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ON THE ESTIMATION OF THE Q-NUMERICAL RANGE OF MONIC MATRIX POLYNOMIALS*

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Abstract. For a given $q \in [0,1]$, the q-numerical range of an $n \times n$ matrix polynomial $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ is defined by $W_q(P) = \{\lambda \in \mathbb{C} : y^* P(\lambda)x = 0,\, x,y \in \mathbb{C}^n,\, x^*x = y^*y = 1,\, y^*x = q\}$. In this paper, an inclusion-exclusion methodology for the estimation of $W_q(P)$ is proposed. Our approach is based on i) the discretization of a region Ω that contains $W_q(P)$, and ii) the construction of an open circular disk, which does not intersect $W_q(P)$, centered at every grid point $\mu \in \Omega \setminus W_q(P)$. For the cases $q = 1$ and $0 < q < 1$, an important difference arises in one of the steps of the algorithm. Thus, these two cases are discussed separately.

Key words. matrix polynomial, eigenvalue, q-numerical range, boundary, inner q-numerical radius, Davis-Wielandt shell.

AMS subject classifications. 15A22, 15A60, 65D18, 65F30, 65F35.

1. Introduction and definitions.

Let $\mathcal{M}_n$ be the algebra of all $n\times n$ complex matrices, and suppose that

$$P(\lambda) \,=\, I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1.1}$$

is an $n \times n$ monic *matrix polynomial*, where $A_j \in \mathcal{M}_n$ $(j = 0, 1, \ldots, m-1)$ and λ is a complex variable. One encounters matrix polynomials for instance when studying systems of ordinary differential equations or difference equations, with constant coefficients. The suggested references are [5, 11, 19].

For a given $q \in [0,1]$, the *q-numerical range* of $P(\lambda)$ in (1.1) is defined by [14, 15]

$$W_q(P) \,=\, \{\lambda \in \mathbb{C} : y^*P(\lambda)x = 0,\, x,y \in \mathbb{C}^n,\, x^*x = y^*y = 1,\, y^*x = q\}. \tag{1.2}$$

Evidently, $W_q(P)$ is always closed and contains the *spectrum* of $P(\lambda)$, that is, the set of all *eigenvalues* of $P(\lambda)$, $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$. For $q = 1$, we have the classical *numerical range* of $P(\lambda)$ [10, 12], namely,

$$W(P) \,\equiv\, W_1(P) \,=\, \{\lambda \in \mathbb{C} : x^*P(\lambda)x = 0,\, x \in \mathbb{C}^n,\, x^*x = 1\}.$$

If $q \in (0,1]$ and $P(\lambda) = I\lambda - q\,A$ for some $A \in \mathcal{M}_n$, then $W_q(P)$ coincides with the q-numerical range of A, $F_q(A) = \{y^*Ax : x, y \in \mathbb{C}^n,\, x^*x = y^*y = 1,\, y^*x = q\}$. For $q = 1$, the numerical range of A is $F(A) \equiv F_1(A) = \{x^*Ax : x \in \mathbb{C}^n,\, x^*x = 1\}$ [4]. Moreover, the *(outer) q-numerical radius* and the *inner q-numerical radius* of A are defined by

$$r_q(A) \,=\, \max\{\, |\mu| : \mu \in F_q(A)\} \quad \text{and} \quad \hat{r}_q(A) \,=\, \min\{\, |\mu| : \mu \in \partial F_q(A)\},$$

respectively. Note also that $r_q(A) \le \|A\|_2$ for every $q \in [0,1]$ [8], where $\|\cdot\|_2$ denotes the norm induced by the standard inner product.

During the last decade, the numerical range $W(P)$ has attracted attention, and several results have been obtained (see, e.g., [2, 6, 7, 10, 12, 13, 16, 18]). These results are helpful

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in investigating and understanding matrix polynomials, and some of them have been generalized to the case of the q-numerical range [3, 14, 15]. Furthermore, applications of the numerical range of matrix polynomials in spectral analysis, factorization and stability of matrix polynomials can be found in [6, 13, 18], respectively. It is easy to see that if $q = 0,$ then $W_q(P) \equiv \mathbb{C}$. Hence, in the remainder of this note, we assume that $0 < q \leq 1$. As a consequence, since the leading coefficient of $P(\lambda)$ in (1.1) is the identity matrix, the q-numerical range $W_q(P)$ in (1.2) is compact and has no more than m connected components [10, 14].

