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COMPARING MULTILEVEL COARSENING STRATEGIES*

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Abstract. We compare several multilevel coarsening strategies by using stable subspace splitting techniques. The obtained condition numbers give an answer on how well the coarsening strategies are suited for solving an anisotropic elliptic boundary value problem.

Key words. Finite elements, multilevel algorithms, semi-coarsening.

AMS subject classifications. 65N30, 65N55, 65N22.

1. Introduction. In discretizing an elliptic boundary value problem adaptively, the so-called "grid of grids" [5] turns out to be a good tool to characterize the adaptive refinement process. Given an adaptively refined grid the question arises how to solve the discretized equation effectively.

We give an answer to the question how to wander through a "grid of grids" within a multilevel setting. For this purpose, we consider an anisotropic model problem and a computational energy space $V_{\mathbf{n}}$ of multilinear finite elements with possibly very different refinement steps $h_1 = 2^{-n_1}, \ldots, h_d = 2^{-n_d}$ per direction. By experience, it is known that the anisotropy of the problem as well as the anisotropy of the refinement should be taken care of in a solver or a preconditioner.

The method of stable subspace splittings (see e.g. [2, 3, 8, 9]) is a powerful tool for comparing multilevel coarsening strategies. To each of these strategies there corresponds a subspace splitting of the computational space with a certain condition number. This condition number is an indicator on how many steps are needed for an multigrid solver (see Theorem 2.2) or for a conjugate gradient solver with a multilevel preconditioner.

For the model problem, one can find robust and stable infinite splittings of the infinite energy space $H_1(\Omega)$ (cf. [4]) with a condition number independent of the anisotropy of the problem.

From this, one can easily find an induced splitting of the computational energy space $V_{\mathbf{n}}$, where the condition number is independent of the anisotropy of the problem and the refinement, respectively.

The question, how other multilevel coarsenings will behave, will be answered for standard refinement, standard coarsening (see Section 5 for the difference) and semi-coarsening. None of these others splittings has a condition number independent of the anisotropy of the problem. We will distinguish between the influences of the anisotropy of the grid and of the anisotropy of the problem. This will allow us to get an idea of a "good" splitting also for elliptic problems with variable coefficients, where the anisotropy of the problem cannot be built into the splitting in an easy way.

2. Stable subspace splittings. For an introduction in stable subspace splittings, we refer to Oswald [8, 9] or Griebel and Oswald [2, 3, 4]. Let $\mathcal{H}$ be a fixed (possibly finite dimensional) Hilbert space with an inner product $(\cdot,\cdot)$ and $b(u,v) = (Bu,v)$ a symmetric, positive definite bilinear form on $\mathcal{H}$. Consider an additive representation of $\mathcal{H}$ by a (possibly finite) number

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of subspaces $\mathcal{H}_j \subset \mathcal{H}$ equipped with bilinear s.p.d. forms $b_j(u,v) = (B_j u, v)$:

$$\{\mathcal{H}, b\} = \sum_j \{\mathcal{H}_j, b_j\}. \tag{2.1}$$

For this splitting, the norm $\|\!|\cdot|\!\|$ on $\mathcal{H}$ is defined by

$$|\!|\!| u |\!|\!|^2 := \inf_{u=\sum u_j} \sum_j b_j(u_j, u_j).$$

If there exist positive and finite values

$$\lambda_{\min} := \inf_{u\in\mathcal{H}\setminus\{0\}} \frac{b(u,u)}{|\!|\!| u |\!|\!|^2}, \qquad \lambda_{\max} := \sup_{u\in\mathcal{H}\setminus\{0\}} \frac{b(u,u)}{|\!|\!| u |\!|\!|^2},$$

the subspace splitting is called stable. The quotient

$$\kappa := \frac{\lambda_{\max}}{\lambda_{\min}}$$

represents the condition number of the splitting.

We want to solve the following problem: For a given function $f \in \mathcal{H}$ find $u \in \mathcal{H}$ such that

$$b(u,v) = (f,v), \qquad \forall v \in \mathcal{H}.$$

When considering iterative methods, we assume $\mathcal{H}$ to be finite-dimensional and the splitting (2.1) to be finite, i.e., to have $J \in \mathbb{N}$ subspaces $\mathcal{H}_j$. Further let $R_j : \mathcal{H} \to \mathcal{H}_j$ be some restriction and $P_j : \mathcal{H}_j \to \mathcal{H}$ denote the natural imbedding. Then, the following iterative solution methods are associated with the above introduced splitting. Let $\omega > 0$ be given. The additive subspace correction method reads as

$$u^{(k+1)} = u^{(k)} - \omega \sum_{j=0}^{J} P_j B_j^{-1} R_j (B u^{(k)} - f), \qquad k = 0, 1, \ldots$$

The multiplicative subspace correction method is given by

$$u^{(k+(j+1)/J)} = u^{(k+j/J)} - \omega P_j B_j^{-1} R_j (B u^{(k+j/J)} - f), \quad j = 0, \ldots, J,\ k = 0, 1, \ldots$$

We follow Griebel and Oswald [3] who showed how the condition number of the subspace splitting influences the convergence of the corresponding additive and multiplicative methods.

THEOREM 2.1 (Additive Schwarz, [3]). *Let $\mathcal{H}$ be finite-dimensional and the splitting (2.1) be finite. The additive method converges for $0 < \omega < 2/\lambda_{\max}$ with the convergence rate $\varrho_a = \max\{|1-\omega\,\lambda_{\min}|, |1-\omega\,\lambda_{\max}|\}$. This bound takes its minimum $\varrho_a^* = 1-2/(1+\kappa)$ for $\omega^* = 2/(\lambda_{\min} + \lambda_{\max})$.*

For many multilevel splittings, we can use the so-called strengthened Cauchy-Schwarz inequalities, i.e., we may assume that there exist a positive constant C and positive constants $\epsilon_{i,j}$ such that $\epsilon_{i,j} = \epsilon_{j,i}$, $\epsilon_{i,i} = 1$ and

$$b(v_i, v_j) \le C\, \epsilon_{i,j} \sqrt{b_i(v_i, v_i)}\, \sqrt{b_j(v_j, v_j)}, \qquad \forall v_i \in \mathcal{H}_i, v_j \in \mathcal{H}_j. \tag{2.2}$$

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We denote $E := (\epsilon_{i,j})_{i,j=1}^{J}$ and its spectral radius by $\varrho(E)$.

THEOREM 2.2 (Multiplicative Schwarz, [3]). *Let $\mathcal{H}$ be finite-dimensional and the splitting (2.1) be finite. Assume the strengthened Cauchy-Schwarz inequalities (2.2) to be satisfied. Then the multiplicative method converges for $0 < \omega < 2/C$. The optimal convergence rate is given by*

$$(\varrho_m^*)^2 \leq 1 - \frac{\lambda_{\min}}{C\,\varrho(E)}.$$

This remains valid for any reordering of the spaces in the splitting.

On the other hand, the additive Schwarz operator

$$\Big(\sum_j P_j B_j^{-1} R_j\Big)\, B \tag{2.3}$$

may be seen as a preconditioned version of the operator B. Its condition number equals the condition number κ of the splitting (2.1).

