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COLLOCATION METHODS FOR CAUCHY SINGULAR INTEGRAL EQUATIONS ON THE INTERVAL*

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Abstract. In this paper we consider polynomial collocation methods for the numerical solution of a singular integral equation over the interval, where the operator of the equation is supposed to be of the form $aI+b\mu^{-1}S\mu I+K$ with S the Cauchy singular integral operator, with piecewise continuous coefficients a and b, and with a Jacobi weight μ. K denotes an integral operator with a continuous kernel function. To the integral equation we apply two collocation methods, where the collocation points are the Chebyshev nodes of the first and second kind and where the trial space is the space of polynomials multiplied by another Jacobi weight. For the stability and convergence of this collocation scheme in weighted L^2-spaces, we derive necessary and sufficient conditions. Moreover, we discuss stability of operator sequences belonging to algebras generated by the sequences of the collocation methods for the above described operators. Finally, the so-called splitting property of the singular values of the sequences of the matrices of the discretized equations is proved.

Key words. Cauchy singular integral equation, polynomial collocation method, stability, singular values, splitting property.

AMS subject classifications. 45L10, 65R20, 65N38.

1. Introduction and preliminaries. The present paper can be considered as an immediate continuation of [7], where the stability of the collocation method with respect to Chebyshev nodes of second kind for Cauchy singular integral equations (CSIEs) is investigated. Here we purpose, firstly, to establish analogous results for collocation with respect to Chebyshev nodes of first kind (and to compare them with the results of [7]) and, secondly, to study the stability of operator sequences belonging to an algebra generated by the sequences of the collocation methods applied to Cauchy singular integral operators (CSIOs). Moreover, we will be able to prove results on the singular value distribution of the respective matrix sequences related to the collocation methods.

A function $a:[-1,1]\longrightarrow\mathbb{C}$ is called piecewise continuous if it has one-sided limits $a(x\pm 0)$ for all $x\in(-1,1)$ and is continuous at ± 1. For definiteness, we assume that the function values coincide with the limits from the left. The set of piecewise continuous functions on $[-1,1]$ is denoted by **PC**.

We analyze polynomial collocation methods for CSIEs on the interval $(-1,1)$ of the type

$$a(x)u(x)+\frac{b(x)}{\mu(x)}\frac{1}{\pi\mathrm{i}}\int_{-1}^{1}\frac{\mu(y)u(y)}{y-x}\,dy+\int_{-1}^{1}k(x,y)u(y)\,dy=f(x)\,, \tag{1.1}$$

where $a,b:[-1,1]\longrightarrow\mathbb{C}$ stand for given piecewise continuous functions, where the weight function μ is of the form $\mu(x)=v^{\gamma,\delta}(x):=(1-x)^{\gamma}(1+x)^{\delta}$ with real numbers $-1<\gamma,\delta<1$, where the kernel $k:(-1,1)\times(-1,1)\longrightarrow\mathbb{C}$ is supposed to be continuous (comp. Lemma 2.10), where the right-hand side function f is assumed to belong to a weighted L^2-space $\mathbf{L}^2_\nu$, and where $u\in\mathbf{L}^2_\nu$ stands for the unknown solution. The Hilbert space $\mathbf{L}^2_\nu$ is defined as the space of all (classes of) functions $u:(-1,1)\longrightarrow\mathbb{C}$ which are square integrable with respect

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to the weight $\nu = v^{\alpha,\beta}$, $-1 < \alpha, \beta < 1$. The inner product in this space is defined by

$$\langle u, v \rangle_\nu := \int_{-1}^{1} u(x)\overline{v(x)}\nu(x)\, dx$$

and the norm by $\|u\|_\nu := \sqrt{\langle u, u \rangle_\nu}$. In short operator notation (1.1) takes the form

$$Au := (aI + b\mu^{-1}S\mu I + K)u = f. \tag{1.2}$$

Here $aI : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ denotes the multiplication operator defined by $(au)(x) := a(x)u(x)$, the operator $S : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is the CSIO given by

$$(Su)(x) := \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{u(y)}{y - x}\, dy,$$

and $K : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ stands for the integral operator with kernel $k(x, y)$. Note that the condition $-1 < \alpha, \beta < 1$ for the exponents of the classical Jacobi weight $\nu(x)$ guarantees that the CSIO $S : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is continuous, i.e. $S \in \mathcal{L}(\mathbf{L}^2_\nu)$ (see [3]).

Let $\sigma(x) = (1 - x^2)^{-\frac{1}{2}}$ and $\varphi(x) = (1 - x^2)^{\frac{1}{2}}$ denote the Chebyshev weights of first and second kind, respectively. For the numerical solution of the CSIE (1.2), we consider the polynomial collocation method

$$a(x^\tau_{jn})u_n(x^\tau_{jn}) + \frac{b(x^\tau_{jn})}{\mu(x^\tau_{jn})} \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{\mu(y)u_n(y)}{y - x^\tau_{jn}}\, dy + \int_{-1}^{1} k(x^\tau_{jn}, y)u_n(y)\, dy = f(x^\tau_{jn}),$$

$j = 1, \ldots, n$, where the collocation points x^τ_{jn} are chosen as the Chebyshev nodes $x^\sigma_{jn} = \cos \frac{2j-1}{2n}\pi$ of first kind or $x^\varphi_{jn} = \cos \frac{j\pi}{n+1}$ of second kind and where the trial function u_n is sought in the space of all functions $u_n = \vartheta p_n$ with a polynomial p_n of degree less than n and with the Jacobi weight $\vartheta = v^{\frac{1}{4}-\frac{\alpha}{2}, \frac{1}{4}-\frac{\beta}{2}}$. We write the above method in operator form as

$$A_n u_n = M_n f\,, \quad u_n \in \operatorname{im} L_n\,. \tag{1.3}$$

Here L_n denotes the orthogonal projection of $\mathbf{L}^2_\nu$ onto the n dimensional trial space $\operatorname{im} L_n$ of all polynomials of degree less than n multiplied by ϑ . By $M_n = M^\tau_n$ we denote the interpolation projection defined by $M_n f \in \operatorname{im} L_n$ and $(M_n f)(x^\tau_{jn}) = f(x^\tau_{jn})$, $j = 1, \ldots, n$. Finally, the discretized integral operator $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ is given by $A_n := M_n A|_{\operatorname{im} L_n}$. In accordance with e.g. [11], we call the collocation method stable if the operators A_n are invertible at least for all sufficiently large n and if the norms of the inverse operators A_n^{-1} are bounded uniformly with respect to n . Of course, the norm is the operator norm in the space $\operatorname{im} L_n$ if the last is equipped with the restriction of the $\mathbf{L}^2_\nu$-norm. We call the method (1.3) convergent if, for any right-hand side $f \in \mathbf{L}^2_\nu$ and for any approximating sequence $\{f_n\}$, $f_n \in \operatorname{im} L_n$, with $\|f - f_n\|_\nu \longrightarrow 0$, the approximate solutions u_n obtained by solving $A_n u_n = f_n$ converge to the exact solution u of (1.2) in the norm of $\mathbf{L}^2_\nu$. Note that the stability implies bounded condition numbers for the matrix representation of A_n in a convenient basis, and, together with the consistency relation $A_n L_n \longrightarrow A$, it implies the convergence.

In all what follows, for the exponents in the weight functions μ and ν , we suppose

$$-1 < \alpha - 2\gamma < 1\,, \quad -1 < \beta - 2\delta < 1\,, \tag{1.4}$$

and

$$\alpha_0 := \gamma + \frac{1}{4} - \frac{\alpha}{2} \neq 0\,, \quad \beta_0 := \delta + \frac{1}{4} - \frac{\beta}{2} \neq 0\,. \tag{1.5}$$

Note that condition (1.4) ensures the boundedness of the integral operator $A \in \mathcal{L}(\mathbf{L}^2_\nu)$ whereas (1.5) is needed to derive strong limits for the discrete operators (see Lemma 3.4).

In the subsequent analysis, we will show that there exist four limit operators $W_\omega\{A_n\}$, $\omega = 1,2,3,4$, introduced in the Lemmata 3.2–3.4. Moreover, we show that the mappings $\{A_n\} \mapsto W_\omega\{A_n\}$ can be extended to *-homomorphisms $W_\omega : \mathcal{A}_0 \longrightarrow \mathcal{L}(\mathbf{L}^2_\nu)$, where $\mathcal{A}_0$ denotes a C^*-algebra of operator sequences including all sequences $\{M_n(aI + b\mu^{-1}S\mu I)L_n\}$, $a, b \in \mathbf{PC}$. The invertibility of $W_\omega\{A_n\}$, $\omega = 1,2,3,4$, will turn out to be necessary and sufficient for the stability of $\{A_n\} \in \mathcal{A}_0$.

2. A C^*-algebra of operator sequences and stability. In this section we will introduce one of the C^*-algebras of operator sequences under consideration here. For $n \geq 0$, let $p_n^\sigma = T_n$ and $p_n^\varphi = U_n$ stand for the orthonormal polynomials of degree n with respect to the weight functions σ and φ, respectively. That means that

$$T_0(x) = \frac{1}{\sqrt{\pi}}, \quad T_n(\cos s) = \sqrt{\frac{2}{\pi}}\cos ns, \quad n \geq 1,\ s \in (0,\pi),$$

and

$$U_n(\cos s) = \sqrt{\frac{2}{\pi}}\frac{\sin(n+1)s}{\sin s}, \quad n \geq 0,\ s \in (0,\pi).$$

We set

$$\widetilde{u}_n(x) := \vartheta(x)U_n(x), \quad n = 0,1,2,\dots,$$

with $\vartheta = \sqrt{\nu^{-1}\varphi} = v^{\frac{1}{4}-\frac{\alpha}{2},\frac{1}{4}-\frac{\beta}{2}}$. Then the solution of (1.3) can be represented by

$$u_n(x) = \sum_{k=0}^{n-1} \xi_{kn}\widetilde{u}_k(x),$$

and, with respect to the orthonormal system $\{\widetilde{u}_n\}_{n=0}^\infty$ in $\mathbf{L}^2_\nu$, the orthogonal projection L_n takes the form

$$L_n u = \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k\rangle_\nu \widetilde{u}_k.$$

The interpolation operator $M_n = M_n^\tau$ can be written as $M_n^\tau = \vartheta L_n^\tau \vartheta^{-1} I$, where L_n^τ denotes the polynomial interpolation operator with respect to the nodes $x_{jn} = x_{jn}^\tau$, $j = 1,\dots,n$. By ℓ^2 we denote the Hilbert space of all square summable sequences $\xi = \{\xi_k\}_{k=0}^\infty$ of complex numbers equipped with the inner product

$$\langle \xi, \eta\rangle_{\ell^2} := \sum_{k=0}^\infty \xi_k \overline{\eta}_k.$$

Finally, we introduce the Christoffel numbers with respect to the weights σ and φ by

$$\lambda_{kn}^\sigma := \frac{\pi}{n},\ \lambda_{kn}^\varphi := \frac{\pi[\varphi(x_{kn}^\varphi)]^2}{n+1}, \quad k = 1,\dots,n,$$

and the discrete weights

$$\omega_{kn}^\sigma := \sqrt{\frac{\pi}{n}} v^{\frac{1}{4}+\frac{\alpha}{2},\frac{1}{4}+\frac{\beta}{2}}(x_{kn}^\sigma),\ \omega_{kn}^\varphi := \sqrt{\frac{\pi}{n+1}} v^{\frac{1}{4}+\frac{\alpha}{2},\frac{1}{4}+\frac{\beta}{2}}(x_{kn}^\varphi), \quad k = 1,\dots,n.$$

The proof of the approximation properties of the interpolation operators M_n is based on the following auxiliary results.

LEMMA 2.1 ([10],Theorem 9.25). *Let μ, ν be classical Jacobi weights with $\mu\nu \in \mathbf{L}^1(-1,1)$ and let $j \in \mathbb{N}$ be fixed. Then for each polynomial q with $\deg q \le jn$,*

$$\sum_{k=1}^{n} \lambda_{kn}^{\mu} |q(x_{kn}^{\mu})|\,|\nu(x_{kn}^{\mu})| \le \text{const} \int_{-1}^{1} |q(x)|\mu(x)\nu(x)\,dx,$$

where the constant does not depend on n and q and where x_{kn}^{μ} and

$$\lambda_{kn}^{\mu} = \int_{-1}^{1} \ell_{kn}^{\mu}(x)\mu(x)\,dx \quad \text{with} \quad \ell_{kn}^{\mu}(x) = \prod_{j=1, j\neq k} \frac{x - x_{jn}^{\mu}}{x_{kn}^{\mu} - x_{jn}^{\mu}}$$

are the nodes and the Christoffel numbers of the Gaussian rule with respect to the weight μ, respectively.

Let Q_n^{μ} denote the Gaussian quadrature rule with respect to the weight μ,

$$Q_n^{\mu} f = \sum_{k=1}^{n} \lambda_{kn}^{\mu} f(x_{kn}^{\mu}),$$

and write $\mathbf{R} = \mathbf{R}(-1,1)$ for the set of all functions $f : (-1,1) \longrightarrow \mathbb{C}$, which are bounded and Riemann integrable on each closed subinterval of $(-1,1)$.

LEMMA 2.2 ([2], Satz III.1.6b and Satz III.2.1). *Let $\mu(x) = (1-x)^{\gamma}(1+x)^{\delta}$ with $\gamma, \delta > -1$. If $f \in \mathbf{R}$ satisfies*

$$|f(x)| \le \text{const}\,(1-x)^{\varepsilon-1-\gamma}(1+x)^{\varepsilon-1-\delta}, \qquad -1 < x < 1,$$

for some $\varepsilon > 0$, then $\lim\limits_{n\to\infty} Q_n^{\mu} f = \int_{-1}^{1} f(x)\mu(x)\,dx$. If even

$$|f(x)| \le \text{const}\,(1-x)^{\varepsilon-\frac{1+\gamma}{2}}(1+x)^{\varepsilon-\frac{1+\delta}{2}}, \qquad -1 < x < 1,$$

then $\lim\limits_{n\to\infty} \|f - L_n^{\mu} f\|_{\mu} = 0$.

COROLLARY 2.3. *Let $f \in \mathbf{R}$ and, for some $\varepsilon > 0$,*

$$|f(x)| \le \text{const}\,(1-x)^{\varepsilon-\frac{1+\alpha}{2}}(1+x)^{\varepsilon-\frac{1+\beta}{2}}, \quad -1 < x < 1.$$

Then $\lim\limits_{n\to\infty} \|f - M_n^{\tau} f\|_{\nu} = 0$ for $\tau = \sigma$ and $\tau = \varphi$.

Proof. Introduce the quadrature rule

$$Q_n f = \int_{-1}^{1} (L_n^{\sigma} f)(x)\varphi(x)\,dx = \sum_{k=1}^{n} \sigma_{kn} f(x_{kn}^{\sigma}),$$

where

$$\sigma_{kn} = \int_{-1}^{1} \ell_{kn}^{\sigma}(x)\varphi(x)\,dx = \int_{-1}^{1} \ell_{kn}^{\sigma}(x)(1-x^2)\sigma(x)\,dx = \frac{\pi}{n}[\varphi(x_{kn}^{\sigma})]^2$$

for $n > 2$. Consequently,

$$Q_n f = \frac{\pi}{n} \sum_{k=1}^{n} [\varphi(x_{kn}^{\sigma})]^2 f(x_{kn}^{\sigma}).$$

Since the nodes x_{kn}^σ of the quadrature rule Q_n are the zeros of $2T_n(x) = U_n(x) - U_{n-2}(x)$, the estimate

$$\int_{-1}^{1} |(L_n^\sigma f)(x)|^2 \varphi(x)\,dx \le 2Q_n|f|^2 \tag{2.1}$$

holds true (see [2, Hilfssatz 2.4, §III.2]). As an immediate consequence we obtain

$$\|M_n^\sigma f\|_\nu^2 = \|L_n^\sigma \vartheta^{-1} f\|_\varphi^2 \le \frac{2\pi}{n}\sum_{k=1}^{n} |\vartheta^{-1}(x_{kn}^\sigma)\varphi(x_{kn}^\sigma) f(x_{kn}^\sigma)|^2 = 2Q_n^\sigma|\vartheta^{-1}\varphi f|^2 . \tag{2.2}$$

Now let $\epsilon > 0$ be arbitrary and p be a polynomial such that $\|\vartheta p - f\|_\nu < \epsilon$. For $n > \deg p$ we have $\|M_n^\sigma f - f\|_\nu^2 \le 2\left(\|M_n^\sigma(\vartheta p - f)\|_\nu^2 + \|\vartheta p - f\|_\nu^2\right)$. Since, in view of Lemma 2.2, $\lim\limits_{n\to\infty} Q_n^\sigma|\vartheta^{-1}\varphi(\vartheta p - f)|^2 = \|\vartheta^{-1}\varphi(\vartheta p - f)\|_\sigma^2 = \|\vartheta p - f\|_\nu^2$, we get in view of (2.2) that $\limsup\limits_{n\to\infty} \|M_n^\sigma f - f\|_\nu^2 < 6\epsilon^2$.

The proof for the case $\tau = \varphi$ is analogous (see also [2, Satz III.2.1]). □

Now we start to prepare the definition of a certain C^*-algebra of operator sequences, which is closely related to the above mentioned four limit operators defined as strong limits

$$W_\omega\{A_n\} := \lim_{n\to\infty} E_n^{(\omega)} A_n (E_n^{(\omega)})^{-1} L_n^{(\omega)}, \quad \omega \in T := \{1,2,3,4\},$$

in some Hilbert spaces $\mathbf{X}_\omega$. Here, $L_n^{(\omega)} : \mathbf{X}_\omega \longrightarrow \mathbf{X}_\omega$ are projections and $E_n^{(\omega)} : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n^{(\omega)}$ are certain operators defined by

$$\mathbf{X}_1 := \mathbf{X}_2 := \mathbf{L}_\nu^2, \quad \mathbf{X}_3 := \mathbf{X}_4 := \ell^2, \quad L_n^{(1)} := L_n^{(2)} := L_n, \ L_n^{(3)} := L_n^{(4)} := P_n,$$

$$E_n^{(1)} := L_n, \ E_n^{(2)} := W_n, \ E_n^{(3)} = E_{n,\tau}^{(3)} := V_n = V_n^\tau, \ E_n^{(4)} = E_{n,\tau}^{(4)} := \widetilde{V}_n = \widetilde{V}_n^\tau,$$

and

$$P_n\{\xi_0,\xi_1,\xi_2,\ldots\} := \{\xi_0,\ldots,\xi_{n-1},0,0,0,\ldots\}, \quad W_n u := \sum_{k=0}^{n-1} \langle u, \widetilde{u}_{n-1-k}\rangle_\nu \widetilde{u}_k,$$

$$V_n^\tau u := \{\omega_{1n}^\tau u(x_{1n}^\tau), \ldots, \omega_{nn}^\tau u(x_{nn}^\tau), 0, 0, \ldots\},$$

$$\widetilde{V}_n^\tau u := \{\omega_{nn}^\tau u(x_{nn}^\tau), \ldots, \omega_{1n}^\tau u(x_{1n}^\tau), 0, 0, \ldots\}.$$

Immediately from the definitions, we conclude that

$$(E_n^{(1)})^{-1} = L_n, \quad (E_n^{(2)})^{-1} = W_n,$$

$$(E_{n,\tau}^{(3)})^{-1}\xi = \sum_{k=1}^{n} \frac{\xi_{k-1}}{\omega_{kn}^\tau}\,\widetilde{\ell}_{kn}^\tau, \quad (E_{n,\tau}^{(4)})^{-1}\xi = \sum_{k=1}^{n} \frac{\xi_{n-k}}{\omega_{kn}^\tau}\,\widetilde{\ell}_{kn}^\tau,$$

where

$$\widetilde{\ell}_{kn}^\tau(x) := \frac{\vartheta(x)}{\vartheta(x_{kn}^\tau)}\,\ell_{kn}^\tau(x) = \frac{\vartheta(x) p_n^\tau(x)}{\vartheta(x_{kn}^\tau)(x - x_{kn}^\tau)(p_n^\tau)'(x_{kn}^\tau)}.$$

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Between the operators V_n and $\widetilde{V}_n$, we have the relations

$$\widetilde{V}_n V_n^{-1} P_n = V_n \widetilde{V}_n^{-1} P_n = \widetilde{W}_n P_n , \tag{2.3}$$

where $\widetilde{W}_n \in \mathcal{L}(\mathrm{im} P_n)$ is defined by

$$\widetilde{W}_n \{\xi_0, \xi_1, \ldots, \xi_{n-1}\} = \widetilde{W}_n^{-1} \{\xi_0, \xi_1, \ldots, \xi_{n-1}\} = \{\xi_{n-1}, \xi_{n-2}, \ldots, \xi_0\} .$$

Furthermore, the operators $E_{n,\sigma}^{(\omega)}$, $\omega \in \{1, 2\}$, and $E_{n,\varphi}^{(\omega)}$, $\omega \in \{1, 2, 3, 4\}$, are unitary operators, i.e.

$$\left(E_{n,\tau}^{(\omega)}\right)^* = \left(E_{n,\tau}^{(\omega)}\right)^{-1} . \tag{2.4}$$

For $E_{n,\sigma}^{(\omega)}$, $\omega \in \{3, 4\}$, we have the following result.