Algorithms for plotting the boundary of the numerical range of the matrix polynomials $A\lambda^{2m_1+m_2}+B\lambda^{m_1+m_2}+C\lambda^{m_2}$ and $A\lambda^{m_1+m_2}+(B+\mathrm{i}\,C)^{m_2}$, where the matrices A, B, C are Hermitian, can be found in [7, 16, 17]. Moreover, the point equation of the boundary of the numerical range of a general matrix polynomial is studied in [2]. The numerical approximation of the q-numerical range of the monic matrix polynomial $P(\lambda)$ in (1.1) is still an open and challenging problem. The "brute force" approach would be to plot the roots of the polynomial $y^*P(\lambda)x$ for a large number of randomly chosen unit vectors $x, y \in \mathbb{C}^n$ satisfying $y^*x = q$. But that would be too costly, and it would probably not accurately depict the boundary of $W_q(P)$. In this paper, a methodology for the estimation of the q-numerical range $W_q(P)$ is proposed. This method is the first method for drawing $W_q(P)$ besides the application of the definition, and can also be used for the approximation of the boundary $\partial W_q(P)$. In the next section, we describe the general inclusion-exclusion algorithm, which is based on the boundedness of $W_q(P)$ and a result on open disks that do not intersect $W_q(P)$ [12, 14], and requires the computation of the inner q-numerical radius of the (fixed) matrix $P(\mu)$ for several $\mu \in \mathbb{C} \setminus W_q(P)$. In Sections 4 and 5, algorithms for the calculation of the inner q-numerical radius of a square complex matrix for the cases $q = 1$ and $0 < q < 1$, respectively, are given. Furthermore, numerical examples are presented to illustrate our results. For all the experiments, the computations were performed in MATLAB 4.2 on a PC Celeron 600.

2. The general algorithm. Consider an $n \times n$ matrix polynomial $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ as in (1.1) and a real $q \in (0, 1]$. Let μ be a complex number, which does not belong to the q-numerical range of $P(\lambda)$, i.e., μ lies in the open set $\mathbb{C} \setminus W_q(P)$. Then, we can construct an open circular disk $S(\mu, \rho_\mu)$ with center at μ and radius $\rho_\mu < 1$ such that $S(\mu, \rho_\mu) \cap W_q(P) = \emptyset$. The closure of $S(\mu, \rho_\mu)$ is denoted by $\overline{S}(\mu, \rho_\mu)$.

THEOREM 2.1. *Suppose $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ and $q \in (0, 1]$, and let $\mu \in \mathbb{C} \setminus W_q(P)$. Then the open disk $S(\mu, \rho_\mu)$ with radius*

$$\rho_\mu \;=\; \frac{\hat{r}_q(P(\mu))}{\hat{r}_q(P(\mu)) + \max_{j=1,\ldots,m} \|\frac{1}{j!}P^{(j)}(\mu)\|_2}$$

does not intersect the q-numerical range $W_q(P)$.

Proof. Consider the matrix polynomial

$$P_\mu(\lambda) \;=\; P(\lambda+\mu) \;=\; I\lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_1\lambda + B_0$$

and denote $B_m = I$. It is well-known that $W_q(P_\mu) \;=\; W_q(P) - \mu$ [10, 14]. Thus, the origin does not belong to $W_q(P_\mu)$, or equivalently, $0 \notin F_q(B_0)$ $(\equiv F_q(P(\mu)))$. By [14, Theorem 1.4] (see also [12, Theorem 3.1]), for every $z \in W_q(P_\mu),$ we have that

$$\frac{\hat{r}_q(B_0)}{\hat{r}_q(B_0) + \max_{j=1,\ldots,m} r_q(B_j)} \;\leq\; |z|.$$

Since $r_q(B_j) \leq \|B_j\|_2$ ($j = 1, 2, \ldots, m$), it follows that

$$\frac{\hat{r}_q(B_0)}{\hat{r}_q(B_0) + \max_{j=1,\ldots,m} \|B_j\|_2} \leq |z|.$$

Hence, $S(0, \rho_\mu) \cap W_q(P_\mu) = \emptyset$, or equivalently, $S(\mu, \rho_\mu) \cap W_q(P) = \emptyset$. Since the coefficients of $P_\mu(\lambda) = P(\lambda + \mu)$ are given by

$$B_j \;=\; \frac{1}{j\,!}\, P^{(j)}(\mu) \;\;;\;\; j = 0, 1, \ldots, m,$$

the proof is complete. □

The q-numerical range $W_q(P)$ is bounded, and thus, we can always find a bounded region Ω in the complex plane that contains $W_q(P)$. For example, it is known that [14, Theorem 1.4]

$$(2.1)\;\; W_q(P) \subseteq \overline{S}\left(0\;,\;1 + \max_{j=0,\ldots,m-1} \frac{r_q(A_j)}{q}\right) \subseteq \overline{S}\left(0\;,\;1 + \max_{j=0,\ldots,m-1} \frac{\|A_j\|_2}{q}\right).$$

Then we can approximate $W_q(P)$ by plotting its complement with respect to Ω.

