; \sigma_A(f)\,.$$

COROLLARY 3.3. *If A, B are symmetric or normal operators in a Hilbert space $\mathcal{H}$, then*

$$\sigma_A(f)\,\sigma_B(f) \;\geq\; \tfrac{1}{2}|\langle [A,B]f,f\rangle| \tag{3.8}$$

and

$$\sigma_A(f)\,\sigma_B(f) \;\geq\; \mathrm{cov}_{AB}(f)\,, \tag{3.9}$$

for all $f \in \mathcal{D}(AB)\cap\mathcal{D}(BA),\ f \neq 0$.

Both (3.8) and (3.9) include uncertainties and, thus, can be called uncertainty principles. They state that the product of uncertainties of two (symmetric or normal) operators in a Hilbert space is bounded from below by the expectation values of their commutator (the "classical" UP) and their anticommutator. The other way round, we have an estimate for the latter ones from above in form of the uncertainty product including the simple fact that the covariance is bounded from above by the square root of the product of variances. Both directions can be improved for symmetric operators as follows from Theorem 3.2, squaring and subtracting the one or the other term from the right-hand side as the case may be.

COROLLARY 3.4. *If A and B are symmetric operators in a Hilbert space $\mathcal{H}$, then*

$$\sigma_A(f)\,\sigma_B(f) \;\geq\; \tfrac{1}{2}\sqrt{|\langle [A,B]f,f\rangle|^2 + 4\mathrm{cov}^2_{AB}(f)} \tag{3.10}$$

for all $f \in \mathcal{D}(AB)\cap\mathcal{D}(BA),\ f \neq 0$. Equality holds if and only if $(A-\tau_A(f))f$ and $(B-\tau_B(f))f$ are scalar multiples of one another.

Note that the right-hand side in (3.10) is greater or equal to

$$|\langle (A-\tau_A(f))f,(B-\tau_B(f))f\rangle| \;=\; |\langle Af,Bf\rangle - \tau_A(f)\overline{\tau_B(f)}\|f\|^2|\,.$$

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4. UP's on the real line and on the circle. Applying (3.8) to the operators from Example 1 we get the classical Heisenberg UP on the real line (see [2, Section 2.8] for the two common versions). Example 2 yields an UP for 2π-periodic functions (see [11, Theorem 2.1]; for an asymptotic connection between the two see [12]). In both cases, in principle, operator A yields an expectation value which is known as first (trigonometric) moment, whereas B stands for the first derivative. An obvious generalization gives uncertainty principles for higher moments and higher derivatives.

First, we look at $\mathcal{H} = L^2(\mathbb{R})$ and take

$$A_k f(x) \;=\; x^k f(x) \quad \text{and} \quad B_\ell f(x) \;=\; (-id/dx)^\ell f(x)$$

for arbitrary fixed $k, \ell \in \mathbb{N}$, with $\mathcal{D}(A_k) = \{f \in L^2(\mathbb{R}) : \int x^{2k}|f(x)|^2\,dx < \infty\}$ and $\mathcal{D}(B_\ell) = \{f \in L^2(\mathbb{R}) \cap C^\ell(\mathbb{R}) : f^{(\ell)} \in L^2(\mathbb{R})\}$. Their commutator is

$$[A_k, B_\ell]f(x) \;=\; -(-i)^\ell \sum_{n=1}^{\min(k,\ell)} \binom{\ell}{n} \frac{k!}{(k-n)!}\, x^{k-n} f^{(\ell-n)}(x)\,.$$

Obviously, A_k and B_ℓ are symmetric operators. (Clearly, we could specify the domains of self-adjointness for both of them.) So, from Corollary 3.3 we obtain

COROLLARY 4.1. *For $f \in D(A_k B_\ell) \cap D(B_\ell A_k)$, the uncertainty principle*

$$\left(\|(\cdot)^k f\|^2 - \frac{|\langle (\cdot)^k f, f\rangle|^2}{\|f\|^2}\right)\left(\|f^{(\ell)}\|^2 - \frac{|\langle f^{(\ell)}, f\rangle|^2}{\|f\|^2}\right) \geq \frac{1}{4}\left|\sum_{n=1}^{\min(k,\ell)} \binom{\ell}{n} \frac{k!}{(k-n)!} \langle (\cdot)^{k-n} f^{(\ell-n)}, f\rangle\right|^2$$

holds.