LEMMA 2.4. *Let $V_n = V_n^\sigma$ and $\widetilde{V}_n = \widetilde{V}_n^\sigma$. Then*

$$\left(V_n^{-1}\right)^* = \frac{1}{2} V_n (L_n + L_{n-1}) , \quad \left(\widetilde{V}_n^{-1}\right)^* = \frac{1}{2} \widetilde{V}_n (L_n + L_{n-1}) ,$$

and, consequently,

$$V_n^* = \left(\left(V_n^{-1}\right)^*\right)^{-1} = (2L_n - L_{n-1}) V_n^{-1} , \ \widetilde{V}_n^* = \left(\left(\widetilde{V}_n^{-1}\right)^*\right)^{-1} = (2L_n - L_{n-1}) \widetilde{V}_n^{-1} .$$

Proof. For symmetry reasons, we may restrict our considerations to the operator $\left(V_n^{-1}\right)^*$. Let $j = 0, 1, \ldots, n-1$. Then

$$\left\langle V_n^{-1} \xi, u \right\rangle_\nu = \left\langle \sum_{k=1}^n \frac{\xi_{k-1}}{\omega_{kn}^\sigma \vartheta(x_{kn}^\sigma)} \, \vartheta \ell_{kn}^\sigma , \vartheta U_j \right\rangle_\nu = \left\langle \sum_{k=1}^n \frac{\xi_{k-1}}{\omega_{kn}^\sigma \vartheta(x_{kn}^\sigma)} \, \ell_{kn}^\sigma , \varphi^2 U_j \right\rangle_\sigma ,$$

and, for $j = 0, \ldots, n-2$, we obtain

$$\begin{aligned} \left\langle V_n^{-1} \xi, \widetilde{u}_j \right\rangle_\nu &= \frac{\pi}{n} \sum_{k=1}^n \frac{\xi_{k-1} [\varphi(x_{kn}^\sigma)]^2}{\omega_{kn}^\sigma \vartheta(x_{kn}^\sigma)} \, U_j(x_{kn}^\sigma) \\ &= \sum_{k=1}^n \xi_{k-1} \omega_{kn}^\sigma \vartheta(x_{kn}^\sigma) U_j(x_{kn}^\sigma) = \langle \xi, V_n \widetilde{u}_j \rangle_{\ell_2} . \end{aligned}$$

For $j = n-1$, using the relation

$$(1 - x^2) U_{n-1}(x) = \frac{1}{2} [\gamma_{n-1} T_{n-1}(x) - \gamma_{n+1} T_{n+1}(x)] , \tag{2.5}$$

where $\gamma_0 = \sqrt{2}$ and $\gamma_n = 1$ for $n \geq 1$, and the fact that

$$T_{n+1}(x_{kn}^\sigma) = -T_{n-1}(x_{kn}^\sigma) , \quad n > 1 , \tag{2.6}$$

we get, for $n > 1$,

$$\left\langle V_n^{-1} \xi, \widetilde{u}_{n-1} \right\rangle_\nu = \frac{1}{2} \left\langle \sum_{k=1}^n \frac{\xi_{k-1}}{\omega_{kn}^\sigma \vartheta(x_{kn}^\sigma)} \, \ell_{kn}^\sigma , T_{n-1} - T_{n+1} \right\rangle_\sigma$$

$$
\begin{aligned}
&= \frac{\pi}{2n}\sum_{k=1}^{n}\frac{\xi_{k-1}}{\omega_{kn}^{\sigma}\vartheta(x_{kn}^{\sigma})}\,T_{n-1}(x_{kn}^{\sigma}) \\
&= \frac{\pi}{2n}\sum_{k=1}^{n}\frac{\xi_{k-1}}{\omega_{kn}^{\sigma}\vartheta(x_{kn}^{\sigma})}[\varphi(x_{kn}^{\sigma})]^2U_{n-1}(x_{kn}^{\sigma}) \\
&= \frac{1}{2}\sum_{k=1}^{n}\xi_{k-1}\omega_{kn}^{\sigma}\vartheta(x_{kn}^{\sigma})U_{n-1}(x_{kn}^{\sigma}) \\
&= \frac{1}{2}\left\langle \xi, V_n\widetilde{u}_{n-1}\right\rangle_{\ell_2}\,.
\end{aligned}
$$

□

LEMMA 2.5. *The sequences* $\left\{E_n^{(\omega_1)}\big(E_n^{(\omega_2)}\big)^{-1}L_n^{(\omega_2)}\right\}$ *converge weakly to zero for all indices* $\omega_1,\omega_2\in T$ *with* $\omega_1\ne\omega_2$.

Proof. The proof for the case $\tau=\varphi$ one can find in [7, Lemma 2.1]. The case $\tau=\sigma$ can be dealt with completely analogous after checking the uniform boundedness of the sequences $\{V_n^{\sigma}\}$, $\{(V_n^{\sigma})^{-1}\}$, and $\{\widetilde{V}_n^{\sigma}\}$, $\{(\widetilde{V}_n^{\sigma})^{-1}\}$. But, this follows, by using Lemma 2.1, relation (2.2), and the notation $u=\vartheta p_n\in\operatorname{im}L_n$, from

$$
\begin{aligned}
\|V_n^{\sigma}u\|_{l^2}^2 &= \frac{\pi}{n}\sum_{k=1}^{n}\varphi^2(x_{kn}^{\sigma})|p_n(x_{kn}^{\sigma})|^2 \\
&\le \text{const}\int_{-1}^{1}\left|\frac{\vartheta(x)p_n(x)}{\vartheta(x)}\right|^2[\varphi(x)]^2\sigma(x)\,dx = \text{const}\,\|u\|_{\nu}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|(V_n^{\sigma})^{-1}\xi\right\|_{\nu}^2 &= \left\|\sum_{k=1}^{n}\xi_{k-1}\sqrt{\frac{n}{\pi}}\,\frac{\vartheta(x_{kn}^{\sigma})}{\varphi(x_{kn}^{\sigma})}\,\widetilde{\ell}_{kn}^{\sigma}\right\|_{\nu}^2 \\
&\le 2Q_n^{\sigma}\left|\sum_{k=1}^{n}\sqrt{\frac{n}{\pi}}\,\xi_{k-1}\widetilde{\ell}_{kn}^{\sigma}(x)\right|^2 = 2\sum_{k=1}^{n}|\xi_{k-1}|^2 = 2\|\xi\|_{\ell^2}^2\,.
\end{aligned}
$$

Analogously we get the uniform boundedness of the sequences $\left\{\widetilde{V}_n^{\sigma}\right\}$ and $\left\{(\widetilde{V}_n^{\sigma})^{-1}\right\}$. □

COROLLARY 2.6. *The sequences* $\left\{\big(E_n^{(\omega_1)}\big)^{-*}\big(E_n^{(\omega_2)}\big)^{*}L_n^{(\omega_2)}\right\}$ *converge weakly to zero for all indices* $\omega_1,\omega_2\in T$ *with* $\omega_1\ne\omega_2$.

Of course, all constructions in what follows depend on the choice of $\tau=\sigma$ or $\tau=\varphi$. Nevertheless, we will omit the subscript τ if there is no possibility of misunderstandings.

By $\mathcal{F}$ we denote the set of all sequences $\{A_n\}=\{A_n\}_{n=1}^{\infty}$ of linear operators $A_n:\operatorname{im}L_n\longrightarrow\operatorname{im}L_n$, for which there exist operators $W_\omega\{A_n\}\in\mathcal{L}(\mathbf{X}_\omega)$ such that, for all $\omega\in T$,

$$
\begin{aligned}
&E_n^{(\omega)}A_n\big(E_n^{(\omega)}\big)^{-1}L_n^{(\omega)}\longrightarrow W_\omega\{A_n\}\,, \\
&\left(E_n^{(\omega)}A_n\big(E_n^{(\omega)}\big)^{-1}L_n^{(\omega)}\right)^{*}\longrightarrow W_\omega\{A_n\}^{*}
\end{aligned}
\tag{2.7}
$$

holds in $\mathbf{X}_\omega$ in the sense of strong convergence for $n\longrightarrow\infty$. If we define, for $\lambda_1,\lambda_2\in\mathbb{C}$,

$$
\lambda_1\{A_n\}+\lambda_2\{B_n\}:=\{\lambda_1A_n+\lambda_2B_n\}\,,
$$

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$$\{A_n\}\{B_n\} := \{A_nB_n\}\,, \quad \{A_n\}^* := \{A_n^*\}\,,$$

and

$$\|\{A_n\}\|_{\mathcal{F}} := \sup\left\{\|A_nL_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} : n = 1, 2, \ldots\right\},$$

then it is not hard to see that $\mathcal{F}$ becomes a Banach algebra with unit element $\{L_n\}$. From Lemma 2.5 and Corollary 2.6 we conclude

COROLLARY 2.7. *For all $\omega \in T$ and all compact operators $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$, the sequences $\{A_n^{(\omega)}\} = \left\{(E_n^{(\omega)})^{-1}L_n^{(\omega)}T_\omega E_n^{(\omega)}\right\}$ belong to $\mathcal{F}$, and for $\omega_1 \neq \omega_2$, we get the strong limits*

$$E_n^{(\omega_1)}A_n^{(\omega_2)}(E_n^{(\omega_1)})^{-1}L_n^{(\omega_1)} \longrightarrow 0\,, \quad \left(E_n^{(\omega_1)}A_n^{(\omega_2)}(E_n^{(\omega_1)})^{-1}L_n^{(\omega_1)}\right)^* \longrightarrow 0\,.$$

COROLLARY 2.8. *The algebra $\mathcal{F}$ is a C^*-algebra and the mappings $W_\omega : \mathcal{F} \longrightarrow \mathcal{L}(\mathbf{X}_\omega)$, $\omega \in T$, are *-homomorphisms.*

Proof. Of course, the mappings $W_\omega : \mathcal{F} \longrightarrow \mathcal{L}(\mathbf{X}_\omega)$, $\omega \in T$, are homomorphisms. Hence, it suffices to show that the operator sequences $\{E_n^{(\omega)}A_n^*(E_n^{(\omega)})^{-1}L_n^{(\omega)}\}$ and the respective sequences of adjoint operators are strongly convergent for all sequences $\{A_n\} \in \mathcal{F}$ and that $W_\omega\{A_n^*\} = (W_\omega\{A_n\})^*$, $\omega \in T$. In case $(E_n^{(\omega)})^{-1} = (E_n^{(\omega)})^*$ this can be easily verified. Consequently, due to (2.4), it remains to consider the case $\tau = \sigma$, $\omega = 3, 4$.

For symmetry reasons we may restrict the proof to the case $\tau = \sigma$, $\omega = 3$. Let $\{A_n\} \in \mathcal{F}$. Using Lemma 2.4, the relation $L_n - L_{n-1} = W_nL_1W_n$, the compactness of $L_1 : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$, and Corollary 2.7, we get

$$\begin{aligned}
&V_nA_n^*V_n^{-1}P_n \\
&= \frac{1}{2}\left[V_n(2L_n - W_nL_1W_n)A_n(L_n + W_nL_1W_n)V_n^{-1}P_n\right]^* \\
&= \left(P_n + V_n^{-1}W_nL_1W_nV_n^{-1}P_n\right)^*\left(V_nA_nV_n^{-1}P_n\right)^*\frac{1}{2}\left(2P_n - V_n^{-1}W_nL_1W_nV_nP_n\right)^* \\
&\longrightarrow (W_3\{A_n\})^*\,.
\end{aligned}$$

The proof for the respective sequence $\{(V_nA_n^*V_n^{-1}P_n)^*\}$ is analogous. ◻

Using Corollary 2.7, we define the subset $\mathcal{J} \subset \mathcal{F}$, of all sequences of the form

$$\sum_{\omega=1}^{4}\left\{(E_n^{(\omega)})^{-1}L_n^{(\omega)}T_\omega E_n^{(\omega)}\right\} + \{C_n\}$$

where $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$ and where $\{C_n\}$ is in the ideal $\mathcal{N} \subset \mathcal{F}$ of all sequences $\{C_n\}$ tending to zero in norm, i.e. of all sequences with $\|C_nL_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \longrightarrow 0$. Now, the following theorem is crucial for our stability and convergence analysis.

THEOREM 2.9 ([11], Theorem 10.33). *The set $\mathcal{J}$ forms a two-sided closed ideal of $\mathcal{F}$. A sequence $\{A_n\} \in \mathcal{F}$ is stable if and only if the operators $W_\omega\{A_n\} : \mathbf{X}_\omega \longrightarrow \mathbf{X}_\omega$, $\omega \in T$, are invertible and if the coset $\{A_n\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$.*

Furthermore, we will need the auxiliary algebra $\mathcal{F}_2$ of sequences $\{A_n\}$ of linear operators $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$, for which (2.7) holds true for $\omega = 1, 2$. Moreover, we define the subset $\mathcal{J}_2 \subset \mathcal{F}_2$ of all sequences of the form

$$\sum_{\omega=1}^{2} \left\{ (E_n^{(\omega)})^{-1} L_n^{(\omega)} T_\omega E_n^{(\omega)} \right\} + \{C_n\}$$

where $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$ and where $\{C_n\}$ is in the ideal $\mathcal{N} \subset \mathcal{F}$. Obviously, the set $\mathcal{J}_2$ forms a two-sided closed ideal of $\mathcal{F}_2$, and $\mathcal{F} \subset \mathcal{F}_2$, $\mathcal{J}_2 \subset \mathcal{J}$.

In addition to the operator sequences corresponding to the collocation method applied to compact operators, the sequences of quadrature discretizations of integral operators with continuous kernels are contained in $\mathcal{J}$, too. Indeed, we can formulate the following lemma.

LEMMA 2.10. *Suppose the function $k(x,y)/\rho(y)$, where $\rho = \sqrt{\nu\varphi} = \vartheta^{-1}\varphi$, is continuous on $[-1,1] \times [-1,1]$ and that K is the integral operator with kernel $k(x,y)$. Then $\{M_n K L_n\} \in \mathcal{J}_2 \subset \mathcal{J}$. Moreover, if the approximations $K_n \in \mathcal{L}(\operatorname{im} L_n)$ are defined by*

$$K_n = (E_n^{(3)})^{-1} \Big(\widetilde{\omega}_n^\tau k(x_{j+1,n}^\tau, x_{k+1,n}^\tau) \rho(x_{j+1,n}^\tau) \vartheta(x_{k+1,n}^\tau) \Big)_{j,k=0}^{n-1} E_n^{(3)} L_n \,,$$

where $\widetilde{\omega}_n^\sigma = \pi/n$ and $\widetilde{\omega}_n^\varphi = \pi/(n+1)$, then the operator norm of $K_n - L_n K|_{\operatorname{im} L_n}$ tends to zero and $\{K_n\}$ is in $\mathcal{J}_2$.

Proof. Consider the case $\tau = \sigma$. Since

$$\int_{-1}^{1} \ell_{kn}^\sigma(y)\varphi(y)\,dy = \int_{-1}^{1} \ell_{kn}^\sigma(y)\varphi^2(y)\sigma(y)\,dy = \frac{\pi}{n}[\varphi(x_{kn}^\sigma)]^2 \,,$$

the operators K_n can be written as $M_n^\sigma \mathbf{K}_n$, where

$$(\mathbf{K}_n u_n)(x) = \int_{-1}^{1} \varphi(y) L_n^\sigma[k(x,\cdot)\varphi^{-1}u_n](y)\,dy \,.$$

Obviously, due to the Arzela-Ascoli theorem the operator $K : \mathbf{L}_\nu^2 \to \mathbf{C}[-1,1]$ is compact. Hence, $\lim\limits_{n\to\infty} \|M_n K L_n - L_n K L_n\|_{\mathcal{L}(\mathbf{L}_\nu^2)} = 0$ (see Corollary 2.3), and it is sufficient to show that $\lim\limits_{n\to\infty} \|\mathbf{K}_n L_n - K L_n\|_{\mathcal{L}(\mathbf{L}_\nu^2, \mathbf{C}[-1,1])} = 0$. To this end, we introduce operators $\widetilde{\mathbf{K}}_n : \operatorname{im} L_n \longrightarrow \mathbf{C}[-1,1]$ by

$$(\widetilde{\mathbf{K}}_n u_n)(x) = \int_{-1}^{1} \varphi(y) L_n^\sigma[k(x,\cdot)\rho^{-1}](y)(\vartheta^{-1}u_n)(y)\,dy \,.$$

Due to the exactness of the Gaussian rule we have, for $j = 0, \ldots, n-2$,

$$\widetilde{\mathbf{K}}_n \widetilde{u}_j = \left\langle L_n^\sigma[k(x,\cdot)\rho^{-1}], \varphi^2 U_j \right\rangle_\sigma = \left\langle L_n^\sigma[k(x,\cdot)\rho^{-1}U_j], \varphi^2 \right\rangle_\sigma = \mathbf{K}_n \widetilde{u}_j \,,$$

and, in view of relations (2.5), (2.6),

$$\begin{aligned} 2\widetilde{\mathbf{K}}_n \widetilde{u}_{n-1} &= \left\langle L_n^\sigma[k(x,\cdot)\rho^{-1}], 2\varphi^2 U_{n-1} \right\rangle_\sigma \\ &= \left\langle L_n^\sigma[k(x,\cdot)\rho^{-1}], T_{n-1} - T_{n+1} \right\rangle_\sigma \\ &= \left\langle L_n^\sigma[k(x,\cdot)\rho^{-1}U_{n-1}], \varphi^2 \right\rangle_\sigma \\ &= \mathbf{K}_n \widetilde{u}_{n-1} \,. \end{aligned}$$

Consequently, $\mathbf{K}_n L_n = \widetilde{\mathbf{K}}_n (2L_n - L_{n-1})$.

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Now, we deal with $\lim_{n\to\infty} \|\widetilde{\mathbf{K}}_n L_n - K L_n\|_{\mathcal{L}(\mathbf{L}^2_\nu, \mathbf{C}[-1,1])}$. We take an arbitrary $u \in \mathbf{L}^2_\nu$ and get $L_n u = \vartheta p_n$, where p_n is a certain polynomial of degree less than n . By $k_n(x,y)$ we refer to the best uniform approximation to $k(x,y)/\rho(y)$ in the space of polynomials with degree less then n in both variables. Using (2.1) we get, for $x \in [-1,1]$,

$$
\begin{aligned}
&|(\widetilde{\mathbf{K}}_n L_n u - K L_n u)(x)| \\
&\quad = \left| \int_{-1}^{1} \varphi(y) \left(L_n^\sigma[k(x,\cdot)\rho^{-1}](y) - k(x,y)/\rho(y) \right) p_n(y)\, dy \right| \\
&\quad \le \left| \int_{-1}^{1} \varphi(y) L_n^\sigma[k(x,\cdot)\rho^{-1} - k_n(x,.)](y) p_n(y)\, dy \right| \\
&\qquad\qquad + \left| \int_{-1}^{1} \varphi(y)[k(x,y)/\rho(y) - k_n(x,y)](y) p_n(y)\, dy \right| \\
&\quad \le \left(\int_{-1}^{1} \left| L_n^\sigma[k(x,\cdot)\rho^{-1} - k_n(x,y)](y) \right|^2 \varphi(y)\, dy \right)^{1/2} \|p_n\|_\varphi \\
&\qquad\qquad + \left(\int_{-1}^{1} \left| k(x,y)/\rho(y) - k_n(x,y) \right|^2 \varphi(y)\, dy \right)^{1/2} \|p_n\|_\varphi \\
&\quad \le \left(\frac{2\pi}{n} \sum_{k=1}^{n} |k(x,x_{kn}^\sigma)/\rho(x_{kn}^\sigma) - k_n(x,x_{kn}^\sigma)|^2 [\varphi(x_{kn}^\sigma)]^2 \right)^{1/2} \|L_n u\|_\nu \\
&\qquad\qquad\qquad\qquad + \|k(\cdot,\cdot)\rho^{-1} - k_n(\cdot,\cdot)\|_\infty \|1\|_\varphi \|L_n u\|_\nu \\
&\quad \le 3 \|k(\cdot,\cdot)\rho^{-1} - k_n(\cdot,\cdot)\|_\infty \|1\|_\varphi \|L_n u\|_\nu \,.
\end{aligned}
$$

Thus, since $\lim_{n\to\infty} \|k(\cdot,\cdot)\rho^{-1} - k_n(\cdot,\cdot)\|_\infty = 0$, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \|\mathbf{K}_n L_n - K L_n\|_{\mathcal{L}(\mathbf{L}^2_\nu, \mathbf{C}[-1,1])} \\
&\quad \le \lim_{n\to\infty} \|\widetilde{\mathbf{K}}_n L_n - K L_n\|_{\mathcal{L}(\mathbf{L}^2_\nu, \mathbf{C}[-1,1])} \|2L_n - L_{n-1}\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \\
&\qquad\qquad + \lim_{n\to\infty} \|K(L_n - L_{n-1})\|_{\mathcal{L}(\mathbf{L}^2_\nu, \mathbf{C}[-1,1])} = 0 \,.
\end{aligned}
$$

The proof in case of $\tau = \varphi$ is similar and can be found in the proof of [7, Lemma 2.4]. □

3. The operator sequence of the collocation method. We will show that the sequence $\{M_n A P_n\}$ corresponding to the singular integral operator $A \in \mathcal{L}(\mathbf{L}^2_\nu)$ (cf. (1.2) belongs to the algebra $\mathcal{F}$, and we will compute $W_\omega\{A_n\}$, $\omega \in T$. We do this separately for multiplication operators, for the singular integral operator $\mu^{-1} S \mu$ with a special weight $\mu = \rho$ (see Lemma 2.10), and for $\mu^{-1} S \mu$ with a general μ .

We will use the well-known relations between the Chebyshev polynomials of first and second kind

(3.1) $$S\varphi U_n = \mathrm{i} T_{n+1}\,,\ S\varphi^{-1} T_n = -\mathrm{i} U_{n-1}\,, \quad n = 0,1,2,\dots\,, \quad U_{-1} \equiv 0\,,$$

and

(3.2) $$T_{n+1} = \frac{1}{2}(U_{n+1} - U_{n-1})\,, \quad n = 0,1,2,\dots\,, \quad U_{-1} \equiv 0\,.$$

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Furthermore, for the description of the occuring strong limits we need the operators

$$J_\nu \in \mathcal{L}(\mathbf{L}^2_\nu, \mathbf{L}^2_\sigma)\,, \quad u \mapsto \sum_{n=0}^\infty \gamma_n \langle u, \widetilde{u}_n\rangle_\nu T_n\,, \tag{3.3}$$

$$J_\nu^{-1} \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^2_\nu)\,, \quad u \mapsto \sum_{n=0}^\infty \frac{1}{\gamma_n} \langle u, T_n\rangle_\sigma \widetilde{u}_n\,, \tag{3.4}$$

$$V \in \mathcal{L}(\mathbf{L}^2_\nu)\,, \quad u \mapsto \sum_{n=0}^\infty \langle u, \widetilde{u}_n\rangle_\nu \widetilde{u}_{n+1}\,, \tag{3.5}$$

with γ_n as in (2.5), and their adjoint operators

$$J_\nu^* \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^2_\nu)\,, \quad u \mapsto \sum_{n=0}^\infty \gamma_n \langle u, T_n\rangle_\sigma \widetilde{u}_n\,,$$

$$J_\nu^{-*} \in \mathcal{L}(\mathbf{L}^2_\nu, \mathbf{L}^2_\sigma)\,, \quad u \mapsto \sum_{n=0}^\infty \frac{1}{\gamma_n} \langle u, \widetilde{u}_n\rangle_\nu T_n\,,$$

$$V^* \in \mathcal{L}(\mathbf{L}^2_\nu)\,, \quad u \mapsto \sum_{n=0}^\infty \langle u, \widetilde{u}_{n+1}\rangle_\nu \widetilde{u}_n\,.$$

Finally, we will use the following special case of Lebesgue's dominated convergence theorem.