Algorithm 1 (The general inclusion-exclusion procedure)
Step I Obtain an open bounded region $\Omega \subset \mathbb{C}$ such that $W_q(P) \subset \Omega$.
Step II Construct a grid $\mathcal{G}_\Omega$ of Ω.
Step III For every grid point $\mu \in \mathcal{G}_\Omega$, repeat the following:
(a) check if $\mu \notin W_q(P)$, or equivalently, if $0 \notin F_q(P(\mu))$,
(b) if $0 \notin F_q(P(\mu))$, then compute the inner q-numerical radius $\hat{r}_q(P(\mu))$ and the matrices

$$B_j \;=\; \frac{1}{j\,!}\, P^{(j)}(\mu) \;\;;\;\; j = 0, 1, \ldots, m,$$

(c) construct the open circular disk $S(\mu, \rho_\mu) \subset \mathbb{C} \setminus W_q(P)$ with radius

$$\rho_\mu \;=\; \frac{\hat{r}_q(B_0)}{\hat{r}_q(B_0) + \max_{j=1,\ldots,m} \|B_j\|_2}.$$

Step IV The set

$$\Omega \setminus \bigcup_{\substack{\mu \,\in\, \mathcal{G}_\Omega \\ 0 \,\notin\, F_q(P(\mu))}} S(\mu, \rho_\mu)$$

is an approximation of $W_q(P)$ and always contains $W_q(P)$.

An important feature of the above methodology is that it does not depend strongly on the degree m of $P(\lambda)$, which appears only in the computation of the matrices B_j ($j = 0, 1, \ldots, m$). Note also that the most expensive part of Algorithm 1 is the calculation of the inner q-numerical radius $\hat{r}_q(P(\mu))$ in Step III (b). This step will be further discussed in Sections 4 and 5.

The two inclusion regions given by (2.1) are not always satisfactory, since they are centered at the origin. By a simple MATLAB code, one can plot the roots of a few polynomials of the form $y^*P(\lambda)x$, where $x, y \in \mathbb{C}^n$ are unit vectors satisfying $y^*x = q$. In this way, a first approach of an open rectangle $\Omega = (u_{\min}, u_{\max}) \times (\mathrm{i}\, v_{\min}, \mathrm{i}\, v_{\max})$

($u_{\min}, u_{\max}, v_{\min}, v_{\max} \in \mathbb{R}$), which contains $W_q(P)$, is obtained. After choosing Ω, we can construct either a constant or a variable grid. In our experiments, we use two grids, where we move rightwards on each grid line and upwards on each grid column. The first grid, denoted by $\mathcal{G}_{\Omega,1}(\xi)$, is formed by a partition of the intervals $(u_{\min}, u_{\max})$ and $(v_{\min}, v_{\max})$ with constant length ξ, where all grid points within the disks generated by Step III (c) are excluded. The second one, denoted by $\mathcal{G}_{\Omega,2}(\xi)$, is formed by a partition of the interval $(u_{\min}, u_{\max})$ with constant length ξ and a partition of the interval $(v_{\min}, v_{\max})$ with variable length $\max\{\xi, \rho_\mu\}$, where ρ_μ is the radius of each disk generated by Step III (c).

3. Approximating the boundary. Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) and let $q \in (0,1]$. For a $\mu \in \mathbb{C} \setminus W_q(P)$, recall the open disk $S(\mu, \rho_\mu)$ in Theorem 2.1 and consider a positive real $M_\mu \geq \max_{j=1,\ldots,m} \|\frac{1}{j!} P^{(j)}(\mu)\|_2$. Then from the relation

$$\frac{\hat{r}_q(P(\mu))}{\hat{r}_q(P(\mu)) + M_\mu} \;\leq\; \rho_\mu \;\leq\; \frac{\hat{r}_q(P(\mu))}{\hat{r}_q(P(\mu)) + 1}\,,$$

it follows that

$$\rho_\mu \;<\; \hat{r}_q(P(\mu)) \;\leq\; \frac{\rho_\mu\, M_\mu}{1-\rho_\mu}\,, \tag{3.1}$$

and it is clear that the radius ρ_μ is small if and only if the inner q-numerical radius of $P(\mu)$ is sufficiently small.