In $\mathcal{H} = L^2_{2\pi}$, we consider

$$A_m f(x) \;=\; e^{imx} f(x) \quad \text{and} \quad B_\ell f(x) \;=\; (-id/dx)^\ell f(x)$$

for any fixed $m \in \mathbb{Z},\ \ell \in \mathbb{N}$, with $\mathcal{D}(A_m) = L^2_{2\pi}$ and $\mathcal{D}(B_\ell) = C^\ell_{2\pi}$. The commutator is

$$[A_m, B_\ell]f \;=\; -A_m \sum_{n=1}^{\ell} \binom{\ell}{n} m^n B_{\ell-n} f\,.$$

Since A_m is a unitary operator and B_ℓ is symmetric, from Corollary 3.3 we obtain a similar result as for $L^2(\mathbb{R})$.

COROLLARY 4.2. *For $f \in C^\ell_{2\pi}$, the uncertainty principle*

$$\left(\|f\|^2 - \frac{|\langle e^{im\cdot} f, f\rangle|^2}{\|f\|^2}\right)\left(\|f^{(\ell)}\|^2 - \frac{|\langle f^{(\ell)}, f\rangle|^2}{\|f\|^2}\right) \;\geq\; \frac{1}{4}\left|\sum_{n=1}^{\ell} \binom{\ell}{n} (im)^n \langle e^{im\cdot} f^{(\ell-n)}, f\rangle\right|^2$$

holds.

Whether these inequalities are known or useful is not clear to the author at the moment. There might be applications for the one or the other special commutator to be estimated from above, or to compute a lower bound for one variance by dividing the inequality by the other variance, respectively.

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5. UP's on the interval. For ultraspherical expansions, an UP has been proved by Rösler and Voit in [14] making use of the Dunkl operator as the angular momentum operator. Here we give a more general UP for a whole class of "position" operators. Furthermore, we obtain a similar UP for the Laplace operator. The question of which operators can be considered as the "natural" position and momentum operators on the interval is not answered, yet.

We consider the weighted Hilbert space $\mathcal{H} = L^2([0,\pi],\omega_\alpha)$, for fixed $\alpha \geq -1/2$, with probability measure

$$d\omega_\alpha(t) \;=\; c_\alpha\,(\sin t)^{2\alpha+1}\,dt\,, \qquad c_\alpha \;=\; \frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+1)^2 2^{2\alpha+1}}\,.$$

The polynomials orthogonal with respect to this measure are the ultraspherical (or Gegenbauer) polynomials

$$p_n^{(\alpha)}(t) \;:=\; {}_2F_1(-n, n+2\alpha+1; \alpha+1; \sin^2\frac{t}{2}) \;=\; P_n^{\alpha,\alpha}(\cos t) \qquad (n \in \mathbb{N}_0)\,.$$

In order to define appropriate operators A, B in $\mathcal{H}$, no general theory seems to be available, at present. In the Heisenberg UP, operator B was a differential operator of first kind being a root of the Laplace operator, at the same time. Ultraspherical polynomials are solutions of the differential equations $L_\alpha p_n^{(\alpha)} \;=\; n(n+2\alpha+1)p_n^{(\alpha)}$, for $n \in \mathbb{N}_0$, where the *Laplace operator* is defined as

$$L_\alpha f(t) \;=\; -\left(f''(t) + (2\alpha+1)\frac{\cos t}{\sin t}f'(t)\right)$$

with domain $\mathcal{D}(L_\alpha) = \{f \in C^2[0,\pi]:\; f'(0) = f'(\pi) = 0\}$. A root (in a generalized meaning) is given in [14] by the corresponding *Dunkl operator*

$$(T_\alpha f)(t) \;:=\; f'(t) + \left(\alpha + \frac{1}{2}\right)\frac{\cos t}{\sin t}(f(t) - f(-t))$$