3. Notation. We first summarize some notation necessary for multilinear finite elements on $(0,1)^d$.

- Multi-integer: $\mathbf{m} := (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d$,
  - $\mathbf{o} := (0, 0, \dots, 0)$, – $\mathbf{e} := (1, 1, \dots, 1)$,
  - $\mathbf{e}_j := (\dots, 0, 1, 0, \dots)$, the j-th unit vector,
  - $|\mathbf{m}| := \sum_{j=1}^d m_j$, – $\mathbf{m} < \mathbf{n} \;\Leftrightarrow\; m_j < n_j\ \forall j = 1, 2, \dots, d$,
  - $\lfloor \mathbf{m} \rfloor := \min\limits_{j=1,\dots,d} m_j$, – $\lceil \mathbf{m} \rceil := \max\limits_{j=1,\dots,d} m_j$.
- Univariate hat function: $\varphi(x) := \max(0, 1 - |x|)$.
- Translates and dilates: $\varphi_{j,k}(x) := \varphi(2^j x - k)$.
- Univariate spaces of piecewise linear functions in $[0,1]$ (sample spaces):

$$V_j := \operatorname{span}\{\varphi_{j,k} \mid k \in \mathbb{N}_0, \operatorname{supp}(\varphi_{j,k}) \subset [0,1]\}.$$

- Univariate wavelet spaces:

$$W_0 := V_0, \qquad W_j := V_j \ominus^{\perp} V_{j-1},\ j \in \mathbb{N}.$$

- Multivariate sample and wavelet spaces:

$$V_{\mathbf{j}} := V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_d},$$
$$W_{\mathbf{j}} := W_{j_1} \otimes W_{j_2} \otimes \cdots \otimes W_{j_d}.$$

4. The model problem. We consider a simple model problem of an anisotropic elliptic equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\nabla^T(\boldsymbol{C}\nabla u) + c_0 u &= f \qquad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{4.1}$$

with constant coefficients $\boldsymbol{C} := \operatorname{diag}(c_1, c_2, \dots, c_d)$, $c_i > 0, i = 1, \dots, d$ and $c_0 \geq 0$ on the cube $\Omega := (0,1)^d$.

The problem (4.1) reads in weak formulation as

$$a(u, v) = (f, v)_2, \qquad \forall v \in H_0^1(\Omega), \tag{4.2}$$

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where $(\cdot,\cdot)_2$ denotes the inner product in $L_2(\Omega)$ and $a(u,v)$ is the bilinear form

$$a(u,v) := \int\limits_{\Omega} (\nabla u)^T (\boldsymbol{C}\nabla v) + c_0 uv\,\mathrm{d}\mathbf{x}. \tag{4.3}$$

The corresponding energy space $\{H;a\}$ is $H := H_0^1(\Omega)$ equipped with the norm $\|\cdot\|_H := \sqrt{a(u,u)}$.

For this space, there exist robust stable subspace splittings; see [4] for the bivariate version.

THEOREM 4.1 (cf. [4]). *The following splittings are stable with a bound of the condition numbers uniform with respect to the coefficients $c_0, c_1, \ldots, c_d$:*

$$\{H,a\} = \sum_{\mathbf{j}\geq\mathbf{o}} \{W_{\mathbf{j}};\ (c_1 2^{2j_1} + c_2 2^{2j_2} + \cdots + c_d 2^{2j_d} + c_0)(\cdot,\cdot)_2\}, \tag{4.4}$$

$$\{H,a\} = \sum_{\ell\geq 0} \{V_{\mathbf{j}_0+\ell\mathbf{e}};\ 2^{2\ell}(\cdot,\cdot)_2\}, \tag{4.5}$$

$$\{H,a\} = \sum_{\ell\geq 0} \{\hat{W}_\ell;\ 2^{2\ell}(\cdot,\cdot)_2\}, \tag{4.6}$$

where $\hat{W}_\ell := V_{\mathbf{j}_0+\ell\mathbf{e}} \ominus^\perp V_{\mathbf{j}_0+(\ell-1)\mathbf{e}}$ for $\ell > 0$ and $\hat{W}_0 := V_{\mathbf{j}_0}$. The multi-index $\mathbf{j}_0$ is defined as follows: Let i indicate the index for which $c_i = \max\limits_{k=1,\ldots,d} c_k$, then

- *if $c_0 > 0$ and $c_i \geq c_0$ or if $c_0 = 0$ and $c_i \geq 1$*
$$j_{0k} := [\log_4(c_i/c_k)] \quad \textit{for } k = 1,\ldots,d,$$
- *if $c_0 > 0$ and $c_i < c_0$*
$$j_{0k} := [\log_4(c_0/c_k)] \quad \textit{for } k = 1,\ldots,d,$$
- *if $c_0 = 0$ and $c_i < 1$*
$$j_{0k} := [\log_4(1/c_k)] \quad \textit{for } k = 1,\ldots,d.$$

Here, $[x]$ is the largest integer $\leq x$.

In particular, we consider two problems for illustration:

2D–*Example.* Our first problem to solve is

$$\begin{aligned} -\frac{1}{10}\frac{\partial^2 u}{\partial x_1^2} - 10\frac{\partial^2 u}{\partial x_2^2} &= f \qquad \text{in } \Omega = (0,1)^2, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{4.7}$$

The index $\mathbf{j}_0$ can be computed as $([\log_4 100], 0) = (3,0)$. So we need the subspaces

$$V_{3,0} \subset V_{4,1} \subset V_{5,2} \subset V_{6,3} \subset V_{7,4} \subset V_{8,5} \subset V_{9,6} \cdots$$

for the stable splitting (4.5).

3D–*Example.* In 3D, we consider the problem

$$\begin{aligned} -100\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{1}{100}\frac{\partial^2 u}{\partial x_3^2} &= f \qquad \text{in } \Omega = (0,1)^3, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{4.8}$$

Then we have the index $\mathbf{j}_0 = (0, [\log_4 100], [\log_4 10000]) = (0,3,6)$. For equation (4.8), the corresponding chain of subspaces for the splitting (4.5) is

$$V_{0,3,6} \subset V_{1,4,7} \subset V_{2,5,8} \subset V_{3,6,9} \subset V_{4,7,10} \subset V_{5,8,11} \subset V_{6,9,12} \cdots .$$

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5. Subspace splittings of a finite dimensional subspace. In practice, we need to approximate the solution in a finite dimensional subspace of the energy space H. Assume that for some reason (e.g. adaptivity), we want to find a numerical solution from the finite element space $V_{\mathbf{n}}$. For this, we introduce several subspace splittings corresponding to different multi-level strategies. By the finiteness of these splittings, all of them are stable. But the condition numbers may or may not depend on $\mathbf{n}$ or on the number of spaces in the splitting or on the anisotropy of the operator.

5.1. The induced splitting. The splittings presented first are induced by the splittings (4.4)–(4.6) of the whole energy space H by the intersection with the computational space $V_{\mathbf{n}}$. We will use this later as a reference.

THEOREM 5.1. *The following splittings are stable with a bound on the condition numbers uniform with respect to the coefficients* $c_0, c_1, \ldots, c_d$ *and independent of* $\mathbf{n}$*:*

$$\{V_{\mathbf{n}}, a\} = \sum_{\mathbf{o} \le \mathbf{j} \le \mathbf{n}} \{W_{\mathbf{j}};\ (c_1 2^{2j_1} + c_2 2^{2j_2} + \cdots + c_d 2^{2j_d} + c_0)(\cdot,\cdot)_2\}, \tag{5.1}$$

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell \ge 0} \{\tilde{V}_{\mathbf{j}_0 + \ell\mathbf{e}};\ 2^{2\ell}(\cdot,\cdot)_2\}, \tag{5.2}$$

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell \ge 0} \{\tilde{W}_{\ell};\ 2^{2\ell}(\cdot,\cdot)_2\}, \tag{5.3}$$

where $\tilde{W}_\ell := \hat{W}_\ell \cap V_{\mathbf{n}}$ *and* $\tilde{V}_{\mathbf{j}_0+\ell\mathbf{e}} := V_{\mathbf{j}_0+\ell\mathbf{e}} \cap V_{\mathbf{n}}$. *The multi-index* $\mathbf{j}_0$ *is chosen as in Theorem 4.1.*

We will use splitting (5.1) as our reference splitting. Denote

$$\Lambda_{\min} := \lambda_{\min,(5.1)}, \qquad \Lambda_{\max} := \lambda_{\max,(5.1)} \quad \text{and} \quad K := \kappa_{(5.1)}.$$

In this splitting, the anisotropy of the problem is “built in”. Starting from a space $\tilde{V}_{\mathbf{j}_0}$, we apply standard refinement, as long as we stay within the space $V_{\mathbf{n}}$ and continue with semi-refinement (possibly in more than one direction) until we reach the full computational space.