REMARK 3.1. *If* $\xi, \eta \in \ell^2$, $\xi^n = \{\xi^n_k\}$, $|\xi^n_k| \le |\eta_k|$ *for all* $n > n_0$, *and if* $\lim\limits_{n\to\infty} \xi^n_k = \xi_k$ *for all* $k = 0, 1, 2, ...$, *then* $\lim\limits_{n\to\infty} \|\xi^n - \xi\|_{\ell^2} = 0$.

LEMMA 3.2. *Let* $a \in \mathbf{PC}$, $A = aI$, $A_n = M_n a L_n$. *Then* $\{A_n\} \in \mathcal{F}$. *In particular,* $W_1\{A_n\} = A$, $W_3\{A_n\} = a(1)I$, $W_4\{A_n\} = a(-1)I$, *and*

$$W_2\{A_n\} = \begin{cases} J_\nu^{-1} a J_\nu & , \quad \tau = \sigma\,, \\ aI = A & , \quad \tau = \varphi\,. \end{cases}$$

Proof. The proof in case of $\tau = \varphi$ is given in [7, Lemma 3.8], and the proof in case of $\tau = \sigma$ is very similar. Thus, here we only pay attention to the proof of the convergence of $(M^\sigma_n a L_n)^*$ and of $W_n M^\sigma_n a W_n$.

We write $M^\sigma_n f = \sum\limits_{j=0}^{n-1} \alpha^\sigma_{jn}(f)\widetilde{u}_j$ and get, for $j = 0, 1, \ldots, n-2$,

$$\begin{aligned} \alpha^\sigma_{jn}(f) = \langle M^\sigma_n f, \widetilde{u}_j\rangle_\nu &= \langle L^\sigma_n \vartheta^{-1} f, \varphi^2 U_j\rangle_\sigma \\ &= \frac{\pi}{n}\sum_{k=1}^n \frac{f(x^\sigma_{kn})}{\vartheta(x^\sigma_{kn})}[\varphi(x^\sigma_{kn})]^2 U_j(x^\sigma_{kn}) \\ &= \frac{\pi}{n}\sum_{k=1}^n f(x^\sigma_{kn})\nu(x^\sigma_{kn})\varphi(x^\sigma_{kn})\widetilde{u}_j(x^\sigma_{kn})\,. \end{aligned}$$

For $j = n-1$, $n \ge 2$, we use relations (2.5) and (2.6) to obtain

$$\alpha^\sigma_{n-1,n}(f) = \langle M^\sigma_n f, \widetilde{u}_{n-1}\rangle_\nu = \langle L^\sigma_n \vartheta^{-1} f, \varphi^2 U_{n-1}\rangle_\sigma$$

$$
\begin{aligned}
&= \frac{1}{2}\langle L_n^\sigma \vartheta^{-1} f, T_{n-1}\rangle_\sigma \\
&= \frac{\pi}{2n}\sum_{k=1}^{n} \frac{f(x_{kn}^\sigma)}{\vartheta(x_{kn}^\sigma)}\, T_{n-1}(x_{kn}^\sigma) \\
&= \frac{\pi}{2n}\sum_{k=1}^{n} \frac{f(x_{kn}^\sigma)}{\vartheta(x_{kn}^\sigma)} [\varphi(x_{kn}^\sigma)]^2 U_{n-1}(x_{kn}^\sigma) \\
&= \frac{\pi}{2n}\sum_{k=1}^{n} f(x_{kn}^\sigma)\nu(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)\widetilde{u}_{n-1}(x_{kn}^\sigma)\,.
\end{aligned}
$$

Hence,

$$
\alpha_{jn}^\sigma(f) = \varepsilon_{jn}\frac{\pi}{n}\sum_{k=1}^{n} f(x_{kn}^\sigma)\nu(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)\widetilde{u}_j(x_{kn}^\sigma)\,, \tag{3.6}
$$

where $\varepsilon_{jn} = 1$ for $j = 0, 1, \ldots, n-2$ and $\varepsilon_{n-1,n} = 1/2$. As an immediate consequence of (3.6) we obtain, for $u, v \in \mathbf{L}^2_\nu$,

$$
\langle M_n^\sigma a L_n u, v\rangle_\nu = \sum_{j=0}^{n-1} \overline{\langle v, \widetilde{u}_j\rangle_\nu} \sum_{l=0}^{n-1} \langle u, \widetilde{u}_l\rangle_\nu \langle M_n^\sigma a \widetilde{u}_l, \widetilde{u}_j\rangle_\nu
$$

$$
= \sum_{j=0}^{n-1} \varepsilon_{jn}\frac{\pi}{n}\sum_{k=1}^{n} a(x_{kn}^\sigma)\sum_{l=0}^{n-1}\langle u, \widetilde{u}_l\rangle_\nu \widetilde{u}_l(x_{kn}^\sigma)\nu(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)\widetilde{u}_j(x_{kn}^\sigma)\overline{\langle v, \widetilde{u}_j\rangle_\nu}
$$

$$
= \sum_{l=0}^{n-1} \overline{\frac{\pi}{n}\sum_{k=1}^{n} \overline{a}(x_{kn}^\sigma)\sum_{j=0}^{n-1}\varepsilon_{jn}\langle v, \widetilde{u}_j\rangle_\nu \widetilde{u}_j(x_{kn}^\sigma)\nu(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)\widetilde{u}_l(x_{kn}^\sigma)}\langle u, \widetilde{u}_l\rangle_\nu
$$

$$
= \frac{1}{2}\langle u, (2L_n - L_{n-1})M_n^\sigma \overline{a}(L_n + L_{n-1})v\rangle_\nu\,.
$$

Thus,

$$
(M_n^\sigma a L_n)^* = \frac{1}{2}(2L_n - L_{n-1})M_n^\sigma \overline{a}(L_n + L_{n-1})\,, \tag{3.7}
$$

whence we have the strong convergence of $(M_n^\sigma a L_n)^*$ to $\overline{a}I$ in $\mathbf{L}^2_\nu$.

We verify the convergence of $W_n M_n^\sigma a W_n \widetilde{u}_m$ for each fixed $m \geq 0$. Let $n > m$. With the help of (3.6), the identity

$$
\begin{aligned}
\widetilde{u}_{n-1-m}(x_{kn}^\sigma) &= \frac{\vartheta(x_{kn}^\sigma)}{\varphi(x_{kn}^\sigma)}\,\varphi(x_{kn}^\sigma)U_{n-1-m}(x_{kn}^\sigma) \\
&= \frac{1}{\rho(x_{kn}^\sigma)}\sqrt{\frac{2}{\pi}}\sin\frac{(n-m)(2k-1)\pi}{2n} \\
&= \frac{(-1)^{k+1}}{\rho(x_{kn}^\sigma)}\,\gamma_m T_m(x_{kn}^\sigma)\,,
\end{aligned} \tag{3.8}
$$

and the formula for the Fourier coefficients of the interpolating polynomial $L_n^\sigma f$,

(3.9) $$L_n^\sigma f = \sum_{j=0}^{n-1} \widetilde{\alpha}_{jn}^\sigma(f) T_j \quad \text{with} \quad \widetilde{\alpha}_{jn}^\sigma(f) = \frac{\pi}{n} \sum_{k=1}^{n} f(x_{kn}^\sigma) T_j(x_{kn}^\sigma)\,,$$

we get, using Lemma 2.2,

$$\begin{aligned} W_n M_n^\sigma a W_n \widetilde{u}_m &= \sum_{j=0}^{n-1} \alpha_{n-1-j,n}^\sigma (a\widetilde{u}_{n-1-m}) \widetilde{u}_j \\ &= \sum_{j=0}^{n-1} \varepsilon_{n-1-j,n} \frac{\pi}{n} \sum_{k=1}^{n} a(x_{kn}^\sigma) \widetilde{u}_{n-1-m}(x_{kn}^\sigma) \nu(x_{kn}^\sigma) \varphi(x_{kn}^\sigma) \widetilde{u}_{n-1-j}(x_{kn}^\sigma) \widetilde{u}_j \\ &= \sum_{j=0}^{n-1} \varepsilon_{n-1-j,n} \frac{\pi}{n} \sum_{k=1}^{n} a(x_{kn}^\sigma) \gamma_m T_m(x_{kn}^\sigma) \gamma_j T_j(x_{kn}^\sigma) \widetilde{u}_j \\ &= \sum_{j=0}^{n-1} \frac{\pi}{n} \sum_{k=1}^{n} a(x_{kn}^\sigma) (J_\nu \widetilde{u}_m)(x_{kn}^\sigma T_j(x_{kn}^\sigma) J_\nu^{-1} T_j \\ &= J_\nu^{-1} L_n^\sigma a J_\nu \widetilde{u}_m \longrightarrow J_\nu^{-1} a J_\nu \widetilde{u}_m \quad \text{in} \quad \mathbf{L}_\nu^2\,. \end{aligned}$$

Thus,

(3.10) $$W_n M_n^\sigma a W_n = J_\nu^{-1} L_n^\sigma a J_\nu L_n \longrightarrow J_\nu^{-1} a J_\nu \quad \text{in} \quad \mathbf{L}_\nu^2\,.$$

□

LEMMA 3.3. *Suppose* $A = \rho^{-1} S \rho I$, *where* $\rho = \vartheta^{-1}\varphi = \sqrt{\nu\varphi}$, *and* $A_n = M_n A L_n$. *Then* $\{A_n\} \in \mathcal{F}$ *and*

$$W_1\{A_n\} = A\,, \quad W_2\{A_n\} = \begin{cases} \mathrm{i} J_\nu^{-1} \rho V^* & , \quad \tau = \sigma\,, \\ -A & , \quad \tau = \varphi\,, \end{cases}$$

and $W_{3/4}\{A_n\} = \pm \mathbf{S}$ *with*

$$\mathbf{S} = \begin{cases} \left(\dfrac{1-(-1)^{j-k}}{\pi \mathrm{i}(j-k)} - \dfrac{1-(-1)^{j+k+1}}{\pi \mathrm{i}(j+k+1)} \right)_{j,k=0}^{\infty} & , \quad \tau = \sigma\,, \\ \left(\dfrac{2(k+1)\left[1-(-1)^{j-k}\right]}{\pi \mathrm{i}\left[(j+1)^2-(k+1)^2\right]} \right)_{j,k=0}^{\infty} & , \quad \tau = \varphi\,. \end{cases}$$

Proof. The case $\tau = \varphi$ is considered in [7, Lemma 3.9]. Thus, let us consider the case $\tau = \sigma$.

From (3.1) it follows that $S\rho u_n$ is a polynomial of degree not greater than n if $u_n \in \operatorname{im} L_n$. Hence, applying (2.2), Lemma 2.1, and the boundedness of the operator $S : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2$, we obtain, for $u_n \in \operatorname{im} L_n$,

$$\begin{aligned} \|M_n^\sigma \rho^{-1} S \rho u_n\|_\nu^2 &\le 2 Q_n^\sigma |S\rho u_n|^2 \\ &\le \text{const} \int_{-1}^{1} |(S\rho u_n)(x)|^2 \sigma(x)\, dx \\ &\le \text{const}\, \|\rho u_n\|_\sigma^2 = \text{const}\, \|u_n\|_\nu^2\,, \end{aligned}$$

which shows the uniform boundedness of $\{A_n\}$. Again with the help of (3.1) as well as with the help of Corollary 2.3 we see that, for $n > m$,

$$M_n^\sigma \rho^{-1} S\rho\widetilde{u}_m = \mathrm{i} M_n^\sigma \rho^{-1} T_{m+1} \to \mathrm{i}\rho^{-1} T_{m+1} = \rho^{-1} S\rho\widetilde{u}_m \quad \text{in} \quad \mathbf{L}^2_\nu\,.$$

Whence, the strong convergence of $\{A_n\}$ to A is proved.

The well-known Poincarè-Bertrand commutation formula implies that, for $u \in \mathbf{L}^2_\nu$ and $v \in \mathbf{L}^2_{\nu^{-1}}$,

$$\langle Su, v\rangle = \langle u, Sv\rangle,$$

where $\langle .,.\rangle$ denotes the $\mathbf{L}^2(-1,1)$ inner product without weight. Consequently, the adjoint operator of $S : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is equal to $\nu^{-1} S\nu : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$. Again, taking into account that $S\rho L_n u$ is a polynomial with a degree $\le n$ (cf. (3.1), we get, for $j = 0, ..., n-2$ and $u \in \mathbf{L}^2_\nu$,

$$\begin{aligned}
\langle M_n^\sigma \rho^{-1} S\rho L_n u, \widetilde{u}_j\rangle_\nu &= \langle L_n^\sigma \varphi^{-1} S\rho L_n u, \varphi^2 U_j\rangle_\sigma \\
&= \frac{\pi}{n}\sum_{k=1}^{n} (S\rho L_n u)(x_{kn}^\sigma)\varphi(x_{kn}^\sigma) U_j(x_{kn}^\sigma) = \langle S\rho L_n u, L_n^\sigma \varphi U_j\rangle_\sigma \\
&= \langle S\rho L_n u, \sigma\nu^{-1} L_n^\sigma \varphi U_j\rangle_\nu = \langle \rho L_n u, \nu^{-1} S\sigma L_n^\sigma \varphi U_j\rangle_\nu \\
&= \langle u, L_n \vartheta S\sigma L_n^\sigma \rho\widetilde{u}_j\rangle_\nu
\end{aligned}$$

and, using relations (2.5) and (2.6),

$$\begin{aligned}
\langle M_n^\sigma \rho^{-1} S\rho L_n u, \widetilde{u}_{n-1}\rangle_\nu &= \langle L_n^\sigma \varphi^{-1} S\rho L_n u, \varphi^2 U_{n-1}\rangle_\sigma \\
&= \frac{\pi}{2n}\sum_{k=1}^{n} \frac{(S\rho L_n u)(x_{kn}^\sigma)}{\varphi(x_{kn}^\sigma)}\, T_{n-1}(x_{kn}^\sigma) \\
&= \frac{1}{2}\langle S\rho L_n u, L_n^\sigma \varphi U_{n-1}\rangle_\sigma \\
&= \frac{1}{2}\langle u, L_n \vartheta S\sigma L_n^\sigma \rho\widetilde{u}_{n-1}\rangle_\nu\,.
\end{aligned}$$

Hence, in view of (3.1)

$$(M_n^\sigma \rho^{-1} S\rho L_n)^* = \frac{1}{2} L_n \vartheta S\sigma L_n^\sigma \rho(L_n + L_{n-1}) = \frac{1}{2}\vartheta S\sigma L_n^\sigma \rho(L_n + L_{n-1})\,.$$

Using Lemma 2.2, we obtain the strong convergence of $(M_n^\sigma \rho^{-1} S\rho L_n)^*$ to $\vartheta S\vartheta^{-1} I$.

In view of (3.1), (3.2), (3.10), (2.5), and Lemma 2.2, we have, for $n > m+1$,

$$\begin{aligned}
W_n M_n^\sigma \rho^{-1} S\rho W_n \widetilde{u}_m &= W_n M_n^\sigma \rho^{-1} S\rho\widetilde{u}_{n-1-m} \\
&= \mathrm{i} W_n M_n^\sigma \rho^{-1} T_{n-m} \\
&= \frac{\mathrm{i}}{2} W_n M_n^\sigma \rho^{-1}\vartheta^{-1}(\widetilde{u}_{n-m} - \widetilde{u}_{n-m-2}) \\
&= -\frac{\mathrm{i}}{2} W_n M_n^\sigma \varphi^{-1} W_n(\widetilde{u}_{m+1} - \widetilde{u}_{m-1}) \\
&= -\frac{\mathrm{i}}{2} J_\nu^{-1} L_n^\sigma \varphi^{-1} J_\nu(\widetilde{u}_{m+1} - \widetilde{u}_{m-1})
\end{aligned}$$

$$= \frac{i}{2} J_\nu^{-1} L_n^\sigma \varphi^{-1} (\gamma_{m-1} T_{m-1} - \gamma_{m+1} T_{m+1})$$
$$= \mathrm{i} J_\nu^{-1} L_n^\sigma \varphi^{-1} \varphi^2 U_{m-1} \to \mathrm{i} J_\nu^{-1} \varphi U_{m-1}$$
$$= \mathrm{i} J_\nu^{-1} \rho \widetilde{u}_{m-1} \, .$$

Obviously, $W_n M_n^\sigma \rho^{-1} S \rho W_n \widetilde{u}_0 = \mathrm{i} W_n M_n^\sigma \rho^{-1} T_n = 0$. Hence, by means of the shift operator V introduced in (3.5) and by using the uniform boundedness of $\{M_n^\sigma A L_n\}$, we can derive the strong convergence

$$W_n M_n^\sigma \rho^{-1} S \rho W_n = \mathrm{i} J_\nu^{-1} L_n^\sigma \rho V^* L_n \to \mathrm{i} J_\nu^{-1} \rho V^* \quad \text{in} \quad \mathbf{L}_\nu^2 \, . \tag{3.11}$$

Using (3.11), we get, for all $u, v \in \mathbf{L}_\nu^2$,

$$\begin{aligned}
\langle W_n M_n^\sigma \rho^{-1} S \rho W_n u, v \rangle_\nu &= \mathrm{i} \langle J_\nu^{-1} L_n^\sigma \rho V^* L_n u, L_n v \rangle_\nu \\
&= \mathrm{i} \langle L_n^\sigma \rho V^* L_n u, J_\nu^{-*} L_n v \rangle_\sigma \\
&= \frac{\mathrm{i}\pi}{n} \sum_{k=1}^n \rho(x_{kn}^\sigma)(V^* L_n u)(x_{kn}^\sigma)(J_\nu^{-*} L_n v)(x_{kn}^\sigma) \\
&= \mathrm{i} \langle \vartheta^{-1} \varphi^2 V^* L_n u, L_n^\sigma \varphi^{-1} J_\nu^{-*} L_n v \rangle_\sigma \\
&= \mathrm{i} \langle V^* L_n u, \nu^{-1} \varphi \vartheta^{-2} M_n^\sigma \rho^{-1} J_\nu^{-*} L_n v \rangle_\nu \\
&= \mathrm{i} \langle u, L_n V M_n^\sigma \rho^{-1} J_\nu^{-*} L_n v \rangle_\nu \, .
\end{aligned}$$

Thus, we have (see Corollary 2.3)

$$(W_n M_n^\sigma \rho^{-1} S \rho W_n)^* = -\mathrm{i} L_n V M_n^\sigma \rho^{-1} J_\nu^{-*} L_n \longrightarrow -\mathrm{i} V \rho^{-1} J_\nu^{-*} \quad \text{in} \quad \mathbf{L}_\nu^2 \, .$$

Now, let us investigate the sequence $\{V_n^\sigma M_n^\sigma A L_n (V_n^\sigma)^{-1} P_n\}$. For $n > m > 0$, we have

$$\begin{aligned}
V_n^\sigma M_n^\sigma \rho^{-1} S \rho (V_n^\sigma)^{-1} e_{m-1} &= V_n^\sigma M_n^\sigma \rho^{-1} S \, \frac{\rho}{\omega_{mn}} \widetilde{\ell}_{mn}^\sigma \\
&= V_n^\sigma \sum_{k=1}^n \frac{1}{\omega_{mn}} \rho^{-1}(x_{kn}^\sigma) \left(S \rho \widetilde{\ell}_{mn}^\sigma \right) (x_{kn}^\sigma) \widetilde{\ell}_{kn}^\sigma \\
&= \left\{ \frac{\omega_{jn}}{\omega_{mn}} \rho^{-1}(x_{jn}^\sigma) \left(S \rho \widetilde{\ell}_{mn}^\sigma \right) (x_{jn}^\sigma) \right\}_{j=1}^n \, .
\end{aligned}$$

We compute, for $x \neq x_{kn}^\sigma$,

$$\begin{aligned}
&(\rho^{-1} S \rho \widetilde{\ell}_{kn})(x) \\
&= \frac{1}{\rho(x) \vartheta(x_{kn}^\sigma) T_n'(x_{kn}^\sigma)} \frac{1}{\pi \mathrm{i}} \int_{-1}^1 \frac{\varphi(y) T_n(y)}{y - x_{kn}^\sigma} \, dy \\
&= \frac{1}{\rho(x) \vartheta(x_{kn}^\sigma) T_n'(x_{kn}^\sigma)} \frac{1}{\pi \mathrm{i}} \frac{1}{x - x_{kn}^\sigma} \int_{-1}^1 \left(\frac{1}{y - x} - \frac{1}{y - x_{kn}^\sigma} \right) \varphi(y) T_n(y) \, dy
\end{aligned}$$

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and, taking into account (3.1),

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^{1}\frac{1}{y-x}\varphi(y)T_n(y)\,dy &= \frac{1}{\pi}\int_{-1}^{1}\frac{1-y^2}{y-x}T_n(y)\sigma(y)\,dy \\ &= \frac{1}{\pi}\int_{-1}^{1}\frac{1-x^2}{y-x}T_n(y)\sigma(y)\,dy - \frac{1}{\pi}\int_{-1}^{1}\frac{y^2-x^2}{y-x}T_n(y)\sigma(y)\,dy \\ &= (1-x^2)U_{n-1}(x) - \frac{1}{\pi}\int_{-1}^{1}(y+x)T_n(y)\sigma(y)\,dy\,,\end{aligned}$$

i.e.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{y-x}\varphi(y)T_n(y)\,dy = (1-x^2)U_{n-1}(x)\,. \tag{3.12}$$

We remark that, for $n > 0$,

$$T_n'(x) = nU_{n-1}(x) \quad \text{and} \quad T_n'(x_{kn}^\sigma) = \sqrt{\frac{2}{\pi}}\frac{n(-1)^{k+1}}{\varphi(x_{kn}^\sigma)}\,. \tag{3.13}$$

In view of $\omega_{jn} = \sqrt{\frac{\pi}{n}}\rho(x_{jn}^\sigma)$ and (3.13), we have, for $j \neq k$,

$$\begin{aligned}\frac{\omega_{jn}}{\omega_{kn}\rho(x_{jn}^\sigma)}\left(S\rho\widetilde{\ell}_{kn}^\sigma\right)(x_{jn}^\sigma) &= \sqrt{\frac{\pi}{2}}(-1)^{k+1}\frac{\varphi(x_{jn}^\sigma)]^2U_{n-1}(x_{jn}^\sigma) - [\varphi(x_{kn}^\sigma)]^2U_{n-1}(x_{kn}^\sigma)}{ni(x_{jn}^\sigma - x_{kn}^\sigma)} \\ &= \frac{\varphi(x_{kn}^\sigma) - (-1)^{j+k}\varphi(x_{jn}^\sigma)}{ni(x_{kn}^\sigma - x_{jn}^\sigma)} =: s_{jk}^{(n)}\,.\end{aligned}$$