with domain $\mathcal{D}(T_\alpha) = \{f \in AC[-\pi,\pi]:\; f' \in L^2[-\pi,\pi],\; f(-\pi) = f(\pi)\}$. Then, iT_α is symmetric (see [14, Lemma 3.1]). This differential-difference operator requires the extension of the original interval to $[-\pi,\pi]$. As in [14], we define the probability measure

$$d\tilde{\omega}_\alpha(t) \;=\; \frac{c_\alpha}{2}\,|\sin t|^{2\alpha+1}\,dt$$

as well as the even extension and the restriction operators

$$\begin{aligned} e:\; & L^2([0,\pi],\omega_\alpha) \to L^2([-\pi,\pi],\tilde{\omega}_\alpha), \qquad & e(f)(t) \;&:=\; f(|t|)\,,\\ r:\; & L^2([-\pi,\pi],\tilde{\omega}_\alpha) \to L^2([0,\pi],\omega_\alpha), \qquad & f \;&\mapsto\; f|_{[0,\pi]}\,, \end{aligned}$$

which are isometric isomorphisms between $L^2([0,\pi],\omega_\alpha)$ and $L^2_e([-\pi,\pi],\tilde{\omega}_\alpha)$ being the subspace of even functions. Then, the Dunkl and the Laplace operators are related by

$$L_\alpha f \;=\; -\,r(T_\alpha^2(e(f))) \qquad \text{for } f \in \mathcal{D}(L_\alpha)\,.$$

5.1. UP's for the Dunkl operator. We will work in $\mathcal{H}' = L^2([-\pi,\pi],\tilde{\omega}_\alpha)$ for a moment and require $f \in \mathcal{H}'$ to be even. Let $h \in AC[-\pi,\pi]$ be fixed, and define the operators $A, B:\; \mathcal{H}' \to \mathcal{H}'$,

$$Af \;=\; hf\,, \qquad Bf \;=\; -\,iT_\alpha f\,.$$

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Their commutator turns out to be

$$[A,B]f(t) \;=\; i\left(h'(t)f(t) + \left(\alpha+\frac{1}{2}\right)\frac{\cos t}{\sin t}(h(t)-h(-t))f(-t)\right).$$

So, for $g \in L^2_e([-\pi,\pi],\tilde{\omega}_\alpha)$, we simply have

$$[A,B]g \;=\; ig\,T_\alpha h\,.$$

In order to guarantee $h \in \mathcal{D}(T_\alpha)$, we have to require $h(\pi) = h(-\pi)$, and then it holds that $\mathcal{D}(AB) = \mathcal{D}(BA) = \mathcal{D}(T_\alpha)$. Clearly, A is a normal operator (allowing complex-valued "positioning" functions h) and B is symmetric. Due to (3.8) for these operators an UP follows easily.

THEOREM 5.1. *Let* $\alpha \geq -1/2$ *. In* $\mathcal{H}' = L^2([-\pi,\pi],\tilde{\omega}_\alpha)$*, we have*

$$\left(\|hf\|^2 - \frac{|\langle hf,f\rangle|^2}{\|f\|^2}\right)^{1/2}\|f'\| \;\geq\; \frac{1}{2}\,|\langle fT_\alpha h,f\rangle|\,, \tag{5.1}$$

for all $f,h \in \mathcal{D}(T_\alpha)$*,* f *even.*

Proof . For even $f \in \mathcal{D}(T_\alpha)$, we have that $T_\alpha f = f'$ is odd. Hence $\langle Bf,f\rangle = 0$ and $\sigma_B(f) = \|f'\|$. Then the assertion follows from (3.8). ∎