Denote now by $d(P,\mu)$ the distance between μ and the compact set $W_q(P)$. By Theorem 2.1, this distance is greater than or equal to ρ_μ. Moreover, we have the following result.

THEOREM 3.1. *For an $n \times n$ matrix polynomial $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ and $q \in (0,1]$, consider $\mu \in \mathbb{C} \setminus W_q(P)$ and the disk $S(\mu, \rho_\mu)$ defined in Theorem 2.1. Suppose that for two unit vectors $x_0, y_0 \in \mathbb{C}^n$ such that $y_0^* x_0 = q$ and $|y_0^* P(\mu) x_0| = \hat{r}_q(P(\mu))$, we have $y_0^* P^{(1)}(\mu) x_0 \neq 0$. Then for sufficiently small ρ_μ,*

$$d(P,\mu) \;\leq\; \frac{2\,\rho_\mu\, M_\mu}{(1-\rho_\mu)\,|y_0^* P^{(1)}(\mu) x_0|}\,.$$

Proof. Consider the two unit vectors $x_0, y_0 \in \mathbb{C}^n$ satisfying

$$y_0^* x_0 \;=\; q, \quad |y_0^* P(\mu) x_0| = \hat{r}_q(P(\mu)), \quad \text{and} \quad y_0^* P^{(1)}(\mu) x_0 \neq 0.$$

Then there is a real $\delta > 0$ such that for any $\lambda \in \overline{S}(\mu, \delta)$,

$$\begin{aligned} y_0^* P(\lambda) x_0 &= y_0^* \{P(\mu) + (\lambda-\mu) P^{(1)}(\mu) + (\lambda-\mu) R(\lambda,\mu)\} x_0 \\ &= y_0^* P(\mu) x_0 + (\lambda-\mu)\left(y_0^* P^{(1)}(\mu) x_0 + y_0^* R(\lambda,\mu) x_0\right), \end{aligned} \tag{3.2}$$

where $\|R(\lambda,\mu)\| = o(1)$ as $|\lambda-\mu| \to 0$. Since $y_0^* P^{(1)}(\mu) x_0 \neq 0$, δ can be chosen so small that $|y_0^* P^{(1)}(\mu) x_0| \geq 2\,|y_0^* R(\lambda,\mu) x_0|$ for every $\lambda \in \overline{S}(\mu,\delta)$. Furthermore, for sufficiently small ρ_μ, by (3.1), we can assume that

$$\hat{r}_q(P(\mu)) \;\leq\; \delta\, |y_0^* P^{(1)}(\mu) x_0 + y_0^* R(\lambda,\mu) x_0|.$$

Then, since (3.2) holds for every $\lambda \in \overline{S}(\mu,\delta)$, there exists a $\lambda_0 \in \overline{S}(\mu,\delta)$ such that $y_0^* P(\lambda_0) x_0 = 0$, i.e., $\lambda_0 \in W_q(P)$. Thus,

$$d(P,\mu) \;\leq\; |\lambda_0 - \mu| \;=\; \frac{\hat{r}_q(P(\mu))}{|y_0^* P^{(1)}(\mu) x_0 + y_0^* R(\lambda_0,\mu) x_0|} \;\leq\; \frac{2\,\hat{r}_q(P(\mu))}{|y_0^* P^{(1)}(\mu) x_0|}\,,$$

and the proof is completed by (3.1). □

Notice that for any $\mu \notin W_q(P)$, the origin does not belong to the q-numerical range of the matrix $P(\mu)$ and there are infinitely many pairs of unit vectors $x_0, y_0 \in \mathbb{C}^n$ such that $y_0^* x_0 = q$ and $|y_0^* P(\mu) x_0| = \hat{r}(P(\mu))$. If one of these pairs also satisfies $y_0^* P^{(1)}(\mu) x_0 \neq 0$, then Theorem 3.1 and the inequality $d(P,\mu) \geq \rho_\mu$ imply that the radius ρ_μ is small if and only if the point μ is sufficiently close to the boundary of $W_q(P)$. Recall also that the function $\max_{j=1,\ldots,m} \|\frac{1}{j!} P^{(j)}(\lambda)\|_2$ is bounded. Hence, if we choose a sufficiently small $\epsilon > 0$, then we can approximate the boundary $\partial W_q(P)$ by using Algorithm 1 and plotting the disks $S(\mu, \rho_\mu) \subset \Omega \setminus W_q(P)$ with radius $\rho_\mu < \epsilon$ (see Examples 4.2 and 5.1 below).