2D–*Example.* For problem (4.7) with $\mathbf{j}_0 = (3,0)$ and $\mathbf{n} = (4,2)$, we illustrate the coarsening strategy in a picture. The coarsening for the induced splitting is given by the arrows between the grids in Figure 5.1.

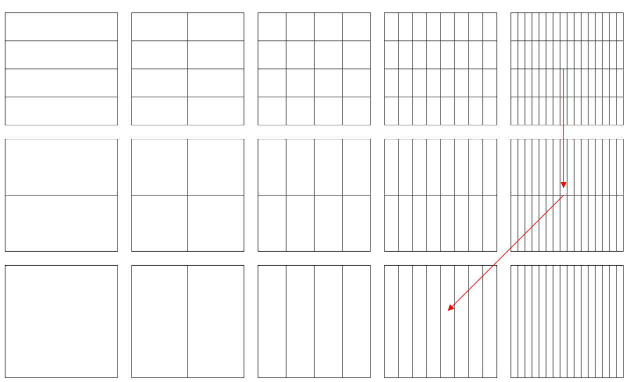

FIG. 5.1. *Coarsening for the induced splitting.*

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3D–*Example.* Assume that we are interested in a solution from $V_{\mathbf{n}} = V_{3,9,7}$ for some reason. Then, the chain of our induced splitting (5.2) looks as

$$V_{0,3,6} \subset V_{1,4,7} \subset V_{2,5,7} \subset V_{3,6,7} \subset V_{3,7,7} \subset V_{3,8,7} \subset V_{3,9,7}.$$

This splitting has a condition number $\kappa_{(5.2)} \leq 12K$.

5.2. Standard refinement. We obtain another splitting if we start our refinement procedure with $V_{\mathbf{o}}$ instead, carrying out standard refinement steps until the refinement of $V_{\mathbf{n}}$ in at least one direction is reached and then continue refining only the remaining directions (as in the previous section). In contrary with the previous splitting, we ignore the anisotropy of the problem. So, we expect a dependency of the condition number on the parameters $c_0, \ldots, c_d$ or more precise on the relation of $c_{\max} := \max_{j=0,\ldots,d} c_j$ and $c_{\min} := \min_{j=0,\ldots,d} c_j$ (if $c_0 = 0$ it is skipped).

THEOREM 5.2. *The following splittings are stable, the condition numbers depend on the coefficients* $c_0, c_1, \ldots, c_d$ *as* $\mathcal{O}(c_{\max}/c_{\min})$ *but are independent of* $\mathbf{n}$*:*

$$\{V_{\mathbf{n}}, a\} = \sum_{\mathbf{o}\leq\mathbf{j}\leq\mathbf{n}} \{W_{\mathbf{j}};\, (2^{2j_1} + 2^{2j_2} + \cdots + 2^{2j_d} + 1)(\cdot,\cdot)_2\}, \tag{5.4}$$

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell=0}^{\lceil\mathbf{n}\rceil} \{\tilde{V}_{\ell\mathbf{e}};\, 2^{2\ell}(\cdot,\cdot)_2\}, \tag{5.5}$$

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell=0}^{\lceil\mathbf{n}\rceil} \{\tilde{W}_{\ell\mathbf{e}};\, 2^{2\ell}(\cdot,\cdot)_2\}, \tag{5.6}$$

with $\tilde{W}_{\ell\mathbf{e}} := W_{\ell\mathbf{e}} \cap V_{\mathbf{n}}$ *and* $\tilde{V}_{\ell\mathbf{e}} := V_{\ell\mathbf{e}} \cap V_{\mathbf{n}}$ *and obvious modifications of* (5.4) *in case of* $c_0 = 0$.

Proof. We restrict ourselves to the case $c_0 > 0$. Because of the L_2-orthogonality between the wavelet spaces $W_{\mathbf{j}}$, the $\|\cdot\|$-norm (corresponding to (5.4)) of an element $u \in V_{\mathbf{n}}$ given in its wavelet decomposition

$$u = \sum_{\substack{\mathbf{o}\leq\mathbf{j}\leq\mathbf{n},\\ w_{\mathbf{j}}\in W_{\mathbf{j}}}} w_{\mathbf{j}}$$

can be written as

$$\|u\|^2_{(5.4)} = \sum_{\mathbf{o}\leq\mathbf{j}\leq\mathbf{n}} (2^{2j_1} + 2^{2j_2} + \cdots + 2^{2j_d} + 1)\|w_{\mathbf{j}}\|_2^2.$$

Analogously, we can handle the $\|\cdot\|$-norm from the splitting (5.1):

$$\|u\|^2_{(5.1)} = \sum_{\mathbf{o}\leq\mathbf{j}\leq\mathbf{n}} (c_1 2^{2j_1} + c_2 2^{2j_2} + \cdots + c_d 2^{2j_d} + c_0)\|w_{\mathbf{j}}\|_2^2.$$

Because

$$\begin{aligned} c_{\min}(2^{2j_1} + 2^{2j_2} + \cdots + 2^{2j_d} + 1) &\leq c_1 2^{2j_1} + c_2 2^{2j_2} + \cdots + c_d 2^{2j_d} + c_0 \\ &\leq c_{\max}(2^{2j_1} + 2^{2j_2} + \cdots + 2^{2j_d} + 1), \end{aligned}$$

we obtain equivalence of the norms

$$c_{\min}\|\cdot\|^2_{(5.4)} \leq \|\cdot\|^2_{(5.1)} \leq c_{\max}\|\cdot\|^2_{(5.4)}$$

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and hence the stability of splitting (5.4) with a condition number

$$\kappa_{(5.4)} \leq \frac{c_{\max}}{c_{\min}} K,$$

where $\lambda_{\min,(5.4)} \geq c_{\min}\Lambda_{\min}$ and $\lambda_{\max,(5.4)} \leq c_{\max}\Lambda_{\max}$.

We find the wavelet spaces $\tilde{W}_{\ell\mathbf{e}}$ by clustering

$$\tilde{W}_{\ell\mathbf{e}} = \bigoplus_{\substack{\mathbf{o}\leq\mathbf{j}\leq\mathbf{n},\\ \lceil\mathbf{j}\rceil=\ell}}^{\perp} W_{\mathbf{j}}.$$

The orthogonality of the wavelet spaces and the inequality

$$2^{2\lceil\mathbf{j}\rceil} \leq 2^{2j_1} + 2^{2j_2} + \cdots + 2^{2j_d} + 1 \leq (d+1)\, 2^{2\lceil\mathbf{j}\rceil} \tag{5.7}$$

yield the equivalence relation

$$\|\!| \cdot |\!\|^2_{(5.6)} \leq \|\!| \cdot |\!\|^2_{(5.4)} \leq (d+1)\, \|\!| \cdot |\!\|^2_{(5.6)}$$

and hence the stability of the splitting (5.6) with a condition number

$$\kappa_{(5.6)} \leq (d+1)\, \frac{c_{\max}}{c_{\min}} K$$

and the parameters $\lambda_{\min,(5.6)} \geq c_{\min}\,\Lambda_{\min}$ and $\lambda_{\max,(5.6)} \leq c_{\max}(d+1)\,\Lambda_{\max}$.