Let us consider special cases for $h \in \mathcal{D}(T_\alpha)$. If h is even, then the expectation value can be reduced to $\langle hf,f\rangle = \langle r(hf),r(f)\rangle_{\omega_\alpha}$, but

$$\langle [A,B]f,f\rangle \;=\; i\langle fT_\alpha h,f\rangle \;=\; i\langle h'f,f\rangle \;=\; 0\,,$$

and so the inequality becomes trivial. For odd h, we have $\langle hf,f\rangle = 0$. In general, $h = h_e + h_o$, where h_e is the even and h_o the odd part of h, respectively, and (omitting r when restricting the integration to $[0,\pi]$, for simplicity)

$$\langle hf,f\rangle \;=\; \langle h_e f,f\rangle \;=\; \langle h_e f,f\rangle_{\omega_\alpha}\,, \tag{5.2}$$

$$\langle fT_\alpha h,f\rangle \;=\; \langle fT_\alpha h_o,f\rangle \;=\; \langle fT_\alpha h_o,f\rangle_{\omega_\alpha}\,. \tag{5.3}$$

Now, let us look at the result by Rösler and Voit. They defined the generalized mean

$$\tau_\alpha(f) \;:=\; \int_0^\pi \cos t\,|f(t)|^2\,d\omega_\alpha(t)$$

and proved the following

THEOREM 5.2. *(*[14, Theorem 2.2]*) Let* $\alpha \geq -1/2$ *and* $f \in L^2([0,\pi],\omega_\alpha)$ *with* $\|f\| = 1$*. Then*

$$(1-|\tau_\alpha(f)|^2)\,\langle L_\alpha f,f\rangle_{\omega_\alpha} \;\geq\; |\tau_\alpha(f)|^2\,(\alpha+1)^2\,,$$

where the constant $(\alpha+1)^2$ *is optimal.*

In fact, we need $f \in \mathcal{D}(L_\alpha)$. Then, this theorem is a special case of Theorem 5.1 for $h(t) = e^{it}$ which yields

$$\begin{aligned}
\|he(f)\| \;&=\; \|e(f)\| \;=\; \|f\|_{\omega_\alpha} \;=\; 1\,,\\
\langle he(f),e(f)\rangle \;&=\; \langle(\cos\cdot)e(f),e(f)\rangle \;=\; \tau_\alpha(f)\,,\\
\langle e(f)T_\alpha h,e(f)\rangle \;&=\; 2(\alpha+1)\langle(\cos\cdot)e(f),e(f)\rangle \;=\; 2(\alpha+1)\tau_\alpha(f)\,.
\end{aligned}$$

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Moreover, since

$$T_\alpha^2 g(t) \;=\; g''(t) + (2\alpha+1)\frac{\cos t}{\sin t} g'(t) \qquad (g \in \mathcal{D}(T_\alpha^2))$$

is even, for even g, and since iT_α is symmetric we have, for $f \in \mathcal{D}(L_\alpha)$,

$$\langle L_\alpha f, f\rangle_{\omega_\alpha} \;=\; \langle (iT_\alpha)^2 e(f), e(f)\rangle_{\tilde{\omega}_\alpha} \;=\; \|T_\alpha e(f)\|^2_{\tilde{\omega}_\alpha} \;=\; \|f'\|^2_{\omega_\alpha}\,.$$

If we use $h(t) = \sin t$ instead, then Theorem 5.1 yields

COROLLARY 5.3. *Let $\alpha \geq -1/2$ and $f \in \mathcal{D}(L_\alpha)$. Then*

$$\|(\sin\cdot)f\|^2_{\omega_\alpha}\,\langle L_\alpha f, f\rangle_{\omega_\alpha} \;\geq\; (\alpha+1)^2\tau_\alpha^2(f)\,,$$

and

$$\|(\sin\cdot)f\|_{\omega_\alpha}\,\|f'\|_{\omega_\alpha} \;\geq\; (\alpha+1)\tau_\alpha(f)\,,$$

the latter being valid also for $f \in AC([0,\pi])$.

For real-valued functions f, these inequalities are sharper than the one in Theorem 5.2 due to

$$(1-|\tau_\alpha(f)|^2)\|f\|^2 \;=\; \|(e^{i\cdot} - \tau_\alpha(f))f\|^2 \;=\; \|(\cos\cdot - \tau_\alpha(f))f\|^2 + \|(\sin\cdot)f\|^2\,.$$

But in general, this is just another example of uncertainty principles in $L^2([0,\pi],\omega_\alpha)$.