With the help of

$$\frac{d}{dx}[(1-x^2)U_{n-1}(x)] = (1-x^2)U_{n-1}'(x) - 2xU_{n-1}(x) = -xU_{n-1}(x) - nT_n(x)$$

we get

$$\frac{\omega_{kn}}{\omega_{kn}\rho(x_{kn}^\sigma)}\left(S\rho\widetilde{\ell}_{kn}^\sigma\right)(x_{kn}^\sigma) = -\frac{x_{kn}^\sigma}{ni\varphi(x_{kn}^\sigma)} =: s_{kk}^{(n)}\,.$$

It follows

$$s_{jk}^{(n)} = \begin{cases} -\dfrac{\cos\frac{k+j-1}{2n}\pi}{ni\,\sin\frac{k+j-1}{2n}\pi} & , \quad j+k \text{ even}, \\[2ex] -\dfrac{\cos\frac{k-j}{2n}\pi}{ni\,\sin\frac{k-j}{2n}\pi} & , \quad j+k \text{ odd}, \end{cases} \tag{3.14}$$

and consequently, for fixed k and $1 \le j \le n$ or for fixed j and $1 \le k \le n$,

$$|s_{jk}^{(n)}| \le \text{const} \begin{cases} \dfrac{1}{k+j-1} & , \quad j+k \text{ even}, \\[2ex] \dfrac{1}{|k-j|} & , \quad j+k \text{ odd}. \end{cases} \tag{3.15}$$

Using Remark 3.1 we find, for fixed $m > 0$,

$$\left\{ s_{1m}^{(n)}, s_{2m}^{(n)}, ..., s_{nm}^{(n)}, 0, ... \right\} \longrightarrow \left\{ \lim_{n\to\infty} s_{jm}^{(n)} \right\}_{j=1}^{\infty} =: \{s_{jm}\} \quad \text{in} \quad \ell^2 ,$$

where $s_{jk} = \lim_{n\to\infty} s_{jk}^{(n)}$, i.e.

$$s_{jk} = \left\{ \begin{array}{ll} -\dfrac{2}{\pi \mathsf{i}(j+k-1)} , & j+k \text{ even}, \\[2ex] \dfrac{2}{\pi \mathsf{i}(j-k)} , & j+k \text{ odd}, \end{array} \right\} = \frac{1-(-1)^{j-k}}{\pi \mathsf{i}(j-k)} - \frac{1-(-1)^{j+k-1}}{\pi \mathsf{i}(j+k-1)} . \tag{3.16}$$

Thus,

$$V_n^\sigma M_n^\sigma A L_n (V_n^\sigma)^{-1} P_n \longrightarrow \mathbf{S} := \left(s_{(j+1)(k+1)} \right)_{j,k=0}^{\infty} \quad \text{in} \quad \ell^2 .$$

Now it is easy to see that, in ℓ^2 ,

$$(V_n^\sigma M_n^\sigma A L_n (V_n^\sigma)^{-1} P_n)^* P_n \longrightarrow \mathbf{S}^* ,$$

$$\widetilde{V}_n^\sigma M_n^\sigma A L_n (\widetilde{V}_n^\sigma)^{-1} P_n = \widetilde{W}_n V_n^\sigma M_n^\sigma A L_n (V_n^\sigma)^{-1} \widetilde{W}_n P_n \longrightarrow -\mathbf{S} ,$$

and

$$((\widetilde{V}_n^\sigma) M_n^\sigma A L_n (\widetilde{V}_n^\sigma)^{-1} P_n)^* P_n \longrightarrow -\mathbf{S}^* .$$

□

Let us turn to the more general operator $\mu^{-1} S \mu I$ and the corresponding sequence of the collocation method.

LEMMA 3.4. *Suppose $A = \mu^{-1} S \mu I$ and $A_n = M_n A L_n$, where $\mu = v^{\gamma,\delta}$ satisfies (1.4) and (1.5). Then $\{A_n\} \in \mathcal{F}$, where $W_1\{A_n\} = A$, $W_2\{A_n\} = W_2\{M_n \rho^{-1} S \rho L_n\}$ (comp. Lemma 3.3), and*

$$W_3\{A_n\} = \mathbf{S} + \mathbf{A}_+^\mu , \quad W_4\{A_n\} = -\mathbf{S} - \mathbf{A}_-^\mu . \tag{3.17}$$

Here $\rho = \vartheta^{-1}\varphi$ and $\mathbf{S}$ are the same as in Lemma 3.3, and

$$\mathbf{A}_\pm^\mu = \mathbf{B}_\pm + \mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1} - \mathbf{A} - \mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1} \mathbf{W} \mathbf{V}_\pm \left\{ \begin{array}{ll} -\mathbf{V}_\pm \mathbf{A}^* \mathbf{W} & , \quad \tau = \sigma , \\ +\mathbf{V}_\pm \mathbf{A} \mathbf{W} & , \quad \tau = \varphi , \end{array} \right. \tag{3.18}$$

with

$$\mathbf{A} := \left\{ \begin{array}{ll} \left(\dfrac{(2k+1)(1-\delta_{j,k})}{\pi \mathsf{i}(k+j+1)(j-k)} \right)_{j,k=0}^{\infty} & , \quad \tau = \sigma , \\[3ex] \left(\dfrac{2(k+1)(1-\delta_{j,k})}{\pi \mathsf{i}[(j+1)^2 - (k+1)^2]} \right)_{j,k=0}^{\infty} & , \quad \tau = \varphi . \end{array} \right. \tag{3.19}$$

$\mathbf{D}_\pm$, $\mathbf{B}_\pm$, $\mathbf{W}$, and $\mathbf{V}_\pm$ are diagonal operators

$$\mathbf{D}_\pm := \left\{ \begin{array}{ll} \left((2k+1)^{2\chi_\pm} \delta_{j,k} \right)_{j,k=0}^{\infty} & , \quad \tau = \sigma , \\ \left((k+1)^{2\chi_\pm} \delta_{j,k} \right)_{j,k=0}^{\infty} & , \quad \tau = \varphi , \end{array} \right. \qquad \mathbf{B}_\pm := \left(b_{k+1}^\pm \delta_{j,k} \right)_{j,k=0}^{\infty} ,$$

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$$\mathbf{W} := \left(\frac{(-1)^{k+1}}{\sqrt{2\pi}} \delta_{j,k} \right)_{j,k=0}^{\infty} , \quad \mathbf{V}_{\pm} := \left(d_{k+1}^{\pm} \delta_{j,k} \right)_{j,k=0}^{\infty} , \tag{3.20}$$

where $\chi_{+} = \frac{1}{4} + \frac{\alpha}{2} - \gamma$, $\chi_{-} = \frac{1}{4} + \frac{\beta}{2} - \delta$, *and, choosing* $\zeta_{\pm} = -\chi_{\pm}$, *the* $b_k^{\pm}$ *and* $d_k^{\pm}$ *are defined by*

$$b_k^{\pm} := \begin{cases} \dfrac{64\,(-1)^{k+1}}{\pi \mathrm{i}} \displaystyle\int_0^{\infty} \frac{\left(\frac{2s}{(2k-1)\pi}\right)^{2\zeta_{\pm}} - 1}{\left([(2k-1)\pi]^2 - [2s]^2\right)^2}\, s^2 \cos s\, ds & , \quad \tau = \sigma , \\ \dfrac{4(-1)^{k+1} k}{\mathrm{i}} \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_{\pm}} - 1}{[(k\pi)^2 - s^2]^2}\, s \,\sin s\, ds & , \quad \tau = \varphi , \end{cases} \tag{3.21}$$

and, in case $\tau = \sigma$,

$$\begin{aligned} d_k^{\pm} := & \sqrt{\frac{2}{\pi}} \frac{16}{(2k-1)\pi} \int_0^{s^*} \frac{\left(\frac{2s}{2k-1}\right)^{2\zeta_{\pm}} - 1}{[(2k-1)\pi]^2 - [2s]^2}\, s^2 \cos s\, ds \\ & + \sqrt{\frac{2}{\pi}} \int_{s^*}^{\infty} \left\{ 512\, s\, \frac{[2s]^{2\zeta_{\pm}} - [(2k-1)\pi]^{2\zeta_{\pm}}}{([(2k-1)\pi]^2 - [2s]^2)^3} + \frac{64}{s} \frac{(1+2\zeta_{\pm})[2s]^{2\zeta_{\pm}} - [(2k-1)\pi]^{2\zeta_{\pm}}}{([(2k-1)\pi]^2 - [2s]^2)^2} \right. \\ & \left. + \frac{4}{s^3} \frac{(4\zeta_{\pm}^2 - 1)[2s]^{2\zeta_{\pm}} - [(2k-1)\pi]^{2\zeta_{\pm}}}{[(2k-1)\pi]^2 - [2s]^2} \right\} \frac{12\cos s + 12 s \sin s - 4s^2 \cos s}{[(2k-1)\pi]^{1+2\zeta_{\pm}}}\, s\, ds \\ & + \sqrt{\frac{2}{\pi}} \frac{12\cos s^* + 12\, s^* \sin s^* - 4(s^*)^2 \cos s^*}{[(2k-1)\pi]^{1+2\zeta_{\pm}}} \left\{ 32\, s^* \frac{[2s^*]^{2\zeta_{\pm}} - [(2k-1)\pi]^{2\zeta_{\pm}}}{([(2k-1)\pi]^2 - [2s^*]^2)^2} \right. \\ & \left. + \frac{4}{s^*} \frac{(1+2\zeta_{\pm})[2s^*]^{2\zeta_{\pm}} - [(2k-1)\pi]^{2\zeta_{\pm}}}{[(2k-1)\pi]^2 - [2s^*]^2} \right\} , \end{aligned} \tag{3.22}$$

where $s^* \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *is the solution of the equation* $\cos s + s \sin s = 0$, *as well as, in case* $\tau = \varphi$,

$$\begin{aligned} d_k^{\pm} = & 2\sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_{\pm}} - 1}{(k\pi)^2 - s^2}\, s \,\sin s\, ds \\ & + 4\sqrt{\frac{2}{\pi}} \int_{\frac{\pi}{2}}^{\infty} \cos s \left\{ \frac{s^2 \left[\left(\frac{s}{k\pi}\right)^{2\zeta_{\pm}} - 1 \right]}{[(k\pi)^2 - s^2]^2} + \frac{\zeta_{\pm} \left(\frac{s}{k\pi}\right)^{2\zeta_{\pm}} + \frac{1}{2} \left[\left(\frac{s}{k\pi}\right)^{2\zeta_{\pm}} - 1 \right]}{(k\pi)^2 - s^2} \right\} ds . \end{aligned}$$

The proof of this lemma in case of $\tau = \varphi$ can be found in [7, Lemma 3.10], in case of $\tau = \sigma$ it is given in the appendix.

4. The operators $W_3\{A_n\}$ and $W_4\{A_n\}$. In this section we show that the operators $W_{3,4}\{A_n\}$ belong to an algebra of Toeplitz matrices. For this we consider the C^*-algebra $\mathcal{L}(\ell^2)$ of linear and bounded operators in ℓ^2. By $alg\ \mathcal{T}(\mathbf{PC})$ we denote the closed C^*-

subalgebra of $\mathcal{L}(\ell^2)$ generated by the Toeplitz matrices $(\hat{g}_{j-k})_{j,k=0}^{\infty}$ with piecewise continuous generating functions

$$g(t) := \sum_{k\in\mathbb{Z}} \hat{g}_k t^k \quad \text{defined on} \quad \mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$$

and continuous on $\mathbb{T} \setminus \{\pm 1\}$.

First we recall some results on the Gohberg-Krupnik symbol for operators belonging to $alg\,\mathcal{T}(\mathbf{PC})$.

LEMMA 4.1 (see [9], Theorem 16.2). *There is a continuous mapping* $\mathbf{smb}$ *from the algebra* $alg\,\mathcal{T}(\mathbf{PC})$ *to a set of functions defined over* $\mathbb{T}\times[0,1]$. *For each* $R \in alg\,\mathcal{T}(\mathbf{PC})$, *the corresponding function* $\mathbf{smb}_R(t,\mu)$ *is called the symbol of* R . *This symbol has the following properties:*

1. *For any fixed point* $(t,\lambda) \in \mathbb{T}\times[0,1]$ *the mapping* $alg\,\mathcal{T}(\mathbf{PC}) \to \mathbb{C}$, $R \mapsto \mathbf{smb}_R(t,\lambda)$ *is a multiplicative functional.*
2. *For any* $t \neq \pm 1$, *the value* $\mathbf{smb}_R(t,\lambda)$ *is independent of* λ , *and the function* $t \mapsto \mathbf{smb}_R(t,0)$ *is continuous on* $\{t \in \mathbb{T} : \Im\mathrm{m}\, t > 0\}$ *and on* $\{t \in \mathbb{T} : \Im\mathrm{m}\, t < 0\}$ *with the limits*

$$\begin{aligned}
\mathbf{smb}_R(1+0,0) &:= \lim_{t\to+1,\Im\mathrm{m}\,t>0} \mathbf{smb}_R(t,0) = \mathbf{smb}_R(1,1)\,,\\
\mathbf{smb}_R(1-0,0) &:= \lim_{t\to+1,\Im\mathrm{m}\,t<0} \mathbf{smb}_R(t,0) = \mathbf{smb}_R(1,0)\,,\\
\mathbf{smb}_R(-1+0,0) &:= \lim_{t\to-1,\Im\mathrm{m}\,t<0} \mathbf{smb}_R(t,0) = \mathbf{smb}_R(-1,1)\,,\\
\mathbf{smb}_R(-1-0,0) &:= \lim_{t\to-1,\Im\mathrm{m}\,t>0} \mathbf{smb}_R(t,0) = \mathbf{smb}_R(-1,0)\,.
\end{aligned}$$

   Moreover, the functions $\lambda \mapsto \mathbf{smb}_R(\pm 1,\lambda)$ *are continuous on* $[0,1]$.
3. *An operator* $R \in alg\,\mathcal{T}(\mathbf{PC})$ *is Fredholm if and only if* $\mathbf{smb}_R(t,\lambda)$ *does not vanish on* $\mathbb{T}\times[0,1]$.
4. *For any Fredholm operator* $R \in alg\,\mathcal{T}(\mathbf{PC})$, *the index of* R *is the negative winding number of the closed curve*

$$\Gamma := \left\{\mathbf{smb}_R(e^{is},0) : 0 < s < \pi\right\} \cup \{\mathbf{smb}_R(-1,s) : 0 \le s \le 1\}$$

$$\cup \left\{\mathbf{smb}_R(-e^{is},0) : 0 < s < \pi\right\} \cup \{\mathbf{smb}_R(1,s) : 0 \le s \le 1\}$$

   with respect to the point zero, where the direction of the curve Γ *is determined by the parametrizations of its definition.*
5. *An operator* $R \in alg\,\mathcal{T}(\mathbf{PC})$ *is compact if and only if its symbol function* $\mathbf{smb}_R(t,\lambda)$ *vanishes on* $\mathbb{T}\times[0,1]$.

For any Toeplitz matrix $T(g) = (\hat{g}_{j-k})_{j,k=0}^{\infty}$ *with piecewise continuous generating function* $g(t) := \sum_{k\in\mathbb{Z}} \hat{g}_k t^k$ *defined on* $\mathbb{T}$ *and continuous on* $\mathbb{T}\setminus\{\pm1\}$, *the symbol is given by*

$$\mathbf{smb}_{T(g)}(t,\lambda) = \begin{cases} g(t) & , \quad t \in \mathbb{T}\setminus\{\pm 1\}\,, \\ \lambda g(t+0) + (1-\lambda) g(t-0) & , \quad t = \pm 1\,. \end{cases}$$

LEMMA 4.2 ([1], Theorem 4.97). *Any Hankel matrix* $H(g) = (\hat{g}_{j+k+1})_{j,k=0}^{\infty}$ *with piecewice continuous generating function* $g(t) := \sum_{k\in\mathbb{Z}} \hat{g}_k t^k$ *defined on* $\mathbb{T}$ *and continuous on*

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$\mathbb{T} \setminus \{\pm 1\}$ *belongs to* $alg\,\mathcal{T}(\mathbf{PC})$ *, and its symbol is defined by*

$$\mathbf{smb}_{H(g)}(t,\lambda) = \begin{cases} 0 & , \quad t \in \mathbb{T} \setminus \{\pm 1\}\,, \\ -t\,\mathrm{i}[g(t+0) - g(t-0)]\sqrt{\lambda(1-\lambda)} & , \quad t = \pm 1\,. \end{cases}$$

LEMMA 4.3 ([11], Lemma 11.4). *Suppose the generating function* $g(t) = \sum_{k\in\mathbb{Z}} \hat{g}_k t^k$ *of the Toeplitz matrix* $(\hat{g}_{j-k})_{j,k=0}^{\infty}$ *is piecewise continuous on* $\mathbb{T}$ *and continuous on* $\mathbb{T} \setminus \{\pm 1\}$ *, and take a complex* z *with* $|\Re\mathrm{e}\, z| < 1/2$ *. Then the matrix*

$$R := ([j+1]^{-z}\delta_{j,k})_{j,k=0}^{\infty}(\hat{g}_{j-k})_{j,k=0}^{\infty}([k+1]^{z}\delta_{j,k})_{j,k=0}^{\infty}$$

belongs to $alg\,\mathcal{T}(\mathbf{PC})$ *, and its symbol is given by*

$$\mathbf{smb}_{R}(t,\lambda) = \begin{cases} g(t) & , \quad t \in \mathbb{T} \setminus \{\pm 1\}\,, \\ \dfrac{\lambda g(t+0) + (1-\lambda) g(t-0) e^{-\mathrm{i}2\pi z}}{\lambda + (1-\lambda)e^{-\mathrm{i}2\pi z}} & , \quad t = \pm 1\,. \end{cases}$$

Furthermore, for any fixed Toeplitz matrix $T(g) = (\hat{g}_{j-k})_{j,k=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$ *with a generating function which is piecewise twice continuously differentiable, the operator valued function*

$$z \mapsto ([j+1]^{-z}\delta_{j,k})_{j,k=0}^{\infty} T(g) ([k+1]^{z}\delta_{j,k})_{j,k=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$$

is continuous on $\{z \in \mathbb{C} : |\Re\mathrm{e}\, z| < 1/2\}$ *in the operator norm.*

From this Lemma one can easily obtain the following result.

COROLLARY 4.4. *Let the generating function* $g(t) = \sum_l \hat{g}_l t^l$ *of the Toeplitz matrix* $(\hat{g}_{j-k})_{j,k=0}^{\infty}$ *be piecewise continuous on* $\mathbb{T}$ *and continuous on* $\mathbb{T} \setminus \{\pm 1\}$ *, and take a complex* z *with* $|\Re\mathrm{e}\, z| < 1/2$ *. Then the matrix*

$$R := \left(\left[j + \frac{1}{2} \right]^{-z} \delta_{j,k} \right)_{j,k=0}^{\infty} (\hat{g}_{j-k})_{j,k=0}^{\infty} \left(\left[k + \frac{1}{2} \right]^{z} \delta_{j,k} \right)_{j,k=0}^{\infty}$$

belongs to $alg\,\mathcal{T}(\mathbf{PC})$ *, and its symbol is given by*

$$\mathbf{smb}_{R}(t,\lambda) = \begin{cases} g(t) & , \quad t \in \mathbb{T} \setminus \{\pm 1\}\,, \\ \dfrac{\lambda g(t+0) + (1-\lambda) g(t-0) e^{-\mathrm{i}2\pi z}}{\lambda + (1-\lambda)e^{-\mathrm{i}2\pi z}} & , \quad t = \pm 1\,. \end{cases}$$

Furthermore, for any fixed Toeplitz matrix $T(g) = (\hat{g}_{j-k})_{j,k=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$ *with a generating function which is piecewise twice continuously differentiable, the operator valued function*

$$z \mapsto \left(\left[j + \frac{1}{2} \right]^{-z} \delta_{j,k} \right)_{j,k=0}^{\infty} T(g) \left(\left[k + \frac{1}{2} \right]^{z} \delta_{j,k} \right)_{j,k=0}^{\infty} \in alg\,\mathcal{T}(\mathbf{PC})$$

is continuous on $\{z \in \mathbb{C} : |\Re\mathrm{e}\, z| < 1/2\}$ *in the operator norm.*

LEMMA 4.5 ([12], Satz 3.3 and [7], Lemma 7.1). *Suppose the Mellin transform*

$$\widehat{m}(z) := \int_0^{\infty} m(\sigma)\sigma^{z-1}\, d\sigma$$

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of the univariate function $m:(0,\infty)\longrightarrow\mathbb{C}$ *is analytic in the strip*

$$1/2-\varepsilon<\Re\mathrm{e}\, z<1/2+\varepsilon$$

for a small $\varepsilon>0$ *. Moreover, assume that*

$$\sup_{z:1/2-\varepsilon<\Re\mathrm{e}\, z<1/2+\varepsilon}\left|\frac{d^k}{dz^k}\widehat{m}(z)(1+|z|)^k\right|<\infty\,,\quad k=0,1,\ldots$$

Then m *is infinitely differentiable on* $(0,\infty)$ *, the operators* $M_{+1},M_{-1}\in\mathcal{L}(\ell^2)$ *defined by*

$$M_{+1}:=\left(m\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)\frac{1}{k+\frac{1}{2}}\right)_{j,k=0}^{\infty}$$

and

$$M_{-1}:=\left((-1)^{j-k}m\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)\frac{1}{k+\frac{1}{2}}\right)_{j,k=0}^{\infty}$$

belong to the algebra $alg\,\mathcal{T}(\mathbf{PC})$ *, and their symbols are given by*

$$\mathbf{smb}_{M_{+1}}(t,\lambda)=\begin{cases}\widehat{m}\left(\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\log\frac{\lambda}{1-\lambda}\right) & , \quad t=1\,,\\ 0 & , \quad t\in\mathbb{T}\setminus\{1\}\,,\end{cases}$$

$$\mathbf{smb}_{M_{-1}}(t,\lambda)=\begin{cases}\widehat{m}\left(\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\log\frac{\lambda}{1-\lambda}\right) & , \quad t=-1\,,\\ 0 & , \quad t\in\mathbb{T}\setminus\{-1\}\,.\end{cases}$$

From Lemma 4.1 and Lemma 4.5 we conclude the following corollary.