By construction, we have $\tilde{W}_{\ell\mathbf{e}} = \tilde{V}_{\ell\mathbf{e}} \ominus^{\perp} \tilde{V}_{(\ell-1)\mathbf{e}}$ for $\ell > 0$ and $\tilde{W}_{\mathbf{o}} = \tilde{V}_{\mathbf{e}}$. Define the numbers

$$\beta_\ell = \left(\sum_{i=\ell}^{\lceil\mathbf{n}\rceil} 2^{-2i}\right)^{-1} = \frac{3}{4}\, 2^{2\ell} \left(1 - 2^{-2(\lceil\mathbf{n}\rceil+\ell+1)}\right)^{-1}.$$

The $\|\!| \cdot |\!\|_{(5.5)}$-norm of an element $u \in V_{\mathbf{n}}$ with the wavelet decomposition

$$u = \sum_{\substack{0\leq\ell\leq\lceil\mathbf{n}\rceil,\\ w_\ell\in\tilde{W}_{\ell\mathbf{e}}}} w_\ell$$

can be computed (cf. [6, 7]) as

$$\|\!| u |\!\|^2_{(5.5)} = \sum_{\ell=0}^{\lceil\mathbf{n}\rceil} \beta_\ell \|w_\ell\|_2^2. \tag{5.8}$$

From the inequality

$$\frac{3}{4} 2^{2\ell} \leq \beta_\ell \leq 2^{2\ell},$$

we find the equivalence

$$\frac{3}{4} \|\!| \cdot |\!\|^2_{(5.6)} \leq \|\!| \cdot |\!\|^2_{(5.5)} \leq \|\!| \cdot |\!\|^2_{(5.6)}.$$

So, the splitting (5.5) is stable with the condition number

$$\kappa_{(5.5)} \leq \frac{4}{3}\, (d+1)\, \frac{c_{\max}}{c_{\min}} K$$

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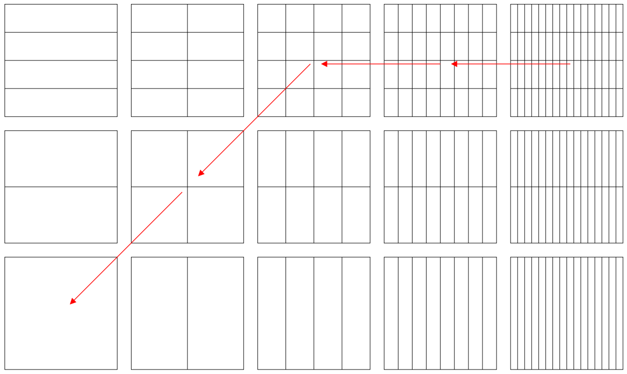

FIG. 5.2. *Coarsening for standard refinement.*

with $\lambda_{\min,(5.5)} \geq c_{\min}\,\Lambda_{\min}$ and $\lambda_{\max,(5.5)} \leq (4/3)c_{\max}(d+1)\,\Lambda_{\max}$. $\square$

Remark. If $c_0 = 0$, the factor $(d+1)$ occurring in the estimates (in this proof and also in the following sections) can of course be reduced to d.

2D–*Example.* The coarsening of $V_{4,2}$ for standard refinement is shown in Figure 5.2.

3D–*Example.* The chain of subspaces from splitting (5.5) reads as

$$V_{0,0,0} \subset V_{1,1,1} \subset V_{2,2,2} \subset V_{3,3,3} \subset V_{3,4,4} \subset V_{3,5,5} \subset V_{3,6,6} \subset V_{3,7,7} \subset V_{3,8,7} \subset V_{3,9,7}.$$

This splitting has a condition number

$$\kappa_{(5.5)} \leq \frac{4}{3}\,3\,\frac{100}{0.01}\,K = 4 \cdot 10^4\,K$$

which is $3 \cdot 10^3$-times higher than the one of the induced splitting in which the anisotropy of the problem (4.8) was "built in".

5.3. Standard coarsening. If we start from the space $V_{\mathbf{n}}$ instead, we obtain another sequence of sample spaces which correspond to the standard coarsening until the 0-level is reached for some direction, then standard coarsening in the other directions and so on. By experience, this procedure is known to be very sensitive to the anisotropies introduced by the choice of the sample space $V_{\mathbf{n}}$, i.e., the different refinement levels for the different directions. On the other hand, this splitting does not take care of the anisotropy of the problem (4.1) to be solved. The next theorem answers how this influences the condition of the corresponding splitting.

THEOREM 5.3. *The following splittings are stable, the condition numbers depend on the coefficients $c_0, c_1, \ldots, c_d$ as $\mathcal{O}(c_{\max}/c_{\min})$ and on $\mathbf{n}$ as $\mathcal{O}(2^{2(\lceil\mathbf{n}\rceil - \lfloor\mathbf{n}\rfloor)})$:*

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell=0}^{\lceil\mathbf{n}\rceil} \{V_{\mathbf{m}(\ell)};\; 2^{2(\lceil\mathbf{n}\rceil-\ell)}(\cdot,\cdot)_2\}, \tag{5.9}$$

$$\{V_{\mathbf{n}}, a\} = \sum_{\ell=0}^{\lceil\mathbf{n}\rceil} \{\check{W}_\ell;\; 2^{2(\lceil\mathbf{n}\rceil-\ell)}(\cdot,\cdot)_2\}, \tag{5.10}$$

with the multi-indices

$$\mathbf{m}(\ell) := (\max\{n_1-\ell,0\}, \max\{n_2-\ell,0\}, \ldots, \max\{n_d-\ell,0\})$$

and the wavelet spaces $\check{W}_\ell := V_{\mathbf{m}(\ell)} \ominus^\perp V_{\mathbf{m}(\ell+1)}$ *for* $\ell = 0, 1, \ldots, \lceil \mathbf{n} \rceil - 1$ *and* $\check{W}_{\lceil \mathbf{n} \rceil} := V_{\mathbf{o}}$.

Proof. The wavelet spaces used for the splitting (5.10) consist of the smaller wavelet parts

$$\check{W}_\ell = \bigoplus_{\mathbf{j} \in \mathcal{J}(\ell)}^{\perp} W_{\mathbf{j}},$$

where the indices are from the index set

$$\mathcal{J}(\ell) := \big\{ \mathbf{j} \mid \mathbf{j} \le \mathbf{m}(\ell), \exists i \in \{1, \ldots, d\} \,:\, j_i = m_i(\ell) \big\}.$$

For such multi-indices, we can estimate the maximum from above and below as

$$\lfloor \mathbf{n} \rfloor - \ell \le \lfloor \mathbf{m}(\ell) \rfloor \le \lceil \mathbf{j} \rceil \le \lceil \mathbf{m}(\ell) \rceil = \lceil \mathbf{n} \rceil - \ell.$$

Using this inequality, inequality (5.7) and the orthogonality between the wavelet spaces, we obtain

$$2^{2(\lfloor \mathbf{n} \rfloor - \lceil \mathbf{n} \rceil)} \, ||| \cdot |||^2_{(5.10)} \le ||| \cdot |||^2_{(5.4)} \le (d+1) \, ||| \cdot |||^2_{(5.10)}.$$

This gives stability of the splitting (5.10) into wavelet spaces with a condition number

$$\kappa_{(5.10)} \le 2^{2(\lceil \mathbf{n} \rceil - \lfloor \mathbf{n} \rfloor)} \, (d+1) \, \frac{c_{\max}}{c_{\min}} \, K$$

and parameters $\lambda_{\min,(5.10)} \ge c_{\min} \, 2^{2(\lfloor \mathbf{n} \rfloor - \lceil \mathbf{n} \rceil)} \, \Lambda_{\min}$ and $\lambda_{\max,(5.10)} \le c_{\max}(d+1) \, \Lambda_{\max}$.