4. The case $q = 1$. Let $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be an $n \times n$ monic matrix polynomial and let $q = 1$. It is clear that for the estimation of the numerical range $W(P)$ $(\equiv W_1(P))$ via the algorithm in Section 2, a method for the computation of the inner numerical radius of an $n \times n$ complex matrix is needed. By [1, Theorem 2.1] (see also [4, Chapter 1.5]), the inner numerical radius of a matrix $A \in \mathcal{M}_n$ is given by

$$\hat{r}(A) \equiv \hat{r}_1(A) = \left| \min_{\theta \in [0,2\pi]} \lambda_{\max} \left(\frac{e^{\mathrm{i}\theta} A + e^{-\mathrm{i}\theta} A^*}{2} \right) \right|,$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a Hermitian matrix. As a consequence, the following procedure produces $\hat{r}(A)$ and classifies the origin as belonging to $F(A) \equiv F_1(A)$ or not.

Algorithm 2

Step I Construct a grid

$$\left\{ \theta = \frac{j\,\pi}{N} : j = 0, 1, \ldots, 2N-1 \right\}$$

of the interval $[0, 2\pi]$ for some positive integer N.

Step II For every choice of θ, compute the largest eigenvalue $\lambda_{\max}(H(\theta))$ of the Hermitian matrix

$$H(\theta) = \frac{1}{2} \left(e^{\mathrm{i}\theta} A + e^{-\mathrm{i}\theta} A^* \right).$$

Step III Find the minimum of $\lambda_{\max}(H(\theta))$ and the corresponding angle θ_0. The absolute value of this minimum is an approximation of the inner numerical radius $\hat{r}(A)$. Moreover, the origin does not belong to the numerical range $F(A)$ if and only if the eigenvalue $\lambda_{\max}(H(\theta_0))$ is negative.

The above algorithm can be used for Step III (a),(b) of Algorithm 1. As a consequence, we have a complete methodology for the approximation of the numerical range $W(P)$, which is illustrated in the following examples.

EXAMPLE 4.1. For the matrix polynomial

$$P_1(\lambda) = I\lambda^3 + \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & \mathrm{i} \\ 1 & -1 \end{bmatrix} \lambda + \begin{bmatrix} 10 & 5 \\ \mathrm{i}\,8 & 7 \end{bmatrix},$$

the roots of a few thousand (randomly chosen) polynomials of the form $x^* P_1(\lambda) x$ ($x \in \mathbb{C}^2$, $x^*x = 1$) are sketched in the left part of Figure 4.1. Observe that we do not have a clear picture of $W(P_1)$, and that three eigenvalues of $P_1(\lambda)$ (marked with '+') appear to lie out of $W(P_1)$. Moreover, it seems that $W(P_1)$ lies in the open rectangle $\Omega = (-3, 3) \times (-\mathrm{i}\,2.5, \mathrm{i}\,3)$.

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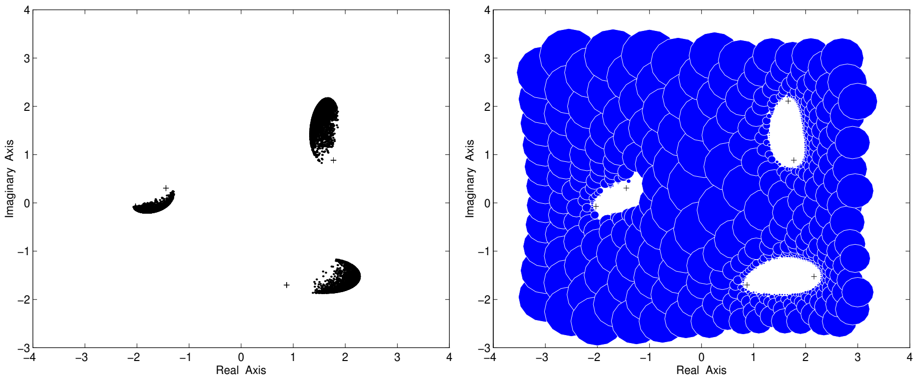

FIG. 4.1. *A numerical range with three connected components.*

For the grid $\mathcal{G}_{\Omega,1}(0.05)$, by Algorithms 1 and 2, an approximation of the numerical range $W(P_1)$ is drawn in the right part of the figure. The number of disks is 1076, and now we can see that $W(P_1)$ has three connected components and contains the spectrum $\sigma(P_1)$ in its interior.