COROLLARY 4.6. *For arbitrary* $\varepsilon>0$ *, an operator* $R\in alg\,\mathcal{T}(\mathbf{PC})$ *admits the representation*

$$R=(\hat{g}_{j-k})_{j,k=0}^{\infty}+M_++M_-+R_c+R_\varepsilon\,, \tag{4.1}$$

where the ℓ^2*-operator norm of* R_ε *is less than* ε *, where* $R_c\in\mathcal{L}(l^2)$ *is a compact operator, where the generating function* g *of the Toeplitz matrix is piecewise continuous on* $\mathbb{T}$ *and continuous on* $\mathbb{T}\setminus\{\pm1\}$ *, and where* $M_\pm\in alg\,\mathcal{T}(\mathbf{PC})$ *are defined by*

$$M_+=\left(m_+\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)\frac{1}{k+\frac{1}{2}}\right)_{j,k=0}^{\infty}$$

and

$$M_-=\left((-1)^{j-k}m_-\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)\frac{1}{k+\frac{1}{2}}\right)_{j,k=0}^{\infty}$$

with suitably chosen functions $m_\pm\in\mathbf{C}^\infty(0,\infty)$ *.*

Now we prove that the operators $W_{3,4}\{A_n\}$ belong to the algebra $alg\,\mathcal{T}(\mathbf{PC})$ and calculate the symbols of these operators.

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LEMMA 4.7. *Let (1.4) and (1.5) hold, and let* $A_n := M_n(aI + b\mu^{-1}S\mu I + K)L_n$. *Then the operators* $W_{3/4}\{A_n\}$ *belong to the algebra* $alg\,\mathcal{T}(\mathbf{PC})$, *and their symbols are given by*

$$\mathbf{smb}_{W_{3/4}\{A_n\}}(t,\lambda)$$

$$= a(\pm 1) \pm b(\pm 1) \begin{cases} 1 & , \quad \Im\mathrm{m}\, t > 0\,, \\ -1 & , \quad \Im\mathrm{m}\, t < 0\,, \\ \mathrm{i}\cot\left(\pi\left[\frac{1}{4} + \chi_\pm + \frac{\mathrm{i}}{4\pi}\log\frac{\lambda}{1-\lambda}\right]\right) & , \quad t = 1\,, \\ \begin{Bmatrix} \mathrm{i}\cot\left(\pi\left[\frac{1}{4} - \frac{\mathrm{i}}{4\pi}\log\frac{\lambda}{1-\lambda}\right]\right) & : \ \tau = \sigma \\ -\mathrm{i}\cot\left(\pi\left[\frac{1}{4} + \frac{\mathrm{i}}{4\pi}\log\frac{\lambda}{1-\lambda}\right]\right) & : \ \tau = \varphi \end{Bmatrix} & , \quad t = -1\,. \end{cases} \tag{4.2}$$

where the numbers $\chi_\pm$ *are defined in Lemma 3.4.*

Proof. The proof in case $\tau = \varphi$ is given in [7, Sect. 8]. So, let us restrict to the case $\tau = \sigma$. For the discretized multiplication operators (see Lemma 3.2), the statements are obvious. It remains to consider the limit operators $\mathbf{S}$ and $\mathbf{A}_\pm^\mu$ (see Lemmata 3.3 and 3.4). Moreover, since the diagonal entries in the diagonal matrices $\mathbf{B}_\pm$ and $\mathbf{V}_\pm$ tend to zero (see (9.19) and (9.29)) and since the compact operators belong to $alg\,\mathcal{T}(\mathbf{PC})$, we only have to show that $\mathbf{S}$, $\mathbf{A}$, and $\mathbf{D}_\pm\mathbf{A}\mathbf{D}_\pm^{-1}$ belong to $alg\,\mathcal{T}(\mathbf{PC})$ (see (3.17) and (3.18)).

For the matrix $\mathbf{S}$, we have the relation $\mathbf{S} = T(\phi) - H(\phi)$, where $T(\phi)$ and $H(\phi)$ are Toeplitz and Hankel matrices, respectively, with the generating function $\phi(t) = \mathrm{sgn}\,(\Im\mathrm{m}\, t)$, $t \in \mathbb{T}$. From Lemma 4.1 and Lemma 4.2 we obtain that $\mathbf{S}$ belongs to $alg\,\mathcal{T}(\mathbf{PC})$ with the symbol

$$\mathbf{smb}_{\mathbf{S}}(t,\lambda) = \begin{cases} 1 & , \quad \Im\mathrm{m}\, t > 0\,, \\ -1 & , \quad \Im\mathrm{m}\, t < 0\,, \\ \pm(2\lambda - 1) + 2\mathrm{i}\sqrt{\lambda(1-\lambda)} & , \quad t = \pm 1 \end{cases}$$

$$= \begin{cases} 1 & , \quad \Im\mathrm{m}\, t > 0\,, \\ -1 & , \quad \Im\mathrm{m}\, t < 0\,, \\ \mathrm{i}\cot\left(\pi\left[\frac{1}{4} \pm \frac{\mathrm{i}}{4\pi}\log\frac{\lambda}{1-\lambda}\right]\right) & , \quad t = \pm 1\,. \end{cases} \tag{4.3}$$

Now, we consider the operators $\mathbf{D}_\pm\mathbf{A}\mathbf{D}_\pm^{-1}$. In [7, Sect. 7], the relation

$$\kappa(x) := (1-x)\frac{2x^{2\chi_\pm}}{1-x^2} = \frac{1}{2}\int_{\{\zeta:\Re\mathrm{e}\,\zeta = \psi\}} x^{-\zeta}\Big\{B(\zeta) - B(\zeta+1)\Big\}\,d\zeta\,, \quad x > 0\,,$$

is proved, where $\max\{-1/2, -2\chi_\pm\} < \psi < 1/4 - \chi_\pm$ and

$$B(\zeta) := -\mathrm{i}\cot\left(\pi\left(\frac{\zeta}{2} + \chi_\pm\right)\right) + i\cot\left(\pi\left(\zeta + \chi_\pm - \frac{1}{4}\right)\right)\,.$$

Consequently, we get, for $j \neq k$,

$$\kappa\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right) = \left(1-\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)^{2\chi_\pm} \frac{2}{1-\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2}}$$

$$= (k-j)\frac{\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)^{2\chi_\pm}(2k+1)}{(k-j)(j+k+1)},$$

such that

$$\mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1}$$

(4.4)

$$= \frac{1}{2}\int_{\{\zeta:\Re e\,\zeta=\psi\}} \left(\frac{1}{\pi \mathrm{i}} \frac{\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)^{-\zeta}(1-\delta_{j,k})}{j-k}\right)_{j,k=0}^{\infty} \Big\{B(\zeta)-B(\zeta+1)\Big\}\, d\zeta\,.$$

Obviosly, the matrix $\left(\frac{1-\delta_{j,k}}{\pi \mathrm{i}(j-k)}\right)_{j,k=0}^{\infty}$ is a Toeplitz matrix, and its generating function $g(e^{i2\pi s}) = \sum_{l\neq 0} \frac{1}{\pi i l}\frac{1}{l} e^{2i\pi ns} = 1-2s$, $0 \le s < 1$, is piecewise continuous on $\mathbb{T}$ and continuous on $\mathbb{T}\setminus\{1\}$. Thus, in view of Corollary 4.4, for any fixed $\zeta \in \mathbb{C}$ with $\Re e\,\zeta = \psi$, the matrix

$$T_\zeta := \left(\frac{1}{\pi \mathrm{i}} \frac{\left(\frac{j+\frac{1}{2}}{k+\frac{1}{2}}\right)^{-\zeta}(1-\delta_{j,k})}{j-k}\right)_{j,k=0}^{\infty}$$

belongs to $alg\,\mathcal{T}(\mathbf{PC})$, and its symbol is given by

$$\mathbf{smb}_{T_\zeta}(t,\lambda) = \begin{cases} 1-2s & , \quad t=e^{2\mathrm{i}\pi s}\in\mathbb{T}\setminus\{1\}\,, \\ \dfrac{\lambda-(1-\lambda)e^{-2\mathrm{i}\pi\zeta}}{\lambda+(1-\lambda)e^{-2\mathrm{i}\pi\zeta}} & , \quad t=1 \end{cases}$$

$$= \begin{cases} 1-2s & , \quad t=e^{2\mathrm{i}\pi s}\in\mathbb{T}\setminus\{1\}\,, \\ -\mathrm{i}\cot\left(\pi\left[\frac{1}{2}+\zeta+\frac{1}{2\pi \mathrm{i}}\log\frac{\lambda}{1-\lambda}\right]\right) & , \quad t=1\,. \end{cases}$$

Note that the integral in (4.4) has to be understood in the sence of Bochner (see [15]). This is possible since the operator function $\{\zeta : \Re\zeta = \psi\} \ni \zeta \mapsto T_\zeta$ is continuous (see Corollary 4.4) and uniformly bounded and since $\{\zeta : \Re\zeta = \psi\} \ni \zeta \mapsto B(\zeta)-B(\zeta+1)$ is a continuous and absolutely integrable function. Consequently, the integral representation (4.4) proves that the operators $\mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1}$ are in $alg\,\mathcal{T}(\mathbf{PC})$ and their symbols are equal to

$$\mathbf{smb}_{\mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1}}(t,\lambda) = \frac{1}{2}\int_{\{\zeta:\Re\zeta=\psi\}} \mathbf{smb}_{T_\zeta}(t,\lambda)\Big\{B(\zeta)-B(\zeta+1)\Big\}\, d\zeta$$

$$= \frac{1}{2}\left(\int_{\{\zeta:\Re\zeta=\psi\}} \mathbf{smb}_{T_\zeta}(t,\lambda)B(\zeta)\, d\zeta - \int_{\{\zeta:\Re\zeta=\psi+1\}} \mathbf{smb}_{T_{\zeta-1}}(t,\lambda)B(\zeta)\, d\zeta\right).$$

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We observe that $\mathbf{smb}_{T_\zeta}$ is 1-periodic with respect to the variable ζ, such that, applying the residue theorem, we arrive at

$$\mathbf{smb}_{\mathbf{D}_\pm \mathbf{A} \mathbf{D}_\pm^{-1}}(t,\lambda)$$

$$= \frac{1}{2}\int_{\{\zeta:\Re\zeta=\psi\}} \mathbf{smb}_{T_\zeta}(t,\lambda)B(\zeta)\,d\zeta - \frac{1}{2}\int_{\{\zeta:\Re\zeta=\psi+1\}} \mathbf{smb}_{T_\zeta}(t,\lambda)B(\zeta)\,d\zeta$$

$$(4.5)\qquad = \mathbf{smb}_{T_{1/4-\chi_\pm}}(t,\lambda) - \begin{cases} 0 & , \quad t\in\mathbb{T}\setminus\{1\}\,, \\ B\left(\dfrac{1}{2}+\dfrac{\mathrm{i}}{2\pi}\log\dfrac{\lambda}{1-\lambda}\right) & , \quad t=1 \end{cases}$$

$$= \begin{cases} 1-2s & , \quad t=e^{2\mathrm{i}\pi s}\in\mathbb{T}\setminus\{1\}\,, \\ \mathrm{i}\cot\left(\pi\left[\dfrac{1}{4}+\chi_\pm+\dfrac{\mathrm{i}}{4\pi}\log\dfrac{\lambda}{1-\lambda}\right]\right) & , \quad t=1\,. \end{cases}$$

In particular, if $\chi_\pm = 0$ we obtain that the operators $\mathbf{D}_\pm\mathbf{A}\mathbf{D}_\pm^{-1}$ are equal to the operator $\mathbf{A}$. Therefore $\mathbf{A}\in alg\,\mathcal{T}(\mathbf{PC})$ and

$$(4.6)\quad \mathbf{smb}_{\mathbf{A}}(t,\lambda) = \begin{cases} 1-2s & , \quad t=e^{2\mathrm{i}\pi s}\in\mathbb{T}\setminus\{1\}\,, \\ \mathrm{i}\cot\left(\pi\left[\dfrac{1}{4}+\dfrac{\mathrm{i}}{4\pi}\log\dfrac{\lambda}{1-\lambda}\right]\right) & , \quad t=1\,. \end{cases}$$

From this we conclude that $W_{3/4}\{A_n\}\in alg\,\mathcal{T}(\mathbf{PC})$. Moreover, since the symbol of the compact operators $\mathbf{B}_\pm$, $\mathbf{D}_\pm\mathbf{A}\mathbf{D}_\pm^{-1}\mathbf{W}\mathbf{V}_\pm$, and $\mathbf{V}_\pm\mathbf{A}^*\mathbf{W}$ are zero, we get (4.2) in view of (4.3), (4.5), (4.6) and (3.17), (3.18). □

5. The subalgebra $\mathcal{A}$ of the algebra $\mathcal{F}$. In this section we prove that further sequences of approximate operators belong to the algebra $\mathcal{F}$. Using these and the operator sequences of the collocation method, we shall form a C^*-algebra which is the basic algebra for the stability analysis of the collocation method.

For $R\in alg\,\mathcal{T}(\mathbf{PC})$, using the projections P_n and the notation Section 2, we define the finite sections $R_n := P_nRP_n \in \mathcal{L}(\mathrm{im}\,P_n)$ and form the operators $R_n^\omega := (E_n^\omega)^{-1}R_nE_n^\omega L_n$, $\omega\in\{3,4\}$, mapping $\mathrm{im}\,L_n$ into $\mathrm{im}\,L_n$. We will show that the sequences $\{R_n^3\}$ and $\{R_n^4\}$ belong to the algebra $\mathcal{F}$.

For $k,n\in\mathbb{Z}$ and $n\ge 1$, let $\widetilde{\varphi}_k^n = \widetilde{\varphi}_{k,\tau}^n$ denote the characteristic function of the interval

$$\left\{ \begin{array}{lc} \left[\dfrac{k-1}{n},\dfrac{k}{n}\right) & , \quad \tau=\sigma\,, \\ \left[\dfrac{k-\frac{1}{2}}{n+1},\dfrac{k+\frac{1}{2}}{n+1}\right) & , \quad \tau=\varphi\,, \end{array} \right\} \quad \text{multiplied by} \quad \begin{cases} \sqrt{n} & , \quad \tau=\sigma\,, \\ \sqrt{n+1} & , \quad \tau=\varphi\,. \end{cases}$$

Then the operators

$$\widetilde{E}_n : \ell^2_{\mathbb{Z}} \longrightarrow \mathbf{L}^2(\mathbb{R})\,,, \quad \{\xi_k\}_{k=-\infty}^\infty \mapsto \sum_{k=-\infty}^\infty \xi_k\widetilde{\varphi}_k^n$$

and

$$(\widetilde{E}_n)^{-1} : \mathrm{im}\,\widetilde{E}_n \longrightarrow \ell^2_{\mathbb{Z}}, \quad \sum_{k=-\infty}^\infty \xi_k\widetilde{\varphi}_k^n \mapsto \{\xi_k\}_{k=-\infty}^\infty$$

act as isometries. By $\widetilde{L}_n$ we denote the orthogonal projection from $\mathbf{L}^2(\mathbb{R})$ onto $\operatorname{im} \widetilde{E}_n$.

LEMMA 5.1 ([4], Prop. 2.10). *For any operator $R \in alg\, \mathcal{T}(\mathbf{PC})$ the sequence*

$$\widetilde{E}_n R(\widetilde{E}_n)^{-1}\widetilde{L}_n : \mathbf{L}^2(\mathbb{R}) \longrightarrow \mathbf{L}^2(\mathbb{R})$$

is strongly convergent.

Let $\widetilde{W} : \ell^2 \longrightarrow \ell^2$, be defined by $\widetilde{W}\xi = \left\{(-1)^k \xi_k\right\}_{k=0}^{\infty}$, and let L_n^0 , W_n^0 , V_n^0 , $\widetilde{V}_n^0$, and M_n^0 refer to the operators L_n , W_n , V_n , $\widetilde{V}_n$, and M_n , respectively, in the special case $\alpha = \beta = -\frac{1}{2}$ (i.e. $\nu = \sigma$). In particular, $V_n^0 : \operatorname{im} L_n^0 \longrightarrow \operatorname{im} P_n$ and $(V_n^0)^{-1} : \operatorname{im} P_n \longrightarrow \operatorname{im} L_n^0$ are given by

$$V_n^0 u = \{\omega_n^\tau u(x_{1n}^\tau), \ldots, \omega_n^\tau u(x_{nn}^\tau), 0, 0, \ldots\} =: \left\{\omega_n^\tau u(x_{k+1,n}^\tau)\right\}_{k=0}^{n-1} ,$$

with

$$\omega_n^\tau = \sqrt{\widetilde{\omega}_n^\tau} = \begin{cases} \sqrt{\frac{\pi}{n}} & , \quad \tau = \sigma , \\ \sqrt{\frac{\pi}{n+1}} & , \quad \tau = \varphi , \end{cases}$$

and

$$(V_n^0)^{-1}\xi = \sum_{k=1}^{n} \frac{\xi_{k-1}}{\omega_n^\tau}\, \widetilde{\ell}_{kn}^{\tau,0} , \qquad \widetilde{\ell}_{kn}^{\tau,0} = \frac{\varphi\, \ell_{kn}^\tau}{\varphi(x_{kn}^\tau)} ,$$

respectively. One can easily check that

$$\begin{gathered} L_n = \rho^{-1} L_n^0 \rho I , \quad W_n = \rho^{-1} W_n^0 \rho I , \\ V_n = V_n^0 \rho I , \quad \widetilde{V}_n = \widetilde{V}_n^0 \rho I , \quad M_n = \rho^{-1} M_n^0 \rho I . \end{gathered} \tag{5.1}$$

LEMMA 5.2. *Let the operators $V_{n,0}$ be defined by*

$$V_{n,0} : \operatorname{im} L_{n,0} \longrightarrow \operatorname{im} P_n \subset \ell^2 , \quad u \mapsto \left\{\omega_n^\sigma u(x_{k+1,n}^\sigma)\right\}_{k=0}^{n-1} ,$$

and

$$L_{n,0} : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2 , \quad u \mapsto \sum_{k=0}^{n-1} \langle u, T_k \rangle_\sigma T_k .$$

Then the sequences $\left\{(V_n^\sigma)^{-1} V_{n,0} J_\nu L_n\right\}$ and $\left\{J_\nu^{-1} V_{n,0}^{-1} V_n^\sigma L_n\right\}$ belong to the algebra $\mathcal{F}_2$.

Proof. The uniform boundedness of these sequences follows from the uniform boundedness of $\{V_n^\sigma\}$, $\left\{(V_n^\sigma)^{-1}\right\}$ (comp. the proof of Lemma 2.5) and of $\{V_{n,0}\}$, $\left\{V_{n,0}^{-1}\right\}$, where, for $u \in \operatorname{im} L_{n,0}$, the equalities

$$\|V_{n,0}u\|_{\ell^2}^2 = \frac{\pi}{n} \sum_{k=1}^{n} |u(x_{kn}^\sigma)|^2 = \int_{-1}^{1} |u(x)|^2 \sigma(x)\, dx = \|u\|_\sigma^2$$

have to be taken into account. Using

$$\begin{aligned} (V_n^\sigma)^{-1} V_{n,0} J_\nu L_n u &= (V_n^\sigma)^{-1} V_{n,0} \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\nu \, \gamma_k T_k \\ &= \sum_{j=1}^{n} \rho^{-1}(x_{jn}^\sigma) \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\nu \, (J_\nu \widetilde{u}_k)(x_{jn}^\sigma) \widetilde{\ell}_{jn}^\sigma \\ &= M_n^\sigma \rho^{-1} J_\nu L_n u \end{aligned}$$

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and

$$(5.2)\quad J_\nu^{-1}V_{n,0}^{-1}V_n^\sigma L_n u = J_\nu^{-1}\sum_{j=1}^{n}\rho(x_{jn}^\sigma)\sum_{k=0}^{n-1}\langle u,\widetilde{u}_k\rangle_\nu\,\widetilde{u}_k(x_{jn}^\sigma)\ell_{jn}^\sigma = J_\nu^{-1}L_n^\sigma\rho L_n u\,,$$

as well as Lemma 2.2 and Corollary 2.3 we get, for $n > m$,

$$(V_n^\sigma)^{-1}V_{n,0}J_\nu L_n\widetilde{u}_m = \gamma_m M_n^\sigma\rho^{-1}T_m \longrightarrow \rho^{-1}\gamma_m T_m = \rho^{-1}J_\nu\widetilde{u}_m$$

and

$$J_\nu^{-1}V_{n,0}^{-1}V_n^\sigma L_n\widetilde{u}_m = J_\nu^{-1}L_n^\sigma\rho\widetilde{u}_m \longrightarrow \rho\widetilde{u}_m \quad \text{in} \quad \mathbf{L}^2_\nu\,.$$

From (3.6), (3.9), and (3.8), for $n > m$, we get

$$\begin{aligned}
&W_n(V_n^\sigma)^{-1}V_{n,0}J_\nu W_n L_n\widetilde{u}_m\\
&\quad = W_n M_n\rho^{-1}\gamma_{n-1-m}T_{n-1-m}\\
&\quad = \sum_{j=0}^{n-1}\alpha_{n-1-j,n}^\sigma(\rho^{-1}\gamma_{n-1-m}T_{n-1-m})\widetilde{u}_j\\
&\quad = \sum_{j=0}^{n-1}\varepsilon_{n-1-j,n}\frac{\pi}{n}\sum_{k=1}^{n}\rho(x_{kn}^\sigma)\gamma_{n-1-m}T_{n-1-m}(x_{kn}^\sigma)\widetilde{u}_{n-1-j}(x_{kn}^\sigma)\widetilde{u}_j\\
&\quad = \sum_{j=0}^{n-1}\varepsilon_{n-1-j,n}\frac{\pi}{n}\sum_{k=1}^{n}\rho(x_{kn}^\sigma)\widetilde{u}_m(x_{kn}^\sigma)\gamma_j T_j(x_{kn}^\sigma)\widetilde{u}_j\\
&\quad = \sum_{j=0}^{n-1}\frac{\pi}{n}\sum_{k=1}^{n}\rho(x_{kn}^\sigma)\widetilde{u}_m(x_{kn}^\sigma)T_j(x_{kn}^\sigma)J_\nu^{-1}T_j = J_\nu^{-1}L_n^\sigma\rho\widetilde{u}_m\,.
\end{aligned}$$

Consequently, due to (5.2),

$$W_n(V_n^\sigma)^{-1}V_{n,0}J_\nu W_n L_n = J_\nu^{-1}L_n^\sigma\rho L_n = J_\nu^{-1}V_{n,0}^{-1}V_n L_n$$

and

$$W_n J_\nu^{-1}V_{n,0}^{-1}V_n W_n L_n = (V_n^\sigma)^{-1}V_{n,0}J_\nu L_n\,,$$

and the strong convergence of these sequences follows from the previous part of the proof.