Here, we proceed as in the proof of Theorem 5.2. The square of the $||| \cdot |||_{(5.9)}$-norm of $u \in V_{\mathbf{n}}$ with the wavelet parts $w_\ell \in \check{W}_\ell$ can be written as in (5.8) as a weighted sum of the squares of L_2-norms of the wavelet parts. The weighting factors are

$$\beta_\ell = \Big(\sum_{i=0}^{\ell} 2^{-2(\lceil \mathbf{n} \rceil - i)} \Big)^{-1} = \frac{3}{4} \, 2^{2(\lceil \mathbf{n} \rceil - \ell)} \, (1 - 2^{-2(\ell+1)})^{-1}$$

and fulfill $(3/4) 2^{2(\lceil \mathbf{n} \rceil - \ell)} \le \beta_\ell \le 2^{2(\lceil \mathbf{n} \rceil - \ell)}$. Hence,

$$\frac{3}{4} ||| \cdot |||^2_{(5.10)} \le ||| \cdot |||^2_{(5.9)} \le ||| \cdot |||^2_{(5.10)}.$$

The splitting (5.9) is stable and has the condition number

$$\kappa_{(5.9)} \le \frac{4}{3} \, 2^{2(\lceil \mathbf{n} \rceil - \lfloor \mathbf{n} \rfloor)} \, (d+1) \, \frac{c_{\max}}{c_{\min}} \, K,$$

where $\lambda_{\min,(5.9)} \ge c_{\min} \, 2^{2(\lfloor \mathbf{n} \rfloor - \lceil \mathbf{n} \rceil)} \, \Lambda_{\min}$ and $\lambda_{\max,(5.9)} \le (4/3) c_{\max}(d+1) \, \Lambda_{\max}$. ◻

2D–*Example.* The coarsening of $V_{4,2}$ for standard coarsening is illustrated by Figure 5.3.

3D–*Example.* For our example, we obtain the subspaces

$$V_{0,0,0} \subset V_{0,1,0} \subset V_{0,2,0} \subset V_{0,3,1} \subset V_{0,4,2} \subset V_{0,5,3} \subset V_{0,6,4} \subset V_{1,7,5} \subset V_{2,8,6} \subset V_{3,9,7}$$

of the splitting (5.9) with a condition

$$\kappa_{(5.9)} \le \frac{4}{3} \, 2^{2(9-3)} \, 3 \, \frac{100}{0.01} \, K = 4 \cdot 10^4 \cdot 2^{12} \, K \le 1.7 \cdot 10^8 \, K.$$

Compared with the standard refinement from the previous section we loose another factor of $4 \cdot 10^3$ because we did not pay enough attention to the anisotropy coming from the computational space $V_{3,9,7}$.

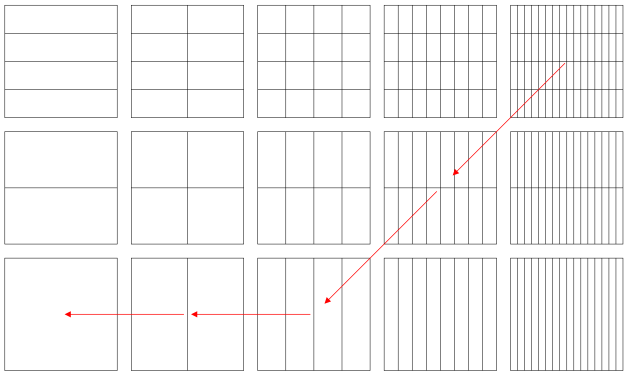

FIG. 5.3. *Coarsening for standard coarsening.*

5.4. Semi-coarsening. Another possibility to avoid the dependency on the anisotropic grid is pure semi-coarsening. Starting with $V_{\mathbf{n}}$, we reduce in one coarsening step only the level of a direction with the maximal level. This will help to overcome the dependency on $\mathbf{n}$ again.

THEOREM 5.4. *Assume without loss of generality the ordering $n_1 \geq n_2 \geq \cdots \geq n_d$ to simplify notation. The following splitting is stable. The condition number depends on the coefficients $c_0, c_1, \ldots, c_d$ as $\mathcal{O}(c_{\max}/c_{\min})$ but is independent of $\mathbf{n}$:*

$$\{V_{\mathbf{n}}, a\} = \{V_{\mathbf{o}}, (\cdot,\cdot)_2\} + \sum_{j_1=1}^{n_1} \sum_{j_2=\min\{n_2,j_1-1\}}^{\min\{n_2,j_1\}} \cdots \sum_{j_d=\min\{n_d,j_1-1\}}^{\min\{n_d,j_1\}} \{V_{\mathbf{j}}, 2^{2j_1}(\cdot,\cdot)_2\}. \tag{5.11}$$

Proof. We obtain the semi-refined splitting (5.11) from the standard refined splitting (5.5) by refinement (cf. [3]), i.e., we split the subspaces $\tilde{V}_{\ell\mathbf{e}}$ further. Fix ℓ with $1 \leq \ell \leq \lceil \mathbf{n} \rceil$. Choose $\mathbf{j}$ such that

$$V_{\mathbf{j}} = \tilde{V}_{\ell\mathbf{e}},$$

especially we have $\ell = j_1 = \lceil \mathbf{j} \rceil$. Then there exists an $m \in \{0, \ldots, d\}$ that

$$\{V_{\mathbf{j}}, 2^{2\ell}(\cdot,\cdot)_2\} = \sum_{k=0}^{m} \{V_{\mathbf{j}-\mathbf{e}_1-\cdots-\mathbf{e}_k}, 2^{2\ell}(\cdot,\cdot)_2\}. \tag{5.12}$$

If we analyze the $\|\cdot\|$-norm of this splitting via the corresponding wavelet spaces we have to look again on coefficients

$$\beta_k = \left(\sum_{i=k}^{m} 2^{-2\ell}\right)^{-1} = 2^{2\ell}(m-k+1)^{-1}.$$

From $1 \leq m-k+1 \leq d-k+1 \leq d+1$ and the orthogonality of the wavelet spaces, we obtain the stability of the subsplitting (5.12) with a condition number $\kappa_{(5.12)} \leq (d+1)$. Hence we obtain the stability of the refined splitting and a condition number

$$\kappa_{(5.11)} \leq \kappa_{(5.5)} \kappa_{(5.12)} \leq \frac{4}{3}(d+1)^2 \frac{c_{\max}}{c_{\min}} K$$

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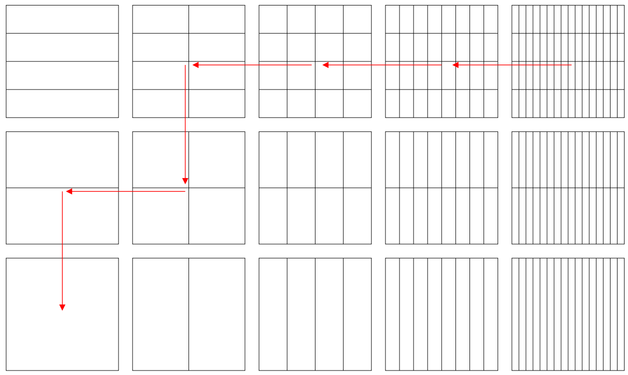

FIG. 5.4. *Coarsening for semi-coarsening.*

with $\lambda_{\min,(5.11)} \geq (c_{\min}/(d+1))\,\Lambda_{\min}$ and $\lambda_{\max,(5.11)} \leq (4/3)c_{\max}(d+1)\,\Lambda_{\max}$. $\square$

2D–*Example.* The semi-coarsening of $V_{4,2}$ is shown in Figure 5.4.