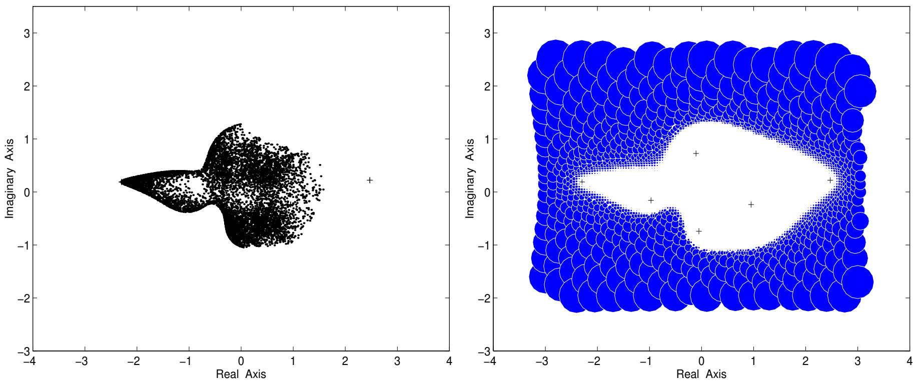

FIG. 4.2. *A connected numerical range.*

EXAMPLE 4.2. For the matrix polynomial

$$P_2(\lambda) = I\lambda^3 + \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & \mathrm{i} \\ 1 & -1 \end{bmatrix} \lambda + \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix},$$

the roots of a few thousand (randomly chosen) polynomials of the form $x^*P_2(\lambda)x$ ($x \in \mathbb{C}^2$, $x^*x = 1$) are sketched in the left part of Figure 4.2. We have a clear picture only for the left part of $W(P_2)$ and an eigenvalue of $P_2(\lambda)$ (marked with '+') appears to lie out of $W(P_2)$. Using our methodology and choosing the grid $\mathcal{G}_{\Omega,1}(0.05)$ of the inclusion domain $\Omega = (-3, 3) \times (-\mathrm{i}\,2, \mathrm{i}\,2.5)$, an estimation of the numerical range $W(P_2)$ is drawn in the right part of the figure and gives a better visualization of $W(P_2)$. The total number of disks is 3812. With respect to the same grid, in Figure 4.3, we plot exactly the open disks $S(\mu, \rho_\mu) \subset \Omega \setminus W(P_2)$ with radius $\rho_\mu < \epsilon$ for $\epsilon = 0.08$ (left part) and $\epsilon = 0.04$ (right part). In this way, we approach the boundary of $W(P_2)$, confirming Theorem 3.1 and the discussion in the previous section.

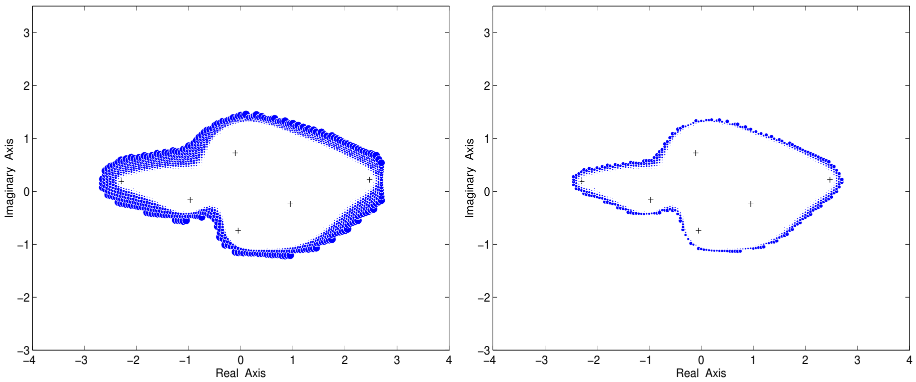

FIG. 4.3. *Approximating the boundary of $W(P_2)$.*

The above method for the estimation of the numerical range of $P(\lambda)$ can be considered reliable when $W(P)$ is a *regular closed set*, i.e., when it coincides with the closure of its interior, and when the boundary $\partial W(P)$ is smooth. If $W(P)$ is not a regular closed set and/or $\partial W(P)$ contains non-differentiable points, then our algorithm may become less satisfactory (locally), as one can see in the next example.