Two other similar UP's in view of higher moments (cp. Section 4) follow from Theorem 5.1 taking $h(t) = \sin 2t$ and $h(t) = e^{2it}$, respectively. Among other interpretations they provide further lower bounds for $\|f'\|_{\omega_\alpha}$.

COROLLARY 5.4. *Let $\alpha \geq -1/2$ and $f \in AC([0,\pi])$. Then*

$$\|(\sin 2\cdot)f\|_{\omega_\alpha}\,\|f'\|_{\omega_\alpha} \;\geq\; \tfrac{1}{2}|\langle fT_\alpha(\sin 2\cdot), f\rangle_{\omega_\alpha}|$$

and

$$\left(\|f\|^2_{\omega_\alpha} - \langle(\cos 2\cdot)f, f\rangle^2_{\omega_\alpha}/\|f\|^2_{\omega_\alpha}\right)^{1/2}\,\|f'\|^2_{\omega_\alpha} \;\geq\; \tfrac{1}{2}|\langle fT_\alpha(\sin 2\cdot), f\rangle_{\omega_\alpha}|$$

with

$$\begin{aligned} |\langle fT_\alpha(\sin 2\cdot), f\rangle_{\omega_\alpha}| \;&=\; |4(\alpha+1)\|(\cos\cdot)f\|^2_{\omega_\alpha} - 2\|(\sin\cdot)f\|^2_{\omega_\alpha}| \\ &=\; |(2\alpha+1)\|f\|^2_{\omega_\alpha} + (2\alpha+3)\langle(\cos 2\cdot)f, f\rangle_{\omega_\alpha}|\,. \end{aligned}$$

Proof . Due to (5.3), for both $h(t) = \sin 2t = 2\sin t\cos t$ and $h(t) = e^{2it}$, we have on the right-hand side of (5.1)

$$|\langle e(f)T_\alpha h, e(f)\rangle| \;=\; |4(\alpha+1)\langle(\cos\cdot)^2 f, f\rangle_{\omega_\alpha} - 2\langle(\sin\cdot)^2 f, f\rangle_{\omega_\alpha}|$$

from which the two forms in the assertion can be easily deduced. The rest follows from (5.2) and Theorem 5.1. ∎

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Last but not least, let us look at B as an operator in $\mathcal{H} = L^2([0,\pi],\omega_\alpha)$ itself. When dealing with even functions $f \in \mathcal{D}(T_\alpha)$, we had $T_\alpha f = f'$. Hence, let $Bf = if'$. For $f, g \in \mathcal{D}(B) = \{f \in C^1[0,\pi] : \ f(0) = f(\pi) = 0\}$, integration by parts gives us

$$\begin{aligned}\langle Bf, g\rangle &= i\int_0^\pi f'(t)\overline{g(t)}\, d\omega_\alpha(t)\\ &= -i\int_0^\pi f(t)\big(\overline{g'(t)} + (2\alpha+1)\frac{\cos t}{\sin t}\overline{g(t)}\big)\, d\omega_\alpha(t),\end{aligned}$$

and so the adjoint is $B^* f(t) = i\big(f'(t) + (2\alpha+1)\frac{\cos t}{\sin t} f(t)\big)$. Thus, except for $\alpha = -1/2$, operator B is not normal since $(BB^* - B^*B)f = (2\alpha+1)f(t)/\sin^2 t$. Hence, we cannot directly gain our results for the first derivative in $\mathcal{H}$ without the "detour" via $\mathcal{H}'$.

5.2. UP's for the Laplace operator. In view of higher derivatives as in Section 4, we want to consider the Laplace operator L_α in $\mathcal{H} = L^2([0,\pi],\omega_\alpha)$. Let

$$Af = hf\,, \qquad Bf = L_\alpha f\,,$$

where $h \in \mathcal{D}(L_\alpha) = \{f \in C^2[0,\pi] : \ f'(0) = f'(\pi) = 0\}$ is fixed. The commutator is in this case

$$[A,B]f = hL_\alpha f - L_\alpha(hf) = 2f'h' - fL_\alpha h\,.$$

Again, Corollary 3.3 provides an uncertainty principle for these operators.