3D–*Example.* Here, we have the splitting (5.11) into the subspaces

$$V_{0,0,0} \subset V_{0,1,0} \subset V_{0,1,1} \subset V_{1,1,1} \subset V_{1,2,1} \subset V_{1,2,2} \subset V_{2,2,2} \subset V_{2,3,2} \subset V_{2,3,3} \subset V_{3,3,3}$$
$$\subset V_{3,4,3} \subset V_{3,4,4} \subset V_{3,5,4} \subset V_{3,5,5} \subset V_{3,6,5} \subset V_{3,6,6} \subset V_{3,7,6} \subset V_{3,8,6} \subset V_{3,9,6}$$

and a condition number

$$\kappa_{(5.11)} \leq \frac{4}{3}\,3^2\,\frac{100}{0.01}\,K = 1.2 \cdot 10^5\,K,$$

which is comparable to the standard refinement. But here we need more subspaces to get the same effect of independence of the computational space.

5.5. Further refinements. Among the splittings presented above, we are interested most in the splittings (5.2), (5.5),(5.9), and (5.11) into the classical sample spaces $V_{\mathbf{i}}$ of multilinear finite elements. For completeness, we want to remind the fact that with further refinement of these splittings, we would arrive at splittings corresponding to multigrid algorithms or multilevel preconditioners.

We can split a subspace $V_{\mathbf{i}}$ further: Because of the L_2-stability of the nodal basis the splitting

$$\{V_{\mathbf{i}}, 2^{2\ell}(\cdot,\cdot)_2\} = \sum_{\mathbf{k}} \{\mathrm{span}\{\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}\}, 2^{2\ell}(\cdot,\cdot)_2\} \tag{5.13}$$

into one-dimensional subspaces spanned by $\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}} := \prod_{m=1}^{d} \varphi_{i_m,k_m}$ is stable. Here ℓ denotes the ℓ for the induced splitting (5.2) or $\lceil \mathbf{i} \rceil$ for the other splittings. By construction, the constants involved of course do not depend on $c_0,\dots,c_d$ or $\mathbf{n}$. Another possibility to split $V_{\mathbf{i}}$ is

$$\{V_{\mathbf{i}}, 2^{2\ell}(\cdot,\cdot)_2\} = \sum_{\mathbf{k}} \{\mathrm{span}\{\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}\}, a\}. \tag{5.14}$$

Here the constants for refining (5.2) do not depend on the $c_0,\dots,c_d$ or $\mathbf{n}$. For refining the other splittings they do depend on $c_0,\dots,c_d$ since we have to use the estimate

$$c_{\min} 2^{2\lceil \mathbf{i} \rceil} \|\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}\|_2^2 \leq a(\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}, \boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}) \leq 3(d+1)c_{\max} 2^{2\lceil \mathbf{i} \rceil} \|\boldsymbol{\varphi}_{\mathbf{i},\mathbf{k}}\|_2^2$$

there.

Applying the additive subspace correction method on one of the splittings (5.2), (5.5),(5.9), or (5.11) with the refinement (5.13), we would obtain a BPX-like preconditioner (see Bramble, Pasciak and Xu [1]). Doing the same with refinement (5.14), we arrive at an MDS preconditioner (multilevel diagonal scaling, see Zhang [11]). The multiplicative subspace correction for the subspace splittings (5.2), (5.5),(5.9), or (5.11) would yield multigrid routines. Refinement (5.14) with an additive subspace correction would give damped Jacobi as a smoother. Using (5.14) with a multiplicative algorithm, we would end up with a Gauß-Seidel smoother.

6. Strengthened Cauchy-Schwarz inequalities. Now we verify the strengthened Cauchy-Schwarz inequalities in the case of our special multilevel spaces and $c_0 = 0$.

We are interested most in multilevel routines based on the sample spaces. That is why we are proving inequalities for these cases. Because of the symmetry of $a(\cdot,\cdot)$ and the construction of our multilevel spaces we may restrict to the case $\mathbf{i} \le \mathbf{j}$.

We start with the univariate case. Let $\Omega = (0,1)$, $u_i \in V_i$, $v_j \in V_j$, $i \le j$ and $I = [2^{-i}k, 2^{-i}(k+1)] \in \Omega$ a part of Ω corresponding to the partition underlying V_i. Simple calculations show that there exists a constant C_1 independent of u_i, v_j and i, j such that

$$(u_i, v_j)_{L_2(I)} \le C_1\, 2^{i+j}\, 2^{(i-j)/2} \|u_i\|_{L_2(I)} \|v_j\|_{L_2(I)}.$$

This can be used to prove the strengthened Cauchy-Schwarz inequalities in the d-variate case. Assume now $\Omega = (0,1)^d$, $u_\mathbf{i} \in V_\mathbf{i}$, $v_\mathbf{j} \in V_\mathbf{j}$, $\mathbf{i} \le \mathbf{j}$ and let $Q = [2^{-i_1}k_1, 2^{-i_1}(k_1+1)] \times \cdots \times [2^{-i_d}k_d, 2^{-i_d}(k_d+1)] \in \Omega$ be a part corresponding to the partition of $V_\mathbf{i}$. With a tensor product argument we have for every Q

$$\begin{aligned}
a(u_\mathbf{i}, v_\mathbf{j})_Q &= \sum_{m=1}^{d} c_m (D_{x_m} u_\mathbf{i}, D_{x_m} v_\mathbf{j})_{L_2(Q)} \\
&\le C_1 \left(\sum_{m=1}^{d} c_m 2^{i_m+j_m}\, 2^{(i_m-j_m)/2} \right) \|u_\mathbf{i}\|_{L_2(Q)} \|v_\mathbf{j}\|_{L_2(Q)} \\
&\le C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(\sum_{m=1}^{d} c_m 2^{i_m+j_m} \right) \|u_\mathbf{i}\|_{L_2(Q)} \|v_\mathbf{j}\|_{L_2(Q)} \qquad (6.1)
\end{aligned}$$

with

$$\epsilon_{\mathbf{i},\mathbf{j}} := \max_{m=1,\ldots,d} 2^{(i_m-j_m)/2}. \qquad (6.2)$$

We sum up the inequalities (6.1) over all small cubes $Q \in \Omega$ and obtain the strengthened Cauchy-Schwarz inequalities on the cube Ω as

$$\begin{aligned}
a(u_\mathbf{i}, v_\mathbf{j}) &\le C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(\sum_{m=1}^{d} c_m 2^{i_m+j_m} \right) \sum_{Q} \|u_\mathbf{i}\|_{L_2(Q)} \|v_\mathbf{j}\|_{L_2(Q)} \\
&\le C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(\sum_{m=1}^{d} c_m 2^{i_m+j_m} \right) \left(\sum_{Q} \|u_\mathbf{i}\|^2_{L_2(Q)} \right)^{1/2} \left(\sum_{Q} \|v_\mathbf{j}\|^2_{L_2(Q)} \right)^{1/2} \\
&= C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(\sum_{m=1}^{d} c_m 2^{i_m+j_m} \right) \|u_\mathbf{i}\|_2 \|v_\mathbf{j}\|_2. \qquad (6.3)
\end{aligned}$$

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With $\epsilon_{\mathbf{i},\mathbf{j}} := \epsilon_{\mathbf{j},\mathbf{i}}$ for $\mathbf{i} \geq \mathbf{j}$, inequality (6.3) remains valid in this case, too. As for other finite elements (see e.g. Xu [10]), the exponential decay of the $\epsilon_{\mathbf{i},\mathbf{j}}$ away from the diagonal yields that $\varrho(E)$ remains bounded independent of the number J of subspaces in the splitting. Now we further use (6.3) to establish the strengthened Cauchy-Schwarz inequalities for our concrete splittings into sample spaces. We start with the induced splitting (5.2).