Taking into account Lemma 2.4 and the fact that $V_{n,0}^* = V_{n,0}^{-1}$ one can easily conclude the strong convergence of the respective sequences of adjoint operators. □

LEMMA 5.3. *For any operator $R \in alg\,\mathcal{T}(\mathbf{PC})$, the sequences $\{R_n^3\}$ and $\{R_n^4\}$ belong to the algebra $\mathcal{F}$. If R is the Toeplitz operator $(\hat{g}_{j-k})_{j,k=0}^\infty$ then*

$$W_3(R_n^3) = W_4(R_n^4) = R\,,\quad W_4(R_n^3) = W_3(R_n^4) = \widetilde{R}\,,\quad \widetilde{R} := (\hat{g}_{k-j})_{j,k=0}^\infty\,.$$

Proof. In case of $\tau = \varphi$ the statements of the lemma have already been proved in [7, Lemma 4.1 (ii)]. Nevertheless, here we give a proof for both cases by other means.

For $k=1,\ldots,n$, define functions $\varphi^n_{k,\tau}:[-1,1]\longrightarrow\mathbb{R}$ by

$$\varphi^n_{k,\sigma}(x)=\begin{cases}\sqrt{\dfrac{n}{\pi}} & , \quad \cos\dfrac{k}{n}\pi\le x<\cos\dfrac{k-1}{n}\pi\,,\\ 0 & , \quad \text{otherwise,}\end{cases}$$

and

$$\varphi^n_{k,\varphi}(x)=\begin{cases}\sqrt{\dfrac{n+1}{\pi}} & , \quad \cos\dfrac{k+\frac12}{n+1}\pi\le x<\cos\dfrac{k-\frac12}{n+1}\pi\,,\\ 0 & , \quad \text{otherwise,}\end{cases}$$

and let by $T^\tau_n, S^\tau_n:\mathbf{L}^2_\sigma\longrightarrow\mathbf{L}^2_\sigma$ refer to the operators

$$T^\tau_n=\frac{1}{\omega^\tau_n}\sum_{k=1}^n\langle u,\varphi^n_{k,\tau}\rangle_\sigma\widetilde{\ell}^{\tau,0}_{kn}=(V^0_n)^{-1}\left\{\langle u,\varphi^n_{k+1,\tau}\rangle_\sigma\right\}_{k=0}^{n-1}\,,\qquad S^\tau_n u=\sum_{k=1}^n\langle u,\varphi^n_{k,\tau}\rangle_\sigma\varphi^n_{k,\tau}\,.$$

Then, in view of the uniform boundedness of $(V^0_n)^{-1}$ (see (2.3) and the proof of Lemma 2.5),

$$\|T^\tau_n u\|^2_\sigma\le C\sum_{k=1}^n|\langle u,\varphi^n_{k,\tau}\rangle_\sigma|^2=C\,\|S^\sigma_n u\|^2_\sigma\le C\,\|u\|^2_\sigma\,,$$

i.e. the sequence $\{T^\tau_n\}\subset\mathcal{L}(\mathbf{L}^2_\sigma)$ is uniformly bounded. Moreover, for the characteristic function $u=\chi_{[x,y]}$ of an interval $[x,y]\subset[-1,1]$, we have

$$\left|\langle u,\varphi^n_{k,\sigma}\rangle_\sigma-\sqrt{\frac{\pi}{n}}u(x^\sigma_{kn})\right|=\left|\sqrt{\frac{n}{\pi}}\int_{\frac{k-1}{n}\pi}^{\frac{k}{n}\pi}\left[u(\cos s)-u\left(\cos\frac{2k-1}{2n}\pi\right)\right]\,ds\right|$$
$$\le\begin{cases}0 & , \quad x,y\notin\left(\cos\dfrac{k}{n}\pi,\cos\dfrac{k-1}{n}\pi\right)\,,\\ \sqrt{\frac{\pi}{n}} & , \quad \text{otherwise,}\end{cases}$$

as well as

$$\left|\langle u,\varphi^n_{k,\varphi}\rangle_\sigma-\sqrt{\frac{\pi}{n+1}}u(x^\varphi_{kn})\right|=\left|\sqrt{\frac{n+1}{\pi}}\int_{\frac{k-\frac12}{n+1}\pi}^{\frac{k+\frac12}{n+1}\pi}\left[u(\cos s)-u\left(\cos\frac{k\pi}{n+1}\right)\right]\,ds\right|$$
$$\le\begin{cases}0 & , \quad x,y\notin\left(\cos\dfrac{k+\frac12}{n+1}\pi,\cos\dfrac{k-\frac12}{n+1}\pi\right)\,,\\ \sqrt{\frac{\pi}{n+1}} & , \quad \text{otherwise,}\end{cases}$$

which implies

$$\|T^\tau_n u-M^0_n u\|^2_\sigma=\left\|(V^0_n)^{-1}\left\{\langle u,\varphi^n_{k+1,\tau}\rangle_\sigma-\omega^\tau_n u(x^\tau_{kn})\right\}_{k=0}^{n-1}\right\|^2_\sigma\le C\,\frac{2\pi}{n}\,.$$

Consequently, $T^\tau_n u\longrightarrow u$ in $\mathbf{L}^2_\sigma$ for all $u\in\mathbf{L}^2_\sigma$. In particular, we get the equivalences ($\xi^n_k\in\mathbb{C}$)

$$\sum_{k=1}^n\xi^n_k\widetilde{\ell}^{\tau,0}_{kn}\longrightarrow u\quad\text{in}\quad\mathbf{L}^2_\sigma\iff\lim_{n\to\infty}\left\|\sum_{k=1}^n\xi^n_k\widetilde{\ell}^{\tau,0}_{kn}-T^\tau_n u\right\|_\sigma=0$$

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$$\iff \lim_{n\to\infty} \sum_{k=1}^{n} \left| \omega_n^\tau \xi_k^n - \langle u, \varphi_{k,\tau}^n \rangle_\sigma \right|^2 = 0$$

$$\iff \lim_{n\to\infty} \left\| \sum_{k=1}^{n} \omega_n^\tau \xi_k^n \varphi_{k,\tau}^n - S_n^\tau u \right\|_\sigma = 0$$

$$\iff \omega_n^\tau \sum_{k=1}^{n} \xi_k^n \varphi_{k,\tau}^n \longrightarrow u \quad \text{in} \quad \mathbf{L}_\sigma^2 \,.$$

Since $T_n^\tau \longrightarrow I$ in $\mathbf{L}_\sigma^2$, the convergence $(V_n^0)^{-1} R_n (V_n^0) L_n^0 u \longrightarrow g$ in $\mathbf{L}_\sigma^2$ for some $u \in \mathbf{L}_\sigma^2$ is equivalent to

$$(V_n^0)^{-1} R_n V_n^0 T_n^\tau u = (V_n^0)^{-1} R_n \left\{ \langle u, \varphi_{k+1,\tau}^n \rangle_\sigma \right\}_{k=0}^{n-1} \longrightarrow g \quad \text{in} \quad \mathbf{L}_\sigma^2$$

and so, due to the previous considerations, equivalent to

$$\sum_{j=1}^{n} \sum_{k=1}^{n} r_{j-1,k-1} \langle u, \varphi_{k,\tau}^n \rangle_\sigma \varphi_{j,\tau}^n \longrightarrow g \quad \text{in} \quad \mathbf{L}_\sigma^2 \,, \tag{5.3}$$

where $R = [r_{jk}]_{j,k=0}^\infty$.

The mapping $T : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}^2(0,1)$ defined by $(Tu)(s) = \sqrt{\pi} u(\cos \pi s)$ is an isometrical isomorphism, whereby $T\varphi_{k,\tau}^n = \widetilde{\varphi}_{k,\tau}^n$, $k = 1, \ldots, n$. Consequently, (5.3) is equivalent to

$$\chi_{[0,1]} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} r_{j-1,k-1} \langle \chi_{[0,1]} Tu, \widetilde{\varphi}_{k,\tau}^n \rangle_{\mathbf{L}^2(\mathbb{R})} \widetilde{\varphi}_{j,\tau}^n \to \chi_{[0,1]} Tg \quad \text{in} \quad \mathbf{L}^2(\mathbb{R}) \,. \tag{5.4}$$

The left-hand side of (5.4) can be written as $\chi_{[0,1]} \widetilde{E}_n R(\widetilde{E}_n)^{-1} \widetilde{L}_n \chi_{[0,1]} Tu$, and Lemma 5.1 guarantees the convergence of this sequence. Hence, we have proved that the strong limit of $(V_n^0)^{-1} R_n V_n^0 L_n^0$ in $\mathbf{L}_\sigma^2$ exists. Since $\rho I : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\sigma^2$ is an isometrical isomorphism, the strong convergence of $V_n^{-1} R_n V_n L_n = \rho^{-1} (V_n^0)^{-1} R_n V_n^0 L_n^0 \rho I$ in $\mathbf{L}_\nu^2$ follows, where we have used (5.1).

To prove the convergence of $\{W_n R_n^3 W_n\}$, we remark that by definitions and by taking into account (3.8) and

$$\widetilde{u}_{n-1-m}(x_{kn}^\varphi) = (-1)^{k+1} \widetilde{u}_m(x_{kn}^\varphi) \,,$$

we find that, for $u \in \mathbf{L}_\nu^2$, the relations

$$\begin{aligned} V_n W_n u &= V_n \left(\sum_{m=0}^{n-1} \langle u, \widetilde{u}_m \rangle_\nu \widetilde{u}_{n-1-m} \right) \\ &= \left\{ \omega_n^\tau \rho(x_{k+1,n}^\tau) \sum_{m=0}^{n-1} \langle u, \widetilde{u}_m \rangle_\nu \widetilde{u}_{n-1-m}(x_{k+1,n}^\tau) \right\}_{k=0}^{n-1} \\ &= \left\{ (-1)^k \omega_n^\tau \sum_{m=0}^{n-1} \langle u, \widetilde{u}_m \rangle_\nu \left\{ \begin{array}{c} (J_\nu \widetilde{u}_m)(x_{k+1,n}^\sigma) \\ \rho(x_{k+1,n}^\tau) \widetilde{u}_m(x_{k+1,n}^\varphi) \end{array} \right\} \right\}_{k=0}^{n-1} \\ &= \left\{ \begin{array}{ll} \widetilde{W} V_{n,0} J_\nu L_n u & , \quad \tau = \sigma \,, \\ \widetilde{W} V_n L_n u & , \quad \tau = \varphi \,, \end{array} \right. \end{aligned} \tag{5.5}$$

are valid, where the operators $V_{n,0}$ are defined in Lemma 5.2. Consequently, for $\xi \in \operatorname{im} P_n$, we have

$$W_n V_n^{-1}\xi = \begin{cases} J_\nu^{-1}(V_{n,0})^{-1}\widetilde{W}\xi & , \quad \tau = \sigma\,, \\ V_n^{-1}\widetilde{W}\xi & , \quad \tau = \varphi\,, \end{cases}$$

and

$$W_n V_n^{-1} R_n V_n W_n = \begin{cases} J_\nu^{-1}(V_{n,0})^{-1} P_n \widetilde{W} R \widetilde{W} P_n V_{n,0} J_\nu L_n & , \quad \tau = \sigma\,, \\ V_n^{-1} P_n \widetilde{W} R \widetilde{W} P_n V_n L_n & , \quad \tau = \varphi\,. \end{cases} \tag{5.6}$$

In case $R = (\hat{g}_{j-k})_{j,k=0}^\infty$ is a Toeplitz matrix with generating function $g(t) = \sum_{k\in\mathbb{Z}} \hat{g}_k t^k$, $t \in \mathbb{T}$, we get $\widetilde{W} R \widetilde{W} = R_-$, where R_- is the Toeplitz matrix with the generating function $g(-t)$. Hence, since $\widetilde{W}^2 = I$, $\widetilde{W} R \widetilde{W} \in alg\,\mathcal{T}(\mathbf{PC})$ if $R \in alg\,\mathcal{T}(\mathbf{PC})$. Thus, by (5.6) and Lemma 5.2 we get the existence of $W_2\{R_n^3\}$.

Obviously,

$$V_n R_n^3 V_n^{-1} P_n = V_n V_n^{-1} P_n R P_n V_n V_n^{-1} P_n = P_n R P_n \longrightarrow R \quad \text{in} \quad \ell^2$$

for each $R \in alg\,\mathcal{T}(\mathbf{PC})$. In view of (2.3) we have

$$\widetilde{V}_n R_n^3 \widetilde{V}_n^{-1} P_n = \widetilde{V}_n V_n^{-1} P_n R P_n V_n \widetilde{V}_n^{-1} P_n = \widetilde{W}_n P_n R \widetilde{W}_n P_n\,.$$

In case of $R = (\hat{g}_{j-k})_{j,k=0}^\infty$ this is equal to $P_n \widetilde{R} P_n$ with $\widetilde{R} = (\hat{g}_{k-j})_{j,k=0}^\infty$. Morover, it is well known that $\widetilde{W}_n P_n R \widetilde{W}_n P_n$ converges strongly in ℓ^2 for each $R \in alg\,\mathcal{T}(\mathbf{PC})$ (comp. [1, Cor. 7.14]).

The strong convergence for the respective sequences of adjoint operators can now be proved with the help of (2.4), Lemma 2.4, and the relations

$$W_n L_{n-1} = (L_n - L_0) W_n\,, \quad L_{n-1} W_n = W_n (L_n - L_0)\,.$$

The proof for $\{R_n^4\}$ is analogous. □

By $\mathcal{A}$ we denote the smallest C^*-subalgebra of $\mathcal{F}$ generated by all sequences of the ideal $\mathcal{J}$, by all sequences $\{R_n^\omega\}$ with $\omega \in \{3,4\}$ and $R \in alg\,\mathcal{T}(\mathbf{PC})$, and by all sequences of the form $\{M_n(aI + b\mu^{-1}S\mu I)L_n\}$, $a, b \in \mathbf{PC}$, where $\mu := v^{\gamma,\delta}$ satisfies (1.4) and (1.5). We shall check the invertibility of the coset $\{A_n\} + \mathcal{J}$ (of the collocation sequence) in $\mathcal{F}/\mathcal{J}$ (see Theorem 2.9) by checking the invertibility in the quotient algebra $\mathcal{A}/\mathcal{J}$. For $\{A_n\} \in \mathcal{F}$, we write $\{A_n\}^o$ for the coset $\{A_n\} + \mathcal{J} \in \mathcal{F}/\mathcal{J}$.

6. Application of the local principle of Allan and Douglas. In this section we show that the set $\mathcal{C} := \{\{M_n f L_n\}^o : f \in \mathbf{C}[-1,1]\}$ forms a subalgebra contained in the center of the quotient algebra $\mathcal{A}/\mathcal{J}$. This result will enable us to apply the local principle of Allan and Douglas in order to prove the invertibility of an element of $\mathcal{A}/\mathcal{J}$. Moreover, by $\mathcal{A}_0$ we will denote the smallest C^*-subalgebra of $\mathcal{F}$ which contains all sequences of the form $\{M_n(aI + b\mu^{-1}S\mu I)L_n\}$, $a, b \in \mathbf{PC}$, $\mu = v^{\gamma,\delta}$ satisfying (1.4) and (1.5), and all sequences from the ideal $\mathcal{J}$.

6.1. A Subalgebra in the center of the quotient algebra $\mathcal{A}/\mathcal{J}$. At first, we prove some auxiliary results.

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LEMMA 6.1. *Suppose* χ^s *and* χ^b *are continuous functions over* $[-1,1]$ *such that, for* $x \in [-1,1]$, $|\chi^s(x)|, |\chi^b(x)| \in [0,1]$, *such that* χ^s *has a small support with* $\operatorname{supp}[\chi^s \circ \cos] \subset [t-\varepsilon^s, t+\varepsilon^s]$, *where* $\cos$ *is considered as a function defined on* $[0,\pi]$, *and such that* χ^b *has a support with* $\operatorname{supp}[\chi^b \circ \cos] \cap [t-\varepsilon^b, t+\varepsilon^b] = \emptyset$. *Then, for any* $R \in alg\,\mathcal{T}(\mathbf{PC})$ *and for any* $\varepsilon > 0$, *there exists a constant* C *such that* $\varepsilon^b/\varepsilon^s > C$ *implies the locality property*

$$\left\| \left(\chi^b(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} R_n \left(\chi^s(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon\,,$$

$$\left\| \left(\chi^s(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} R_n \left(\chi^b(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon\,.$$

Moreover, if the support of χ^s *satisfies* $\operatorname{supp}[\chi^s \circ \cos] \subset [t-\varepsilon^s, t+\varepsilon^s] \subset [0, \pi-\varepsilon^b]$, *then we get*

$$\left\| (I-P_n)RP_n \left(\chi^s(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon\,,$$

$$\left\| \left(\chi^s(x^\tau_{j+1,n})\delta_{j,k}\right)_{j,k=0}^{n-1} P_nR(I-P_n) \right\|_{\mathcal{L}(\ell^2)} \le \varepsilon\,.$$

The proof is independently of the choice of τ and can be found in the proof of [7, Lemma 4.1 (i)].

LEMMA 6.2. *Let*

$$T(\phi) = \left(\frac{1-(-1)^{j-k}}{\pi\mathrm{i}(j-k)} \right)_{j,k=0}^{\infty}$$

be the Toeplitz matrix with the generating function $\phi = \operatorname{sgn} \Im\mathrm{m}\, t$, $t \in \mathbb{T}$, *and let* χ, $\widetilde{\chi}$ *be continuous functions with* $\operatorname{supp}\chi, \operatorname{supp}\widetilde{\chi} \subset (-1,1)$. *Then the sequence*

$$\{M_n^\sigma \chi L_n [M_n^\sigma \rho^{-1} S \rho L_n - [T(\phi)]_n^3] M_n^\sigma \widetilde{\chi} L_n\}$$

belongs to the ideal $\mathcal{J}_2 \subset \mathcal{J}^\sigma$.

Proof. In view of (3.14) we have

$$M_n^\sigma \rho^{-1} S \rho L_n - [T(\phi)]_n^3 = (V_n^\sigma)^{-1} \left\{ \frac{1-(-1)^{j+k}}{2} \left(\frac{2}{\pi\mathrm{i}\,(j-k)} - \frac{\cos\dfrac{j-k}{2n}\pi}{n\mathrm{i}\,\sin\dfrac{j-k}{2n}\pi} \right)_{j,k=0}^{n-1} \right.$$

(6.1)

$$\left. - \frac{1+(-1)^{j+k}}{2} \left(\frac{\cos\dfrac{j+k-1}{2n}\pi}{n\mathrm{i}\,\sin\dfrac{j+k-1}{2n}\pi} \right)_{j,k=0}^{n-1} \right\} V_n^\sigma L_n\,.$$

Now we define functions $k^1(t,s)$ and $k^2(t,s)$ on $[0,\pi]^2$ by

$$k^1(t,s) := \frac{\chi(\cos t)\widetilde{\chi}(\cos s)}{\pi\mathrm{i}\,\rho(\cos t)\vartheta(\cos s)} \left[\frac{1}{t-s} - \frac{\cos\dfrac{t-s}{2}}{\sin\dfrac{t-s}{2}} \right]$$

and

$$k^2(t,s) := \frac{\chi(\cos t)\widetilde{\chi}(\cos s)}{\pi \mathrm{i}\, \rho(\cos t)\vartheta(\cos s)} \frac{\cos \dfrac{t+s}{2}}{\sin \dfrac{t+s}{2}}\,.$$

Clearly, these functions are continuous, and the integral operators K^1 and K^2 with the kernels $k^1(\arccos x, \arccos y)$ and $k^2(\arccos x, \arccos y)$, respectively, can be approximated by quadrature methods $K_n^1, K_n^2 \in \mathcal{L}(\mathrm{im}\, L_n)$ such that

$$K_n^1 = M_n^\sigma \chi L_n (V_n^\sigma)^{-1} \left(\frac{2}{\pi \mathrm{i}(j-k)} - \frac{\cos \dfrac{j-k}{2n}\pi}{n\mathrm{i} \sin \dfrac{j-k}{2n}\pi} \right)_{j,k=0}^{n-1} V_n^\sigma L_n M_n^\sigma \widetilde{\chi} L_n\,, \tag{6.2}$$

$$K_n^2 = M_n^\sigma \chi L_n (V_n^\sigma)^{-1} \left(\frac{\cos \dfrac{j+k-1}{2n}\pi}{n\mathrm{i} \sin \dfrac{j+k-1}{2n}\pi} \right)_{j,k=0}^{n-1} V_n^\sigma L_n M_n^\sigma \widetilde{\chi} L_n\,, \tag{6.3}$$

and $\{K_n^1\}, \{K_n^2\} \in \mathcal{J}_2$ (see Lemma 2.10). Furthermore, in view of (5.5), we obtain

$$P_n \widetilde{W} P_n = V_{n,0} J_\nu W_n V_n^{-1} P_n \quad \text{and} \quad P_n \widetilde{W} P_n = V_n W_n J_\nu^{-1} V_{n,0}^{-1} P_n\,.$$

Using these relations together with (6.1), (6.2), and (6.3), we can write

$$M_n^\sigma \chi L_n [M_n \rho^{-1} S \rho L_n - [T(\phi)]_n^3] M_n \widetilde{\chi} L_n$$

$$= \frac{1}{2}[K_n^1 - K_n^2] - \frac{1}{2}(V_n^\sigma)^{-1} V_{n,0} J_\nu W_n [K_n^1 + K_n^2] W_n J_\nu^{-1} V_{n,0}^{-1} V_n^\sigma L_n\,.$$

Now, the assertion of the lemma follows immediately from Lemma 5.2. □

LEMMA 6.3. *Let $M_{\pm 1}$ be the operators defined in Lemma 4.5, and let $\chi, \widetilde{\chi}$ be continuous functions with* $\mathrm{supp}\,\chi, \mathrm{supp}\,\widetilde{\chi} \subset (-1,1)$. *Then the sequences*

$$\{M_n^\sigma \chi L_n [M_{\pm 1}]_n^3 M_n^\sigma \widetilde{\chi} L_n\}$$

belong to the ideal $\mathcal{J}_2 \subset \mathcal{J}^\sigma$.

Proof. Setting

$$k(x,y) = \frac{\chi(x)\widetilde{\chi}(y)}{\rho(x)\vartheta(y)}\, m\left(\frac{\arccos x}{\arccos y}\right) \frac{1}{\arccos y},$$

the operator $\{M_n^\sigma \chi L_n [M_{+1}]_n^3 M_n^\sigma \widetilde{\chi} L_n\}$ takes the form K_n of Lemma 2.10 and, consequently, $\{M_n^\sigma \chi L_n [M_{+1}]_n^3 M_n^\sigma \widetilde{\chi} L_n\} \in \mathcal{J}_2 \subset \mathcal{J}^\sigma$.

The proof for M_{-1} is analogous. □

In case of $\tau = \varphi$, the following lemma can be found in [7, Lemma 5.1]. Taking into account the previous results of the present section, the proof in case $\tau = \sigma$ is completely analogous.