THEOREM 5.5. *Let* $\alpha \ge -1/2$. *In* $\mathcal{H} = L^2([0,\pi],\omega_\alpha)$, *we have*

$$\left(\|hf\|^2 - \frac{|\langle hf, f\rangle|^2}{\|f\|^2}\right)\left(\|L_\alpha f\|^2 - \frac{|\langle L_\alpha f, f\rangle|^2}{\|f\|^2}\right) \ge \frac{1}{4}\,|\langle 2f'h' - fL_\alpha h, f\rangle|^2\,,$$

for all $f, h \in \mathcal{D}(L_\alpha)$.

Proof. Since A is normal and B is symmetric, we can apply (3.8) to the operators A, B defined above. ∎

This choice of the operators is questionable. Let us look at the following example. For real-valued f and $h(t) = e^{ikt}$ or $h(t) = p_n^\alpha(t)$, $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$, the right-hand side vanishes, i.e.,

$$\langle 2f'h' - fL_\alpha h, f\rangle = 0\,,$$

with the exception $\langle 2ike^{ik\cdot} f' - fL_{-1/2}e^{ik\cdot}, f\rangle = ik\big((-1)^k f^2(\pi) - f^2(0)\big)/\pi$. So, for these operators A and real-valued functions the UP's above become trivial.

6. UP's on the sphere. We consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and the Hilbert space $\mathcal{H} = L^2(\mathbb{S}^2)$. In [10], Narcowich and Ward gave an UP on the sphere using the multiplication with the surface variable $\eta \in \mathbb{S}^2$ as position operator and the *angular momentum operator* $\Omega = -iL^\star = -i\eta \times \nabla^\star$ as momentum operator where $\nabla^\star$ denotes the *surface gradient* and $L^\star$ the *surface curl gradient*. Note that both are roots of the *Laplace-Beltrami operator* $\Delta^\star$ *of the unit sphere* in the sense that

$$\Delta^\star = L^\star L^\star = \nabla^\star\nabla^\star$$

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which is the tangential part of the *Laplace operator in* $\mathbb{R}^3$

$$\Delta \;=\; \frac{\partial^2}{\partial\eta_1^2}+\frac{\partial^2}{\partial\eta_2^2}+\frac{\partial^2}{\partial\eta_3^2} \;=\; \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta^\star\,.$$

See [6] for more details. Let $\|\cdot\|_{\mathbb{R}^3}$ denote the Euclidean norm in $\mathbb{R}^3$.

THEOREM 6.1. *([10, Theorem 5.1]) If $f(\eta)$ is a twice-continuously differentiable complex-valued function on $\mathbb{S}^2$, and if*

$$\begin{aligned}\int_{\mathbb{S}^2}|f(\eta)|^2\,d\omega(\eta) \;&=\; 1\,,\\ \boldsymbol{\tau}_f \;&:=\; \int_{\mathbb{S}^2}\eta|f(\eta)|^2\,d\omega(\eta)\,,\\ \textit{and}\quad \boldsymbol{\omega}_f \;&:=\; \int_{\mathbb{S}^2}\overline{f(\eta)}\Omega f(\eta)\,d\omega(\eta)\,,\end{aligned}$$

then

$$\|\boldsymbol{\tau}_f\|^2_{\mathbb{R}^3} \;\leq\; (1-\|\boldsymbol{\tau}_f\|^2_{\mathbb{R}^3})\mathrm{var}_\Omega(f)\,, \tag{6.1}$$

where

$$\mathrm{var}_\Omega(f) \;:=\; \int_{\mathbb{S}^2}\|(\Omega-\boldsymbol{\omega}_f)f(\eta)\|^2_{\mathbb{R}^3}\,d\omega(\eta)\,.$$

For real-valued functions, another proof can be found in [5, Theorem 1.3.1].