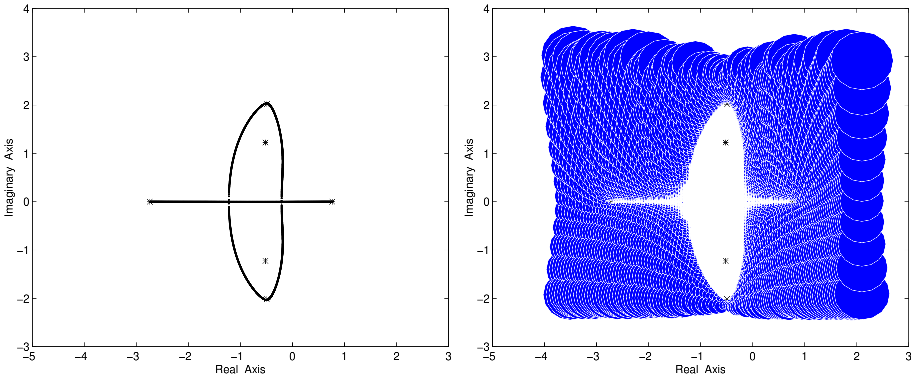

FIG. 4.4. *A numerical range with line segments.*

EXAMPLE 4.3. The boundary of the numerical range of the quadratic matrix polynomial

$$P_3(\lambda) = I\lambda^2 + \begin{bmatrix} 1 & \mathrm{i} & 0 \\ -\mathrm{i} & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

(with Hermitian coefficients) is accurately sketched in the left part of Figure 4.4 by an algorithm described in [7]. In the right part of the same figure, choosing $\Omega = (-3.5, 2.5) \times (-\mathrm{i}\,2.5, \mathrm{i}\,3)$ and its grid $\mathcal{G}_{\Omega,2}(0.03)$, $W(P_3)$ is approximated by Algorithms 1 and 2 (the number of disks is 8356). In both parts of the figure, the eigenvalues of $P_3(\lambda)$ are marked with asterisks, and we remark that the non-real part of the boundary of $W(P_3)$ intersects the real axis orthogonally [7, Theorem 5.2]. Clearly, the existence of real intervals and non-differentiable points on the boundary $\partial W(P_3)$ affects the accuracy of the methodology.

5. The case $0 < q < 1$. In this section, we assume that $0 < q < 1$. In [9], Li and Nakazato describe an algorithm for drawing the q-numerical range of an $n \times n$ complex matrix A. Their method is based on the construction of the (convex) surface of the *Davis-Wielandt shell* of A, that is,

$$WD(A) = \{(x^*Ax, x^*A^*Ax) \in \mathbb{C} \times \mathbb{R} : x \in \mathbb{C}^n,\ x^*x = 1\},$$

and the relation

$$F_q(A) = \bigcup \left\{ \overline{S}\left(q\,z, \sqrt{(1-q^2)(h-|z|^2)}\right) : (z,h) \in \partial WD(A) \right\}, \tag{5.1}$$

and can be used for the calculation of the inner q-numerical radius $\hat{r}_q(A)$ and for verifying whether the origin belongs to $F_q(A)$ or not.

Algorithm 3

Step I Construct a grid on the unit sphere in $\mathbb{R}^3$ using spherical coordinates

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

with

$$\theta = \frac{\pi}{N}, \frac{2\pi}{N}, \dots, \frac{(N-1)\pi}{N}, \pi \quad \text{and} \quad \phi = \frac{\pi}{S}, \frac{2\pi}{S}, \dots, \frac{(2S-1)\pi}{S}, 2\pi$$

for some positive integers N and S.