THEOREM 6.1. *For the induced splitting* (5.2)*, there hold the strengthened Cauchy-Schwarz inequalities* (2.2) *with* $\epsilon_{\mathbf{i},\mathbf{j}}$ *as in* (6.2) *and a constant* $C_{(5.2)} \leq \hat{C}\, C_1$ *independent of* $c_1, \ldots, c_d$ *and* $\mathbf{n}$*.*

Proof. The index $\mathbf{j}_0$ is chosen in a way that there exists a constant $\hat{C}$ independent of $c_1 \ldots, c_d$ and $\mathbf{n}$ such that for all ℓ holds that

$$c_1 2^{2(j_{01}+\ell)} + c_2 2^{2(j_{02}+\ell)} + \cdots + c_d 2^{2(j_{0d}+\ell)} \leq \hat{C} 2^{2\ell}.$$

Fix $s, t \in \mathbb{N}_0$. Let $\mathbf{i}$ and $\mathbf{j}$ be chosen such that

$$V_{\mathbf{i}} = \tilde{V}_{\mathbf{j}_0 + s\mathbf{e}} \qquad \text{and} \qquad V_{\mathbf{j}} = \tilde{V}_{\mathbf{j}_0 + t\mathbf{e}}.$$

Then we can estimate

$$\begin{aligned} \sum_{m=1}^{d} c_m 2^{i_m + j_m} &\leq \Big(\sum_{m=1}^{d} c_m 2^{2i_m} \Big)^{1/2} \Big(\sum_{m=1}^{d} c_m 2^{2j_m} \Big)^{1/2} \\ &\leq \Big(\sum_{m=1}^{d} c_m 2^{2(j_{0m}+s)} \Big)^{1/2} \Big(\sum_{m=1}^{d} c_m 2^{2(j_{0m}+t)} \Big)^{1/2} \\ &\leq \hat{C} 2^s 2^t \end{aligned}$$

which yields with (6.3)

$$a(u_{\mathbf{i}}, v_{\mathbf{j}}) \leq \hat{C}\, C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(2^s \, \|u_{\mathbf{i}}\|_2\right) \left(2^t \|v_{\mathbf{j}}\|_2\right).$$

This completes the proof. $\square$

Now we investigate the other splittings.

THEOREM 6.2. *For the splitting* (5.5) *for standard refinement, the splitting* (5.9) *for standard coarsening, and the splitting* (5.11) *for semi-coarsening, there hold the strengthened Cauchy-Schwarz inequalities* (2.2) *with constants* $\epsilon_{\mathbf{i},\mathbf{j}}$ *as in* (6.2) *and a constant* $C_{(5.5),(5.9),(5.11)} \leq c_{\max}\, d\, C_1$ *independent of* $\mathbf{n}$*.*

Proof. We start with (6.3) and compute

$$\begin{aligned} a(u_{\mathbf{i}}, v_{\mathbf{j}}) &\leq c_{\max}\, d\, C_1\, \epsilon_{\mathbf{i},\mathbf{j}}\, 2^{\lceil \mathbf{i}+\mathbf{j} \rceil} \, \|u_{\mathbf{i}}\|_2 \, \|v_{\mathbf{j}}\|_2 \\ &\leq c_{\max}\, d\, C_1\, \epsilon_{\mathbf{i},\mathbf{j}} \left(2^{\lceil \mathbf{i} \rceil} \, \|u_{\mathbf{i}}\|_2\right) \left(2^{\lceil \mathbf{j} \rceil} \|v_{\mathbf{j}}\|_2\right). \qquad \square \end{aligned}$$

7. Numerical Examples. We present two numerical examples in 3D. We show the convergence of a preconditioned conjugate gradient iteration with a BPX preconditioner based on the splittings discussed before.

In the Figures 7.1–7.6, the convergence history of the preconditioned conjugate gradient iteration is displayed for the relative residuals using the following symbols to characterize the preconditioners corresponding to the different splittings:

- for the induced splitting (5.2): ——
- for standard refinement (5.5): ▫- - ▫- - ▫
- for standard coarsening (5.9): ∘- - ∘- - ∘
- for semi-coarsening (5.11): +- - +- - +

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For the *first example*, we fix the solution of (4.1) as

$$u(\mathbf{x}) = \sin \pi x_1 \, \sin \pi x_2 \, \sin \pi x_3 + \sin 8\pi x_1 \, \sin 8\pi x_2 \, \sin 8\pi x_3$$

in $\Omega = (0,1)^3$ and construct the corresponding right-hand side $f(\mathbf{x})$ for different matrices $\boldsymbol{C}$. We are looking for a solution of our homogeneous Dirichlet problem in the full computational energy space $V_{2,7,4}$.

We compare the convergence for three different values of the coefficient matrix $\boldsymbol{C}$ responsible for the anisotropy of the problem:

a) isotropic: $\boldsymbol{C} = \mathrm{diag}(1,1,1)$, $c_0 = 0$,
b) anisotropic: $\boldsymbol{C} = \mathrm{diag}(10,1,1/10)$, $c_0 = 0$,
c) anisotropic: $\boldsymbol{C} = \mathrm{diag}(100,1,1/100)$, $c_0 = 0$.

From the Figures 7.1–7.3, we can see big differences in the performance of the algorithms. As discussed before, the multilevel preconditioner using standard coarsening (5.9) behaves worst due to the anisotropy of the computational space. In the anisotropic cases, the algorithm with the induced splitting (5.2) is the best. Semi-coarsening (5.11) and the standard refinement (5.5) are somewhere in between, both preconditioners giving results that get worse with the anisotropy of the problem.

For our *second example*, we fix a solution

$$u(\mathbf{x}) = e^{x_1^2} + x_2 + \frac{1}{50} \sin \pi x_3$$

in $\Omega = (0,1)^3$ for the inhomogeneous pendant of (4.1). This means, that we construct from u the Dirichlet boundary conditions and the right–hand sides f for different values of the coefficient matrix $\boldsymbol{C}$ (as in a) – c)). Given the respective right–hand side we look for a solution on an adaptive grid corresponding to a subspace of $V_{7,1,4}$.

The results are shown in Figures 7.4–7.6. Again, standard coarsening (5.9) is the worst choice. The differences arising from the anisotropy of the computational space can be seen best in the isotropic case. In the anisotropic cases, the algorithm using the preconditioner for the induced splitting (5.2) delivers good results in all cases as expected. The other splittings, semi-coarsening (5.11) and standard refinement (5.5), are again somewhere in between and the corresponding algorithms perform worse depending on the anisotropy of the problem.

8. Conclusions. We have compared different subspace splittings corresponding to different coarsening strategies. The result is as follows:

If ever possible, one should build in the anisotropy of the problem into the coarsening as in (5.2). The condition number of all the others splittings depend on the anisotropy of the problem.

The second "best" to do is standard refinement (5.9) or semi–coarsening (5.11). Both can be used also in case of variable coefficients where the anisotropy of the problem can not be built into the splitting easily. Standard refinement is the same as standard coarsening for an isotropic computational energy space $V_{\mathbf{n}}$ with $n_1 = n_2 = \cdots = n_d$. It tends to semi-coarsening for strongly anisotropic $V_{\mathbf{n}}$. The algorithm using semi-coarsening seems to be most flexible, it needs some more operations per iteration but performs quite well in all our examples. In both cases, the splittings take the anisotropy of the grid into account.