LEMMA 6.4. *For $f \in \mathbf{C}[-1,1]$, the coset $\{M_n f L_n\}^o$ belongs to the center of $\mathcal{A}/\mathcal{J}$.*

Due to the last lemma the set $\mathcal{C} = \{\{M_n f L_n\}^o : f \in \mathbf{C}[-1,1]\}$ forms a C^*-subalgebra of the center of $\mathcal{A}/\mathcal{J}$. This subalgebra is *-isomorphic to $\mathbf{C}[-1,1]$ via the isomorphism $\{M_n f L_n\}^o \mapsto f$, and, consequently, the maximal ideal space of $\mathcal{C}$ is equal to $\{\mathcal{I}_\tau : \tau \in [-1,1]\}$ with $\mathcal{I}_t := \{\{M_n f L_n\}^o : f \in \mathbf{C}[-1,1], f(t) = 0\}$. By $\mathcal{J}_t$ we denote the smallest closed ideal of $\mathcal{A}/\mathcal{J}$ which contains $\mathcal{I}_t$, i.e. $\mathcal{J}_t$ is equal to

$$\operatorname{clos}_{\mathcal{A}/\mathcal{J}} \left\{ \sum_{j=1}^m \{A_n^j M_n f_j L_n\}^o : \{A_n^j\} \in \mathcal{A},\ f_j \in \mathbf{C}[-1,1],\ f_j(t) = 0, m = 1,2,\dots \right\}.$$

The local principle of Allan and Douglas claims the following.

THEOREM 6.5. *The ideal $\mathcal{J}_t$ is a proper ideal in $\mathcal{A}/\mathcal{J}$ for all $t \in [-1,1]$. An element $\{A_n\}^o$ of $\mathcal{A}/\mathcal{J}$ is invertible if and only if $\{A_n\}^o + \mathcal{J}_t$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_t$ for all $t \in [-1,1]$.*

6.2. The local invertibility at the points $t \in (-1,1)$. This section is devoted to the invertibility of $\{A_n\}^o + \mathcal{J}_\tau$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_t$ for t in the interior of the interval $[-1,1]$ (see Theorem 6.5).

LEMMA 6.6. *Let $\{A_n\} \in \mathcal{A}_0$. If the limit operator $W_1\{A_n\} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is Fredholm then the coset $\{A_n\}^o + \mathcal{J}_t$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_t$ for $t \in (-1,1)$.*

Proof. The case $\tau = \varphi$ is considered in [7, Section 6]. Since the proof of the lemma in case of $\tau = \sigma$ is completely analogous we give only an outline of it. We fix a $t \in (-1,1)$ and set

$$h_t(x) := \begin{cases} 0 & \text{if} \quad -1 \le x \le t, \\ 1 & \text{if} \quad t < x \le 1. \end{cases}$$

The subalgebra of $(\mathcal{A}/\mathcal{J})/\mathcal{J}_t$ containing all cosets $\{M_n^\sigma(aI + b\mu^{-1}S\mu I)L_n\}^o + \mathcal{J}_t$ is generated by $e = \{L_n\}^o + \mathcal{J}_t$,

$$p := \frac{1}{2}\left(\{L_n\}^o + \{M_n^\sigma \rho^{-1} S \rho L_n\}^o\right) + \mathcal{J}_t, \quad \text{and} \quad q := \{M_n^\sigma h_t L_n\}^0 + \mathcal{J}_t.$$

Obviously, q is a selfadjoint projection. We prove that the same is true for p. We have ([7, (6.4)])

$$\rho^{-1} S \rho \varphi \rho^{-1} S \rho I = \varphi I + K_0, \quad K_0 u = -\frac{1}{\sqrt{2}} \langle u, \widetilde{u}_0 \rangle_\nu \rho^{-1} T_0. \tag{6.4}$$

Due to (3.1) we can write

$$\begin{aligned} M_n^\sigma \varphi \rho^{-1} S \rho L_n u &= M_n^\sigma \varphi \rho^{-1} S \rho \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\nu \widetilde{u}_k \\ &= \mathrm{i} M_n^\sigma \vartheta \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\nu T_{k+1} = \mathrm{i} M_n^\sigma \vartheta \sum_{k=0}^{n-2} \langle u, \widetilde{u}_k \rangle_\nu T_{k+1} \\ &= \varphi \rho^{-1} S \rho L_{n-1} u = \varphi \rho^{-1} S \rho (L_n - W_n L_1 W_n) u. \end{aligned}$$

Together with (6.4), we get the identity

$$M_n^\sigma \rho^{-1} S \rho L_n M_n^\sigma \varphi \rho^{-1} S \rho L_n = M_n^\sigma (\varphi I + K_0)(L_n - W_n L_1 W_n)$$

and, consequently,

$$\{M_n^\sigma \rho^{-1} S\rho L_n\}^o \{M_n^\sigma \rho^{-1} S\rho L_n\}^o + \mathcal{J}_t$$

$$= \frac{1}{\varphi(\tau)}\{M_n^\sigma \rho^{-1} S\rho L_n\}^o \{M_n^\sigma \varphi\rho^{-1} S\rho L_n\}^o + \mathcal{J}_t$$

$$= \frac{1}{\varphi(\tau)}\{M_n^\sigma (\varphi I + K_0)(L_n - W_n L_1 W_n)\}^o + \mathcal{J}_t$$

$$= \frac{1}{\varphi(\tau)}\{M_n^\sigma \varphi L_n\}^o + \mathcal{J}_t = \{L_n\}^o + \mathcal{J}_t\,.$$

Hence, $p^2 = p$. Now, the proof of $p^* = p$ and also the proof of the fact that the spectrum of pqp in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_t$ coincides with the interval $[-1,1]$ are the same as in the case of $\tau = \varphi$. It remains to apply the so-called two-projections lemma (comp. [7, Lemma 7.1]) and the Fredholm criteria for singular integral operators with piecewise continuous coefficients (see [3]). □

6.3. The local invertibility for $t = \pm 1$. In this section we analyze the invertibility of $\{A_n\}^o + \mathcal{J}_{\pm 1}$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_{\pm 1}$ (see Theorem 6.5) and show that the invertibility of the operators $W_3\{A_n\}$ and $W_4\{A_n\}$ imply the invertibility of $\{A_n\}^o + \mathcal{J}_{+1}$ and $\{A_n\}^o + \mathcal{J}_{-1}$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_{\pm 1}$, respectively. For symmetry reasons, we may restrict our considerations to the invertibility of $\{A_n\}^o + \mathcal{J}_1$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_1$.

The proof of the following lemma does not depend on the choice of τ and can be found in [7, Lemma 7.2, i)].

LEMMA 6.7. *Suppose $R \in alg\,\mathcal{T}(\mathbf{PC})$ is invertible and consider the sequence R_n^3, then the coset $\{[R^{-1}]_n^3\}^o + \mathcal{J}_1$ is the inverse of $\{R_n^3\}^o + \mathcal{J}_1$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_1$.*

Let $\mathbf{C}_1$ denote the class of continuous functions $f : [-1,1] \to [0,1]$ satisfying $f(1) = 1$.

LEMMA 6.8. *Let a sequence $\{C_n\} \in \mathcal{A}$ be the sum of two sequences $\{A_n\}$ and $\{B_n\}$ and assume that*

$$\inf_{f\in\mathbf{C}_1}\ \inf_{\{J_n\}\in\mathcal{J}}\ \sup_{n\in\mathbb{N}} \left\| [M_n f L_n] A_n [M_n f L_n] + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0$$

and

$$\inf_{f\in\mathbf{C}_1}\ \inf_{\{J_n\}\in\mathcal{J}}\ \sup_{n\in\mathbb{N}} \left\| [M_n f L_n] B_n [M_n f L_n] + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0\,.$$

Then $\{C_n\}^o \in \mathcal{J}_1$.

Proof. Due to the assumptions we have that, for each $\varepsilon > 0$ there are functions $f_{A,\varepsilon}, f_{B,\varepsilon} \in \mathbf{C}_1$ and sequences $\{J_n^{A,\varepsilon}\}, \{J_n^{B,\varepsilon}\} \in \mathcal{J}$, such that, for all $n \in \mathbb{N}$,

$$\left\| [M_n f_{A,\varepsilon} L_n] A_n [M_n f_{A,\varepsilon} L_n] + J_n^{A,\varepsilon} L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \le \varepsilon\,,$$

$$\left\| [M_n f_{B,\varepsilon} L_n] B_n [M_n f_{B,\varepsilon} L_n] + J_n^{B,\varepsilon} L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \le \varepsilon\,.$$

For $n \in \mathbb{N}$, it follows

$$\Big\| [M_n f_{A,\varepsilon} f_{B,\varepsilon} L_n](A_n + B_n)[M_n f_{A,\varepsilon} f_{B,\varepsilon} L_n]$$

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$$+[M_n f_{B,\varepsilon} L_n] J_n^{A,\varepsilon} [M_n f_{B,\varepsilon} L_n] + [M_n f_{A,\varepsilon} L_n] J_n^{B,\varepsilon} [M_n f_{A,\varepsilon} L_n] \Big\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \le \text{ const } \varepsilon \,,$$

Consequently,

$$\inf_{f \in \mathbf{C}_1} \left\| \{M_n f L_n\}^o \{C_n\}^o \{M_n f L_n\}^o \right\|_{\mathcal{A}/\mathcal{J}} = 0$$

and $\{C_n\}^o \in \mathcal{J}_1$. □

LEMMA 6.9. *Suppose (1.4) and (1.5) to be fulfilled and consider $A_n = M_n[aI + b\mu^{-1}S\mu I + K]L_n$ as well as $R := W_3\{A_n\}$. Then the cosets $\{R_n^3\}^o + \mathcal{J}_1$ and $\{A_n\}^o + \mathcal{J}_1$ coincide. In particular, $\{A_n\}^o + \mathcal{J}_1$ is invertible if R is invertible.*

Proof. The proof of this lemma in case of $\tau = \varphi$ is given in [7, Lemma 7.2, iii)]. The case $\tau = \sigma$ can be treated very analogous.

We have to prove that $\{R_n^3 - A_n\}^o$ belongs to $\mathcal{J}_1$. In view of Lemma 6.8, it is enough to show that (see Lemmata 3.2, 3.3, and 3.4)

$$\left\{ [a(1)I]_n^3 - M_n a L_n \right\}^o \in \mathcal{J}_1 \,, \tag{6.5}$$

$$\left\{ [\mathbf{S}]_n^3 - M_n \rho^{-1} S \rho L_n \right\}^0 \in \mathcal{J}_1 \,, \tag{6.6}$$

$$\inf_{f \in \mathbf{C}_1} \inf_{\{J_n\} \in \mathcal{J}} \sup_{n \in \mathbb{N}} \left\| M_n f L_n V_n^{-1} \mathbf{B}_n V_n M_n f L_n + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0 \,, \tag{6.7}$$

$$\inf_{f \in \mathbf{C}_1} \inf_{\{J_n\} \in \mathcal{J}} \sup_{n \in \mathbb{N}} \left\| M_n f L_n V_n^{-1} \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1} \mathbf{W}_n \mathbf{V}_n V_n M_n f L_n + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0 \,, \tag{6.8}$$

$$\inf_{f \in \mathbf{C}_1} \inf_{\{J_n\} \in \mathcal{J}} \sup_{n \in \mathbb{N}} \left\| M_n f L_n V_n^{-1} \mathbf{V}_n \mathbf{A}_n^* \mathbf{W}_n V_n M_n f L_n + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0 \,, \tag{6.9}$$

$$\inf_{f \in \mathbf{C}_1} \inf_{\{J_n\} \in \mathcal{J}} \sup_{n \in \mathbb{N}} \left\| M_n f L_n V_n^{-1} [\mathbf{A}_n - P_n \mathbf{A} P_n] V_n M_n f L_n + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0, \tag{6.10}$$

$$\inf_{f \in \mathbf{C}_1} \inf_{\{J_n\} \in \mathcal{J}} \sup_{n \in \mathbb{N}} \left\| M_n f L_n V_n^{-1} [\mathbf{F}_n - P_n \mathbf{F} P_n] V_n M_n f L_n + J_n L_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = 0 \tag{6.11}$$

with $\mathbf{F}_n = \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1}$ and $\mathbf{F} = \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1}$, since the operators $\mathbf{B}_+$, $\mathbf{V}_+ \mathbf{A}^* \mathbf{W}$, and $\mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} \mathbf{W} \mathbf{V}_+$ are compact (see (9.19), (9.29) and comp. the beginning of the proof of Lemma 4.7).

Due to $\lim_{t \to 1} a(t) = a(1)$ we have

$$\inf_{f \in \mathbf{C}_1} \left\| \left\{ M_n f L_n \right\}^o \left\{ [a(1)I]_n^3 - M_n a L_n \right\}^o \right\|_{\mathcal{A}/\mathcal{J}}$$

$$\le C \inf_{f \in \mathbf{C}_1} \sup_{n \in \mathbb{N}} \left\| \left(f(x^\sigma_{k+1,n}) [a(1) - a(x^\sigma_{k+1,n})] \delta_{j,k} \right)_{j,k=0}^{n-1} \right\|_{\mathcal{L}(\ell^2)} = 0 \,,$$

and (6.5) is proved.

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To show (6.6) we introduce the bounded function

$$g(s) := \frac{\cos s}{\sin s} - \frac{1}{s}\,, \quad s \in \left[-\frac{3}{4}\pi, \frac{3}{4}\pi\right].$$

In view of the defition of $\mathbf{S}$ and due to (3.14), the entries $r_{j,k}^n$ of

$$V_n\big([\mathbf{S}]_n^3 - M_n\rho^{-1}S\rho L_n\big)V_n^{-1}\,,$$

$0 \le j \le \frac{n}{2}$, $0 \le k < n$, can be estimated by

$$\begin{aligned} |r_{j,k}^n| &= \left|\frac{1-(-1)^{j-k}}{i\pi}\frac{1}{(j-k)} - \frac{1-(-1)^{j+k+1}}{i\pi}\frac{1}{j+k+1} - \right. \\ &\qquad \left. \frac{1-(-1)^{j-k}}{2ni}\frac{\cos\frac{j-k}{2n}\pi}{\sin\frac{j-k}{2n}\pi} + \frac{1-(-1)^{j+k+1}}{2ni}\frac{\cos\frac{j+k+1}{2n}\pi}{\sin\frac{j+k+1}{2n}\pi}\right| \\ &= \left|\frac{1-(-1)^{j+k+1}}{2ni}g\left(\frac{j+k+1}{2n}\pi\right) - \frac{1-(-1)^{j-k}}{2ni}g\left(\frac{j-k}{2n}\pi\right)\right| \le \frac{C}{n}\,. \end{aligned}$$

Consequently, due to $\left(f_n := \big(f(x_{k+1,n}^\sigma)\delta_{j,k}\big)_{j,k=0}^{n-1}\right)$

$$\begin{aligned} &\inf_{f\in\mathbf{C}_1}\left\|\{M_n f L_n\}^o\{[\mathbf{S}]_n^3 - M_n\rho^{-1}S\rho L_n\}^o\{M_n f L_n\}^0\right\|_{\mathcal{A}/\mathcal{J}} \\ &\quad \le \text{const}\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|f_n V_n\big([\mathbf{S}]_n^3 - M_n\rho^{-1}S\rho L_n\big)V_n^{-1}f_n\right\|_{\mathcal{L}(\ell^2)} \\ &\quad \le \text{const}\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|\big(f(x_{j+1,n}^\sigma)r_{j,k}^n\big)_{j,k=0}^{n-1}\right\|_{\mathcal{L}(\ell^2)}\,, \end{aligned}$$

we get, using a Frobenius norm estimate and choosing $f \in \mathbf{C}_1$ with $\operatorname{supp}(f\circ\cos) \subset [0,\varepsilon]$, a bound less than $\text{const}\,\varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily small.

Now, let us introduce the function $\Phi(s) = \cos\sqrt{s}$, $s \in \left[0, \frac{\pi^2}{4}\right]$. Then the function

$$h : \left[0, \frac{\pi^2}{4}\right]^2 \longrightarrow \mathbb{R}\,, \quad (s,t) \mapsto \frac{\Phi'(s)}{\Phi(s)-\Phi(t)} - \frac{1}{s-t}$$

is bounded and, for $s,t \in \left[0,\frac{\pi}{2}\right]$, we have

$$\frac{\sin s}{\cos t - \cos t} - \frac{2s}{s^2-t^2} = \frac{2s\Phi'(s^2)}{\Phi(s^2)-\Phi(t^2)} - \frac{2s}{s^2-t^2} = 2s\,h(s^2,t^2)\,.$$

Hence, we get, for $j,k \le \frac{n-1}{2}$,

$$\begin{aligned} &\left|\frac{\varphi(x_{k+1,n}^\sigma)}{ni(x_{k+1,n}^\sigma - x_{j+1,n}^\sigma)} - \frac{2k+1}{\pi i(k+j+1)(j-k)}\right| \\ &\quad = \left|\frac{1}{ni}\frac{\sin\frac{2k+1}{2n}\pi}{\cos\frac{2k+1}{2n}\pi - \cos\frac{2j+1}{2n}\pi} - \frac{1}{ni}\frac{2\frac{2k+1}{2n}\pi}{\left(\frac{2j+1}{2n}\pi\right)^2 - \left(\frac{2k+1}{2n}\pi\right)^2}\right| \qquad (6.12) \\ &\quad = \left|\frac{2}{ni}\frac{2k+1}{n}h\left(\left(\frac{2k+1}{2n}\pi\right)^2, \left(\frac{2j+1}{2n}\pi\right)^2\right)\right| \le C\frac{k}{n^2}\,. \end{aligned}$$

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Furhermore, the entries of $P_n\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}P_n - \mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1}$ can be written in the form

$$\left(\frac{2j+1}{2k+1}\right)^{2\chi_+}\frac{(2k+1)(1-\delta_{j,k})}{\pi i(k+j+1)(j-k)} - \frac{\chi(x^\sigma_{j+1,n})}{\chi(x^\sigma_{k+1,n})}\frac{\varphi(x^\sigma_{k+1,n})(1-\delta_{j,k})}{ni(x^\sigma_{k+1,n}-x^\sigma_{j+1,n})}$$

$$= \frac{\chi(x^\sigma_{j+1,n})}{\chi(x^\sigma_{k+1,n})}\left[\frac{2k+1}{\pi i(j+k+1)(j-k)} - \frac{\varphi(x^\sigma_{k+1,n})}{ni(x^\sigma_{k+1,n}-x^\sigma_{j+1,n})}\right](1-\delta_{j,k})$$

$$+\left[1-\frac{\chi(x^\sigma_{j+1,n})}{4^{\chi_-}\left(\frac{2j+1}{2n}\pi\right)^{2\chi_+}}\right]\left(\frac{2j+1}{2k+1}\right)^{2\chi_+}\frac{(2k+1)(1-\delta_{j,k})}{\pi i(k+j+1)(j-k)}\frac{4^{\chi_-}\left(\frac{2k+1}{2n}\pi\right)^{2\chi_+}}{\chi(x^\sigma_{k+1,n})}$$

$$+\left[1-\frac{4^{\chi_-}\left(\frac{2k+1}{2n}\pi\right)^{2\chi_+}}{\chi(x^\sigma_{k+1,n})}\right]\left(\frac{2j+1}{2k+1}\right)^{2\chi_+}\frac{(2k+1)(1-\delta_{j,k})}{\pi i(k+j+1)(j-k)}\,.$$

Denoting the first addend on the right-hand side by $\widetilde{r}^n_{jk}$, using (6.12), and taking into account (9.13), we obtain the Frobenius norm estimate

$$\sup_{n\in\mathbb{N}}\left\|\left(f(x^\sigma_{j+1,n})\widetilde{r}^n_{jk}f(x^\sigma_{k+1,n})\right)_{j,k=0}^{n-1}\right\|_{\mathcal{L}(\ell^2)}$$

$$\leq \frac{\mathrm{const}}{n^2}\sqrt{\sum_{\substack{j=0\\2j+1\leq 2\varepsilon}}^{n-1}\sum_{\substack{k=0\\2k+1\leq 2\varepsilon}}^{n-1}(2j+1)^{4\chi_+}(2k+1)^{2-4\chi_+}}$$

$$\leq \frac{\mathrm{const}}{n^2}\sqrt{\sum_{1\leq j\leq 2n\varepsilon}j^{4\chi_+}}\sqrt{\sum_{1\leq k\leq 2n\varepsilon}k^{2-4\chi_+}}$$

$$\leq \frac{\mathrm{const}\sqrt{(n\varepsilon)^{4\chi_++1}}\sqrt{(n\varepsilon)^{3-4\chi_+}}}{n^2} = \mathrm{const}\,\varepsilon^2$$

for any $f\in\mathbf{C}_1$ with $\mathrm{supp}(f\circ\cos)\subset[0,\varepsilon]$. Furthermore we get

$$\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|M_nfL_nV_n^{-1}[\mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1}-P_n\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}P_n]V_nM_nfL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}$$

$$\leq \inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|M_nfL_nV_n^{-1}(\widetilde{r}^n_{j,k})_{j,k=0}^{n-1}V_nM_nfL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}$$

$$+\mathrm{const}\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|P_n\left(f(x^\sigma_{j+1,n})\left[1-\frac{\chi(x^\sigma_{j+1,n})}{4^{\chi_-}\left(\frac{2j+1}{2n}\pi\right)^{2\chi_+}}\right]\delta_{j,k}\right)_{j,k=0}^{n-1}P_n\right\|_{\mathcal{L}(l^2)} *$$

$$(6.13)\qquad * \left\|P_n\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}P_n\right\|_{\mathcal{L}(l^2)}\left\|P_n\left(f(x^\sigma_{k+1,n})\frac{4^{\chi_-}\left(\frac{2k+1}{2n}\pi\right)^{2\chi_+}}{\chi(x^\sigma_{k+1,n})}\delta_{j,k}\right)_{j,k=0}^{n-1}P_n\right\|_{\mathcal{L}(l^2)}$$

$$+\mathrm{const}\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|P_n\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}P_n\right\|_{\mathcal{L}(l^2)} *$$

$$* \left\|P_n\left(f(x^\sigma_{k+1,n})\left[1-\frac{4^{\chi_-}\left(\frac{2k+1}{2n}\pi\right)^{2\chi_+}}{\chi(x^\sigma_{k+1,n})}\right]\delta_{j,k}\right)_{j,k=0}^{n-1}P_n\right\|_{\mathcal{L}(l^2)} = 0\,,$$

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since $\frac{4^{\chi_-}x^{2\chi_+}}{\chi(\cos x)} \longrightarrow 1$ if $x \longrightarrow 0$ and since the operator $\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}$ is bounded. Consequently, (6.11) is true. Completely analogous we get that (6.10) holds and that

$$\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|M_nfL_nV_n^{-1}[\mathbf{A}_n^*-P_n\mathbf{A}^*P_n]V_nM_nfL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}=0\,. \tag{6.14}$$

For fixed k_0, the projection $P_{k_0}\in\mathcal{L}(\ell^2)$ is a compact operator. Hence the sequence $\{V_n^{-1}P_{k_0}\mathbf{V}_nP_{k_0}P_n\mathbf{A}^*\mathbf{W}P_nV_nL_n\}$ belongs to $\mathcal{J}$ and, in view of (6.14) and (9.29), we arrive at

$$\begin{aligned}
&\inf_{f\in\mathbf{C}_1}\inf_{J_n\in\mathcal{J}}\sup_{n\in\mathbb{N}}\left\|[M_nfL_n]V_n^{-1}\mathbf{V}_n\mathbf{A}_n^*\mathbf{W}_nV_n[M_nfL_n]+J_nL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}\\
&\le\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|V_n^{-1}\mathbf{V}_nV_nM_nfL_nV_n^{-1}[\mathbf{A}_n^*-P_n\mathbf{A}^*P_n]V_nM_nfL_nV_n^{-1}\mathbf{W}_nV_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}\\
&\qquad+\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|M_nfL_nV_n^{-1}(I-P_{k_0})\mathbf{V}_nV_nV_n^{-1}P_n\mathbf{A}^*\mathbf{W}P_nV_nM_nfL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}\\
&\le\text{const}\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|P_n(I-P_{k_0})\left(f(x^\sigma_{j+1,n})d^n_{j+1}\delta_{j,k}\right)_{j,k=0}^{n-1}P_n\right\|_{\mathcal{L}(\ell^2)}\\
&\le\text{const}\sup_{n\in\mathbb{N}}\sup_{k_0\le k\le n/4}\frac{1}{k^\varepsilon}=\frac{\text{const}}{k_0^\varepsilon}\,,
\end{aligned}$$

for some $\varepsilon>0$. Consequently, we have proved (6.9). Similarly, we can show that (6.8) is true.