Step II For every choice of $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, repeat the following:

(a) compute the largest eigenvalue $\lambda_{\max}(H(\theta,\phi))$ of the Hermitian matrix

$$H(\theta,\phi) = (\sin\theta\cos\phi)\left(\frac{A+A^*}{2}\right) + (\sin\theta\sin\phi)\left(\frac{A-A^*}{2\,\mathrm{i}}\right) + (\cos\theta)(A^*A),$$

a corresponding unit eigenvector $y_{\theta,\phi} \in \mathbb{C}^n$ and the scalars

$$z(\theta,\phi) = y_{\theta,\phi}^* A\, y_{\theta,\phi} \quad \text{and} \quad h(\theta,\phi) = y_{\theta,\phi}^* A^* A\, y_{\theta,\phi},$$

(b) consider the closed circular disk

$$D(\theta,\phi) = \overline{S}\left(q\,z(\theta,\phi), \sqrt{(1-q^2)\,(h(\theta,\phi) - |z(\theta,\phi)|^2)}\right),$$

and notice that if $|z(\theta,\phi)|^2 \le \frac{1-q^2}{q^2}\left(h(\theta,\phi) - |z(\theta,\phi)|^2\right)$, then the origin belongs to $F_q(A)$.

Step III By (5.1), the union $\bigcup_{\theta,\phi} D(\theta,\phi)$ is an approximation of $F_q(A)$. If $0 \notin \bigcup_{\theta,\phi} D(\theta,\phi)$, then we say that $0 \notin F_q(A)$ and

$$\hat{r}_q(A) \cong \min_{\theta,\phi}\left\{ q\,|z(\theta,\phi)| - \sqrt{(1-q^2)\,(h(\theta,\phi) - |z(\theta,\phi)|^2)} \right\}.$$

If $P(\lambda)$ is a monic matrix polynomial as in (1.1) and $0 < q_2 < q_1 \le 1$, then it is known that $W_{q_1}(P) \subseteq W_{q_2}(P)$ [15, Theorem 10]. This result is confirmed by comparing Figure 4.1 with the figure in the following example.

EXAMPLE 5.1. Recall the matrix polynomial $P_1(\lambda)$ in Example 4.1. In the left part of Figure 5.1, the roots of a few thousand (randomly chosen) polynomials of the form $y^*P(\lambda)x$ ($x, y \in \mathbb{C}^2$, $x^*x = y^*y = 1$, $y^*x = 0.7$) are plotted. Observe the gaps around two eigenvalues of $P_1(\lambda)$ (marked with '+') and that $W_{0.7}(P_1)$ appears to lie in the open

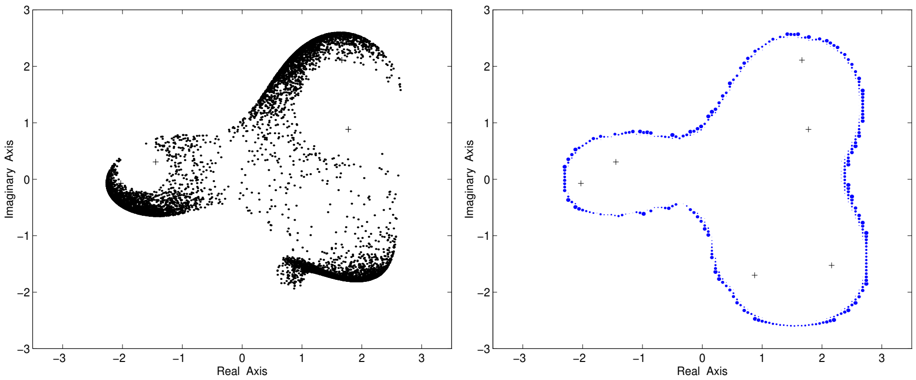

FIG. 5.1. *The* 0.7*-numerical range of* $P_1(\lambda)$.

square $\Omega = (-3,3) \times (-\mathrm{i}\,3, \mathrm{i}\,3)$. In the right part of Figure 5.1, using Algorithms 1 and 3, we sketch an approximation of the boundary of $W_{0.7}(P_1)$ by drawing 638 open disks $S(\mu,\rho_\mu) \subset \Omega \setminus W_{0.7}(P_1)$ with centers in $\mathcal{G}_{\Omega,2}(0.06)$ and radius $\rho_\mu < 0.04$. It is easy to verify that $W_{0.7}(P_1)$ is connected with smooth boundary and contains the numerical range $W(P_1)$ in Figure 4.1.

Algorithm 1 is simple and robust, but it is also expensive (even for medium sized matrix polynomials) because of the required computations in Step III (b). Furthermore, Algorithm 2 is based on polar rotations (two dimensional) rather than spherical rotations (three dimensional), which are required by Algorithm 3. Consequently, the cost of Algorithm 3 is much higher than the cost of Algorithm 2. So, the problem of the design of less expensive algorithms for the computation of the inner q-numerical radius of a general square matrix is still challenging.

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