In case of anisotropic computational energy spaces $V_{\mathbf{n}}$, one should definitely avoid standard coarsening. The dependency of the condition of this splitting (and so of the condition of the preconditioned operator) on the anisotropy of the grid is large.

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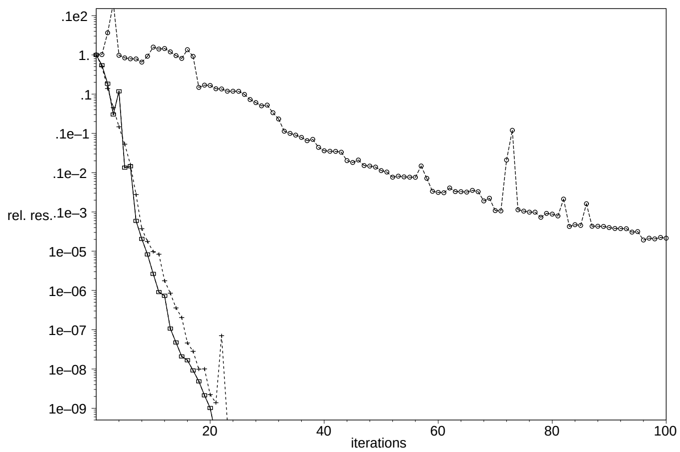

FIG. 7.1. *a) First Example: Results for the isotropic equation with* $\boldsymbol{C} = \mathrm{diag}(1, 1, 1)$

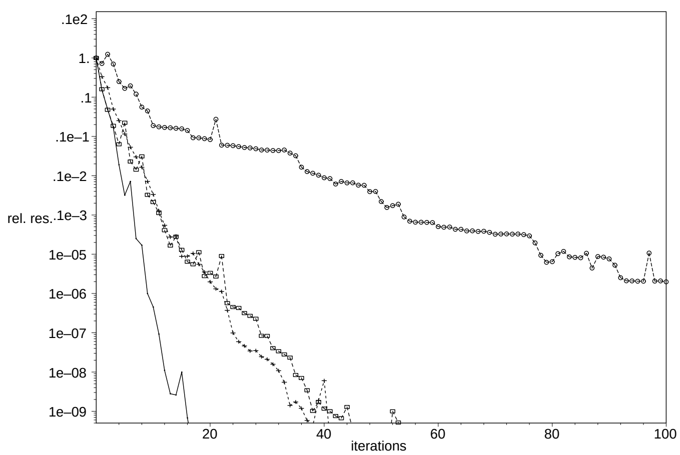

FIG. 7.2. *b) First Example: Results for the anisotropic equation with* $\boldsymbol{C} = \mathrm{diag}(10, 1, 1/10)$

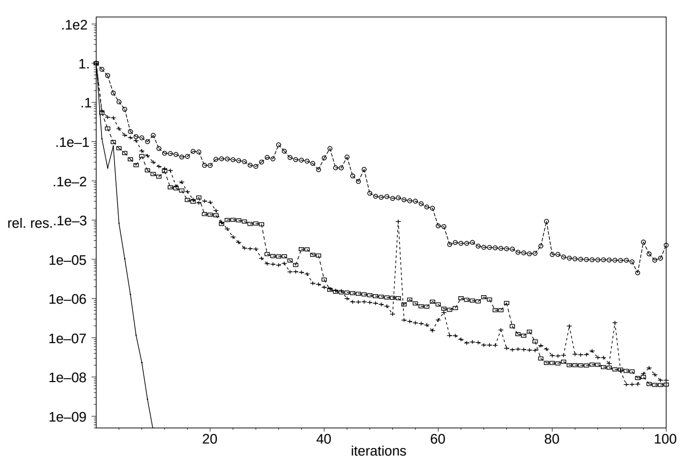

FIG. 7.3. *c) First Example: Results for the anisotropic equation with* $\boldsymbol{C} = \mathrm{diag}(100, 1, 1/100)$

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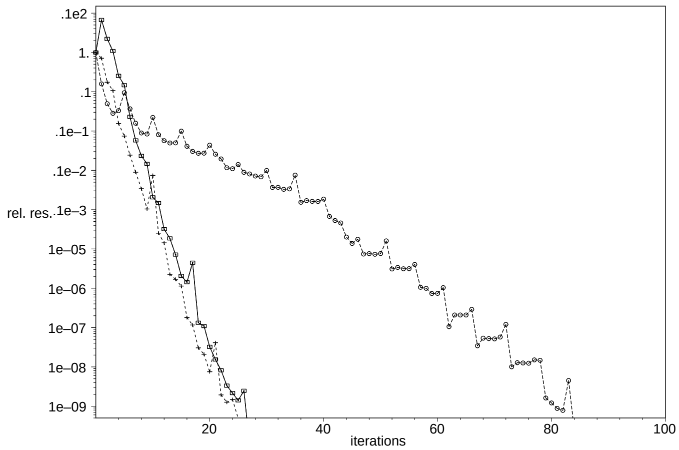

FIG. 7.4. *a) Second Example: Results for the isotropic equation with* $\boldsymbol{C} = \mathrm{diag}(1, 1, 1)$

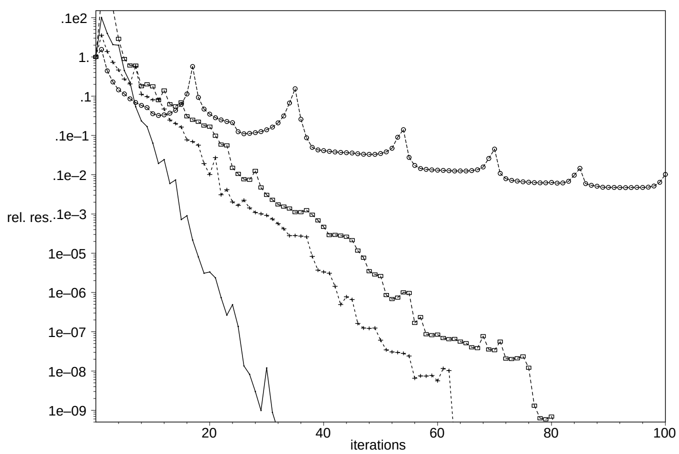

FIG. 7.5. *b) Second Example: Results for the anisotropic equation with* $\boldsymbol{C} = \mathrm{diag}(10, 1, 1/10)$

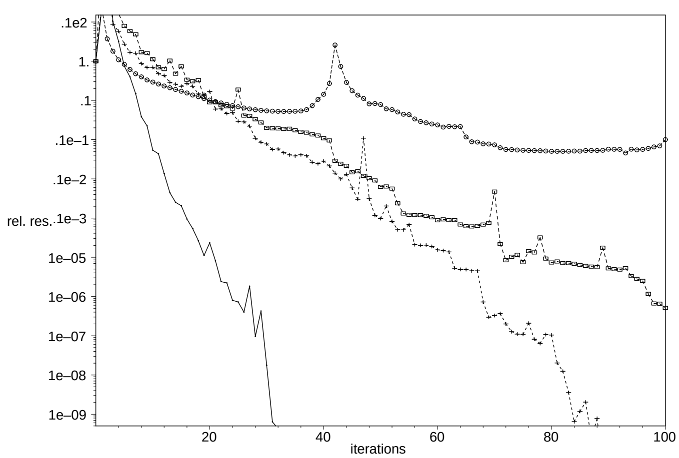

FIG. 7.6. *b) Second Example: Results for the anisotropic equation with* $\boldsymbol{C} = \mathrm{diag}(100, 1, 1/100)$

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The results in this paper have been proved for *full* grids or computational spaces $V_{\mathbf{n}}$. Numerical experiments on *locally refined* grids nevertheless show that they seem to remain true in this case, too.

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