If we set

$$Af(\eta) \;=\; \eta f(\eta)\,, \qquad Bf(\eta) \;=\; -\,iL^\star f(\eta) \;=\; -\,i(\eta\times\nabla^\star)f(\eta)\,, \tag{6.2}$$

with $D(A)=\mathcal{H}$ and $D(B)=\{f\in AC(\mathbb{S}^2):\ \nabla^\star f\in\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\}$, we notice that A,B are vector-valued operators which do not fit in our scheme in Section 3. In particular, the commutator is not easily definable. It will have to be studied because it does not suffice to consider the single operator components $A_k,B_\ell:\mathcal{H}\to\mathcal{H}$, for $k,\ell=1,2,3$, as we will see in the sequel.

We define the (normalized) *expectation vector* of any vector-valued operator B by

$$\boldsymbol{\tau}_B(f) \;:=\; \frac{1}{\|f\|^2}\int_{\mathbb{S}^2} \overline{f(\eta)} B f(\eta)\, d\omega(\eta)\,.$$

Then, the *variance* of such a vector-valued operator can be written, for any $f \in \mathcal{D}(B)$, as

$$\begin{aligned}
\operatorname{var}_B(f) \;&:=\; \int_{\mathbb{S}^2} \|(B - \boldsymbol{\tau}_B(f))f(\eta)\|^2_{\mathbb{R}^3}\, d\omega(\eta)\\
&=\; \sum_{\ell=1}^{3} \|(B_\ell - \tau_{B_\ell}(f))f\|^2 \;=\; \sum_{\ell=1}^{3} \sigma^2_{B_\ell}(f) \qquad (6.3)\\
&=\; \sum_{\ell=1}^{3} \|B_\ell f\|^2 - \|f\|^2 \|\boldsymbol{\tau}_B(f)\|^2_{\mathbb{R}^3}\,.
\end{aligned}$$

Furthermore, for our choice of A in (6.2) we have

$$\sum_{k=1}^{3} \|A_k f\|^2 \;=\; \sum_{k=1}^{3} \int_{\mathbb{S}^2} |\eta_k|^2 |f(\eta)|^2\, d\omega(\eta) \;=\; \int_{\mathbb{S}^2} \|\eta\|^2_{\mathbb{R}^3} |f(\eta)|^2\, d\omega(\eta) \;=\; \|f\|^2\,,$$

and hence,

$$\operatorname{var}_A(f) \;=\; \|f\|^2 \big(1 - \|\boldsymbol{\tau}_A(f)\|^2_{\mathbb{R}^3}\big)\,.$$

We can easily prove the following:

THEOREM 6.2. *If A, B are vector-valued operators with symmetric or normal components A_k, B_ℓ acting in a Hilbert space $\mathcal{H}$, then*

$$\operatorname{var}_A(f)\, \operatorname{var}_B(f) \;\geq\; \frac{1}{4} \sum_{k,\ell} |\langle [A_k, B_\ell] f, f\rangle|^2\,,$$

$$\sum_k \sigma_{A_k}(f) \sum_\ell \sigma_{B_\ell}(f) \;\geq\; \frac{1}{2} \sum_{k,\ell} |\langle [A_k, B_\ell] f, f\rangle|\,,$$

for all $f \in (\cap_{k,\ell} \mathcal{D}(A_k B_\ell)) \cap (\cap_{k,\ell} \mathcal{D}(B_\ell A_k))$.

Proof . Apply (3.8) to each summand of the right-hand side, and then, in the first inequality, use (6.3). ☐

Our operators A and B defined in (6.2) obviously have symmetric components. We compute

$$[A_1, B] = \begin{pmatrix} 0 \\ \eta_3 \\ -\eta_2 \end{pmatrix}, \quad [A_2, B] = \begin{pmatrix} -\eta_3 \\ 0 \\ \eta_1 \end{pmatrix}, \quad [A_3, B] = \begin{pmatrix} \eta_2 \\ -\eta_1 \\ 0 \end{pmatrix}.$$

This yields

$$\sum_{k,\ell=1}^{3} |\langle [A_k, B_\ell] f, f\rangle| \;=\; 2 \sum_{k=1}^{3} |\langle A_k f, f\rangle| \;=\; 2\|f\|^2 \sum_{k=1}^{3} |\tau_{A_k}(f)|\,,$$

$$\sum_{k,\ell=1}^{3} |\langle [A_k, B_\ell] f, f\rangle|^2 \;=\; 2 \sum_{k=1}^{3} |\langle A_k f, f\rangle|^2 \;=\; 2\|f\|^4 \|\boldsymbol{\tau}_A(f)\|^2_{\mathbb{R}^3}\,.$$

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Hence, from Theorem 6.2, we obtain in this case

$$\text{var}_A(f)\,\text{var}_B(f) \;=\; \|f\|^2\big(1-\|\boldsymbol{\tau}_A(f)\|^2_{\mathbb{R}^3}\big)\,\text{var}_B(f) \;\geq\; \tfrac{1}{2}\|f\|^4\|\boldsymbol{\tau}_A(f)\|^2_{\mathbb{R}^3} \qquad , \tag{6.4}$$

where the constant in the lower bound is only half of the constant in (6.1). The second inequality of Theorem 6.2 gives us another inequality, now for uncertainties instead of variances. Here, as in (6.1), the constant should be optimal.

COROLLARY 6.3. *The operators A, B defined in* (6.2) *satisfy*

$$\sum_k \sigma_{A_k}(f)\,\sum_\ell \sigma_{B_\ell}(f) \;\geq\; \|f\|^2\sum_{k=1}^{3}|\tau_{A_k}(f)|\,,$$

for all $f \in D(B)$.

Note that $\boldsymbol{\tau}_A(f)$ is the center of mass of the density $|f|^2/\|f\|^2$ distributed around the sphere and, hence, is located in the unit ball. So, there are a lot of functions f with $\boldsymbol{\tau}_A(f)=0$ for which the inequalities above become trivial. The second problem is that, for real-valued f,

$$\boldsymbol{\tau}_B(f) \;=\; -\,i\int_{\mathbb{S}^2} f(\eta)L^\star f(\eta)\,d\omega(\eta) \;=\; 0,$$

which raises interpretation problems in terms of frequency localization on the sphere (mentioned in [7]).

Alternatively, we propose to consider

$$Af(\eta) \;=\; \eta f(\eta)\,, \qquad Bf(\eta) \;=\; -\,i\nabla^\star f(\eta) \tag{6.5}$$

the components of which are also symmetric and have commutators

$$[A_k, B_\ell]f \;=\; if\,\delta_{k,\ell}\,.$$

From Theorem 6.2 we conclude

COROLLARY 6.4. *For A, B defined in* (6.5)*, we have*

$$\text{var}_A(f)\,\text{var}_B(f) \;\geq\; \frac{3}{4}\|f\|^4$$

and

$$\sum_k \sigma_{A_k}(f)\,\sum_\ell \sigma_{B_\ell}(f) \;\geq\; \frac{3}{2}\|f\|^2\,,$$

for all $f \in D(B)$.

These inequalities using $\nabla^\star$ are obviously nontrivial, and their right-hand sides remember us of the Heisenberg UP. The author suggests studying $\nabla^\star$ as operator for frequency analysis. From the discrepancy between (6.1) and (6.4), we can guess that it should be possible to prove that $\text{var}_A(f)\,\text{var}_{i\nabla^\star}(f) \geq \frac{9}{4}\|f\|^4$, and that probably this bound is sharp. An appropriate generalization of the UP's for vector-valued operators will likely resolve this issue.

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On the other hand, we have Theorem 6.1 and Corollary 6.3 using the angular momentum operator $L^\star$ as momentum operator. The role of this operator should be reconsidered within the framework of time-frequency analysis.

Note that for B in both (6.2) and (6.5) and for $f \in C^2(\mathbb{S}^2)$, we have

$$\sum_{\ell=1}^{3} \|B_\ell f\|^2 \;=\; \sum_{\ell=1}^{3} \langle B_\ell^2 f, f\rangle \;=\; -\langle \Delta^\star f, f\rangle$$

holding two different lower bounds by Theorem 6.1 and by Corollary 6.4.

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