It remains to prove (6.7). We have $k_0,\{V_n^{-1}P_{k_0}\mathbf{B}_nV_nL_n\}\in\mathcal{J}$ for fixed k_0. Consequently, in view of (9.19), we get

$$\begin{aligned}
&\inf_{f\in\mathbf{C}_1}\inf_{\{J_n\}\in\mathcal{J}}\sup_{n\in\mathbb{N}}\left\|[M_nfL_n]V_n^{-1}\mathbf{B}_nV_n[M_nfL_n]+J_nL_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}\\
&\qquad\le\inf_{f\in\mathbf{C}_1}\sup_{n\in\mathbb{N}}\left\|P_n(I-P_{k_0})\left(f^2(x^\sigma_{j+1,n})b^n_{j+1}\delta_{j,k}\right)_{j,k=0}^{n-1}P_n\right\|_{\mathcal{L}(\ell^2)}\\
&\qquad\le\text{const}\sup_{n\in\mathbb{N}}\sup_{k_0\le k\le n/4}\frac{1}{k^\varepsilon}=\frac{\text{const}}{k_0^\varepsilon}
\end{aligned}$$

for some $\varepsilon>0$, and (6.7) is shown. □

7. Stability of the collocation methods. At first let us study the stability of sequences from $\mathcal{A}_0$.

THEOREM 7.1. *A sequence* $\{A_n\}\in\mathcal{A}_0$ *is stable if and only if all operators* $W_\omega\{A_n\}:\mathbf{X}_\omega\longrightarrow\mathbf{X}_\omega$, $\omega=1,2,3,4$, *are invertible.*

Proof. The necessity of the conditions follows from Theorem 2.9. To prove that the conditions are also sufficient we have, due to the same theorem, to show that the invertibility of $W_\omega\{A_n\}$ implies the invertibility of the coset $\{A_n\}^o$ in $\mathcal{F}/\mathcal{J}$. By Lemma 4.7 and by the fact that the mappings $W_{3/4}:\mathcal{F}\longrightarrow\mathcal{L}(\ell^2)$ are *-homomorphisms we conclude that $W_{3/4}\{A_n\}\in alg\,\mathcal{T}(\mathbf{PC})$. This, together with Lemma 6.7, Lemma 6.9, and the relation (see [7, (7.1)])

$$[R_1]_n^{3/4}[R_2]_n^{3/4}-[R_1R_2]_n^{3/4}\in\mathcal{J}_{\pm1}\,,\quad R_1,R_2\in alg\,\mathcal{T}(\mathbf{PC})\,,$$

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implies that the cosets $\{R_n^{3/4}\}^o + \mathcal{J}_{\pm 1}$ and $\{A_n\}^o + \mathcal{J}_{\pm 1}$ coincide in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_{\pm 1}$ for $R = W_{3/4}\{A_n\}$. In particular, $\{A_n\}^o + \mathcal{J}_{\pm 1}$ is invertible if $W_{3/4}\{A_n\}$ is invertible. The invertibility of $\{A_n\}^o + \mathcal{J}_t$ for $t \in (-1,1)$ follows from Lemma 6.6 and the invertibility of $W_1\{A_n\}$. It remains to refer to Theorem 6.5. □

The remaining part of this section is devoted to the case $A_n = M_n(aI + \mu^{-1} b\mu S + K)L_n$, $a, b \in \mathbf{PC}$, which is associated to equation (1.1) or, which is the same, to equation (1.2). At first we recall the Fredholm conditions for the operator $aI + bS : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ (see [3, Theorem 9.4.1]). Define $c(x) = \dfrac{a(x)+b(x)}{a(x)-b(x)}$, and, for $(x,\lambda) \in [-1,1] \times [0,1]$,

$$\mathbf{c}(x,\lambda) = \begin{cases} (1-\lambda)c(x-0) + \lambda c(x+0) & , \quad x \in (-1,1), \\ c(1) + [1 - c(1)]\mathbf{f}_\alpha(\lambda) & , \quad x = 1, \\ 1 + [c(-1) - 1]\mathbf{f}_\beta(\lambda) & , \quad x = -1, \end{cases}$$

where

$$\mathbf{f}_\alpha(\lambda) = \begin{cases} \dfrac{\sin \pi\alpha\lambda}{\sin \pi\alpha} e^{-\mathrm{i}\pi\alpha(\lambda-1)} & , \quad \alpha \in (-1,1) \setminus 0, \\ \lambda & , \quad \alpha = 0. \end{cases}$$

Note that, for $z_1, z_2 \in \mathbb{C}$, the image of the function $z_1 + (z_2 - z_1)\mathbf{f}_\alpha(\lambda)$, $\lambda \in [0,1]$, describes the circular arc from z_1 to z_2 such that the straight line segment $[z_1, z_2]$ is seen from the points of the arc under the angle $\pi(1+\alpha)$, i.e., in case $\alpha \in (-1,0)$, the arc lies on the right of the segment $[z_1, z_2]$ and, in case $\alpha \in (0,1)$, on the left. Thus, if the numbers $c(x \pm 0)$ are finite for $x \in [-1,1]$ the image of $\mathbf{c}(x,\lambda)$ is a closed curve in the complex plane which possesses a natural orientation. By $\mathrm{wind}\, \mathbf{c}(x,\lambda)$ we denote the winding number of this curve w.r.t. the origin. Furthermore, note that, for $-\frac{1}{2} < \kappa < \frac{1}{2}$,

$$\{\mathbf{f}_{-2\kappa}(\lambda) : \lambda \in [0,1]\} = \{f_\kappa(\lambda) : \lambda \in [0,1]\}, \tag{7.1}$$

where

$$f_\kappa(\lambda) = \frac{\lambda}{\lambda + (1-\lambda)e^{2\pi \mathrm{i}\kappa}}.$$

LEMMA 7.2 ([3], Theorem 9.4.1). *Let $a, b \in \mathbf{PC}$. Then the operator $aI + bS : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is Fredholm if and only if $a(x \pm 0) - b(x \pm 0) \neq 0$ for all $x \in [-1,1]$ and $\mathbf{c}(x,\lambda) \neq 0$ for all $(x,\lambda) \in [-1,1] \times [0,1]$. In this case, the operator is one-sided invertible and its Fredholm index is equal to $\mathrm{ind}\,(aI + bS) = -\mathrm{wind}\, \mathbf{c}(x,\lambda)$.*

Define $\mathbf{d}(x,\lambda)$ in the same way as $\mathbf{c}(x,\lambda)$ by using $\alpha - 2\gamma$ and $\beta - 2\delta$ instead of α and β, respectively.

COROLLARY 7.3. *Since the multiplication operator $\mu I : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_{v^{\alpha-2\gamma,\beta-2\delta}}$ is an isometric isomorphism, the operator $A = aI + \mu^{-1} bS\mu : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is invertible if and only if $a(x \pm 0) - b(x \pm 0) \neq 0$ for all $x \in [-1,1]$, if $\mathbf{d}(x,\lambda) \neq 0$ for all $(x,\lambda) \in [-1,1] \times [0,1]$, and if $\mathrm{wind}\, \mathbf{d}(x,\lambda) = 0$.*

LEMMA 7.4. *The operator $W_2\{M_n^\sigma(aI + \mu S\mu I)L_n\}$ is invertible in $\mathbf{L}^2_\nu$ if and only if the operator $aI + bS : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is invertible.*

Proof. Let $A_n = M_n^\sigma(aI + bS)L_n$. Due to Lemma 3.2, Lemma 3.3 and Lemma 3.4 we have that the operator $W_2\{A_n\}$ is equal to $J_\nu^{-1}(aJ_\nu + \mathrm{i}b\rho V^*)$ the invertibility of which in $\mathbf{L}^2_\nu$

is equivalent to the invertibility of the operator $B : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ with $B = \rho^{-1}(aJ_\nu + \mathrm{i}b\rho V^*)$. With the help of (3.1), (3.2), and the three-term recurrence relations

$$T_{k+1}(x) = 2xT_k(x) - \gamma_{k-1}T_{k-1}(x)\,,\; U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)\,,\; k = 1, 2, \ldots\,,$$

we find that

$$J_\nu = \rho(\varphi I - \mathrm{i}\psi\rho^{-1}S\rho I)\,, \quad \text{and} \quad V^* = \psi I + \mathrm{i}\varphi\rho^{-1}S\rho I\,,$$

where $\psi(x) = x$. Hence, the operator B is a singular integral operator the invertibility of which is equivalent to the Fredholmness of B with index zero or to the Fredholmness of BV with index -1. With the help of (6.4) we get

$$\begin{aligned} BV &= a(\varphi I + \mathrm{i}\psi\rho^{-1}S\rho I)(\psi I - \mathrm{i}\varphi\rho^{-1}S\rho I) + \mathrm{i}bI \\ &= -\mathrm{i}a\varphi^2\rho^{-1}S\rho I - \mathrm{i}a\psi^2\rho^{-1}S\rho I + \mathrm{i}bI \\ &= \mathrm{i}(bI - a\rho^{-1}S\rho I) + K \end{aligned}$$

with a compact operator $K : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$, and the assertion follows from $\frac{b-a}{b+a} = -\left(\frac{a+b}{a-b}\right)^{-1}$, Lemma 7.2, and the fact that $\rho I : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\sigma$ is an isometric isomorphism. □

LEMMA 7.5. *Let $A_n = M_n(aI + bS)L_n$. Then the operator $W_2\{A_n\}$ is invertible in $\mathbf{L}^2_\nu$ if the operators $W_\omega\{A_n\} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$, $\omega = 1, 3, 4$, are invertible.*

Proof. We consider the case $\tau = \sigma$. (The case $\tau = \varphi$ is dealt with in [7, Section 8].) Let the operators $W_j\{A_n\}$, $j = 1, 3, 4$, be invertible in $\mathbf{L}^2_\nu$. Then, due to Lemma 7.2, Lemma 4.1, and Lemma 4.7, the curves

$$\Gamma_1 := \left\{ \left\{ \begin{array}{lcl} (1-\lambda)c(x-0) + \lambda c(x+0) & , & x \in (0,1)\,, \\ c(1) + [1 - c(1)]f_{\gamma-\alpha/2}(\lambda) & , & x = 1\,, \\ 1 + [c(-1) - 1]f_{\delta-\beta/2}(\lambda) & , & x = -1 \end{array} \right\} : (x,\lambda) \in [-1,1] \times [0,1] \right\},$$

$$\begin{aligned} \Gamma_3 := &\left\{ a(1) + b(1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2} + \frac{\alpha}{2} - \gamma + \mathrm{i}\lambda\right]\right) : -\infty \le \lambda \le \infty \right\} \\ &\cup \left\{ a(1) + b(1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2} - \frac{1}{4} + \mathrm{i}\lambda\right]\right) : \infty \ge \lambda \ge -\infty \right\}, \end{aligned}$$

and

$$\begin{aligned} \Gamma_4 := &\left\{ a(-1) - b(-1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2} + \frac{\beta}{2} - \delta + \mathrm{i}\lambda\right]\right) : -\infty \le \lambda \le \infty \right\} \\ &\cup \left\{ a(-1) - b(-1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2} - \frac{1}{4} + \mathrm{i}\lambda\right]\right) : \infty \ge \lambda \ge \infty- \right\} \end{aligned}$$

do not run through the zero point, and their winding numbers vanish. For $e^{2\pi\lambda} = \frac{\lambda_1}{1-\lambda_1}$, $\lambda_1 \in [0,1]$, and $-\frac{1}{2} < \kappa < \frac{1}{2}$, we get

$$-\mathrm{i}\cot\left(\pi\left[\frac{1}{2} + \kappa + \mathrm{i}\lambda\right]\right) = \frac{1 - \lambda_1 - \lambda_1 e^{-\mathrm{i}2\pi\kappa}}{1 - \lambda_1 + \lambda_1 e^{-\mathrm{i}2\pi\kappa}}\,,$$

(7.2) $$\frac{a(1)+b(1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2}+\kappa+\mathrm{i}\lambda\right]\right)}{a(1)-b(1)} = c(1)+[1-c(1)]f_{-\kappa}(1-\lambda_1)\,,$$

and

$$\frac{a(-1)-b(-1)\mathrm{i}\cot\left(\pi\left[\frac{1}{2}+\kappa+\mathrm{i}\lambda\right]\right)}{a(-1)-b(-1)} = 1+[c(-1)-1]f_{-\kappa}(1-\lambda_1)\,.$$

Thus, if $W_3\{A_n\}$ and $W_4\{A_n\}$ are invertible in $\mathbf{L}^2_\nu$, then the invertibility of $W_1\{A_n\} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is equivalent to the absence of zero on the curve

$$\Gamma_2 := \left\{ \left\{ \begin{array}{lll} (1-\lambda)c(x-0)+\lambda c(x+0) & , & x\in(0,1)\,, \\ c(1)+[1-c(1)]f_{1/4}(\lambda) & , & x=1\,, \\ 1+[c(-1)-1]f_{1/4}(\lambda) & , & x=-1 \end{array} \right\} : (x,\lambda)\in[-1,1]\times[0,1] \right\}$$

and its vanishing winding number, since zero is not contained in the domains enclosed by the curves Γ_3 and Γ_4 . It remains to apply Lemma 7.4. □

Let $a_0, b_0 \in \mathbb{C}$ with $a_0 \pm b_0 \neq 0$, set $c_0 = \frac{a_0+b_0}{a_0-b_0}$, and consider the arc

$$G_\kappa(a_0,b_0) := \left\{ a_0 + b_0\mathrm{i}\cot\left(\pi\left[\frac{1}{2}+\kappa+\mathrm{i}\lambda\right]\right) : -\infty\le\lambda\le\infty \right\}\,,$$

where $-\frac{1}{2}<\kappa<\frac{1}{2}$. The point zero does not lie in the convex hull of this arc if and only if

$$\lambda_1(a_0-b_0)+(1-\lambda_1)\left[a_0+b_0\mathrm{i}\cot\left(\pi\left[\frac{1}{2}+\kappa+\mathrm{i}\lambda\right]\right)\right]\neq 0$$

for all $(\lambda_1,\lambda)\in[0,1]\times[-\infty,\infty]$ or, which is the same (comp. (7.2)),

$$\lambda_1+(1-\lambda_1)[f_{-\kappa}(1-\lambda)+c_0 f_\kappa(\lambda)]\,,\quad 0\le\lambda_1,\lambda\le 1\,.$$

This condition is equivalent to

$$\lambda_1+(1-\lambda_1)c_0 \neq -\frac{f_{-\kappa}(1-\lambda)}{f_\kappa(\lambda)} = -\frac{1-\lambda}{\lambda}\,e^{\mathrm{i}2\pi\kappa}\,,\quad 0\le\lambda_1,\lambda\le 1\,.$$

The last condition can be written in the form

$$\lambda_1+(1-\lambda_1)c_0\notin e^{\mathrm{i}2\pi\kappa}[-\infty,0]\,,\quad 0\le\lambda_1\le 1\,.$$

This means that c_0 can be represented in the form

(7.3) $$c_0 = |c_0|e^{\mathrm{i}2\pi\kappa_0}$$

with

(7.4) $$-\frac{1}{2}+\kappa<\kappa_0<\frac{1}{2}\quad\text{if}\quad\kappa>0$$

and

(7.5) $$-\frac{1}{2}<\kappa_0<\frac{1}{2}+\kappa\quad\text{if}\quad\kappa<0\,.$$

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Moreover, the point zero is contained in the interior of the convex hull of the arc $G_\kappa(a_0, b_0)$ if and only if c_0 is of the form (7.3) with

$$-\frac{1}{2} < \kappa_0 < -\frac{1}{2} + \kappa \quad \text{if} \quad \kappa > 0$$

and

$$\frac{1}{2} - \kappa < \kappa_0 < \frac{1}{2} \quad \text{if} \quad \kappa < 0\,. \tag{7.6}$$

Now, assume that the operator $A = aI + b\mu^{-1}S\mu I : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ with $a, b \in \mathbf{PC}$ is invertible. Then, due to Corollary 7.3,

$$\frac{1}{2\pi}\arg c(1) \neq \frac{1}{2} + \left(\frac{\alpha}{2} - \gamma\right) + k\,, \quad k \in \mathbb{Z}$$

and

$$\frac{1}{2\pi}\arg c(-1) \neq -\frac{1}{2} - \left(\frac{\beta}{2} - \delta\right) + k\,, \quad k \in \mathbb{Z}\,.$$

Hence we can define two numbers

$$\kappa_+ = -\frac{1}{2\pi}\arg c(1) \in \left(-\frac{1}{2} - \left(\frac{\alpha}{2} - \gamma\right), \frac{1}{2} - \left(\frac{\alpha}{2} - \gamma\right)\right) \tag{7.7}$$

and

$$\kappa_- = \frac{1}{2\pi}\arg c(-1) \in \left(-\frac{1}{2} - \left(\frac{\beta}{2} - \delta\right), \frac{1}{2} - \left(\frac{\beta}{2} - \delta\right)\right)\,.$$

LEMMA 7.6. *Let the operator $A = aI + b\mu^{-1}S\mu I : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ be invertible, $a, b \in$ $\mathbf{PC}$, and set $A_n = M_n A L_n$. Then the operators $W_{3/4}\{A_n\} : \ell^2 \longrightarrow \ell^2$ are Fredholm with index zero if and only if*

$$\left|\kappa_\pm - \frac{1}{4}\right| < \frac{1}{2} \quad \text{if} \quad \omega = \sigma$$

and

$$\left|\kappa_\pm + \frac{1}{4}\right| < \frac{1}{2} \quad \text{if} \quad \omega = \varphi\,.$$

Proof. Let $\omega = \sigma$. In this case the operator $W_3\{A_n\} : \ell^2 \longrightarrow \ell^2$ is Fredholm with index zero if and only if the point zero is not at the curve Γ_3 or in its interior. Since Γ_3 is the union of the two arcs $G_{\frac{\alpha}{2}-\gamma}(a(1), b(1))$ and $G_{-\frac{1}{4}}(a(1), b(1))$, this holds true if and only if eighter

(a) zero is not contained in the convex hulls of the arcs $G_{\frac{\alpha}{2}-\gamma}(a(1), b(1))$ and $G_{-\frac{1}{4}}(a(1), b(1))$, or
(b) zero is contained in the interior of both convex hulls, or
(c) if both arcs are located on the same side of the straight line from $a(1) + b(1)$ to $a(1) - b(1)$, i.e. if $\frac{\alpha}{2} - \gamma < 0$, zero is on this straight line.

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Condition (a) is equivalent to (see (7.4) and (7.5))

$$-\frac{1}{2} < \kappa_+ < \frac{1}{2} - \left(\frac{\alpha}{2} - \gamma\right) \quad \text{if} \quad \frac{\alpha}{2} - \gamma > 0\,,$$

$$-\frac{1}{2} - \left(\frac{\alpha}{2} - \gamma\right) < \kappa_+ < \frac{1}{2} \quad \text{if} \quad \frac{\alpha}{2} - \gamma < 0\,,$$

and

$$-\frac{1}{2} + \frac{1}{4} < \kappa_+ < \frac{1}{2}\,,$$

i.e., taking into account (7.7), equivalent to

$$-\frac{1}{2} + \frac{1}{4} < \kappa_+ < \frac{1}{2}\,. \tag{7.8}$$

Condition (b) can be written as (see (7.6))

$$\frac{\alpha}{2} - \gamma < 0\,, \quad -\frac{1}{2} < \kappa_+ - 1 < -\frac{1}{2} - \left(\frac{\alpha}{2} - \gamma\right)$$

and

$$\frac{1}{2} < \kappa_+ < \frac{1}{2} + \frac{1}{4}\,,$$

which is, due to (7.7), equivalent to

$$\frac{1}{2} < \kappa_+ < \frac{1}{2} + \frac{1}{4}\,. \tag{7.9}$$

Finally, condition (c) is equivalent to

$$\frac{\alpha}{2} - \gamma < 0 \quad \text{and} \quad \kappa_+ = \frac{1}{2}\,. \tag{7.10}$$

Summarizing (7.8), (7.9), and (7.10) we get

$$\left|\kappa_+ - \frac{1}{4}\right| < \frac{1}{2}\,.$$

The proof for $W_4\{A_n\}$ is completely analogous, and the proof in case of $\omega = \varphi$ is given in [7, Section 8]. □

8. Splitting property of the singular values. The singular values of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ are the nonnegative square roots of the eigenvalues of $\mathbf{A}^*\mathbf{A}$. In this section we study the asymptotic behaviour of the singular values of operator sequences $\{A_n\} \in \mathcal{A}_0$, where an operator $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ is identified with one of its matrix representations, for example in the basis $\{\widetilde{u}_k\}_{k=0}^{n-1}$ or in the basis $\{\widetilde{\ell}_{kn}\}_{k=1}^{n}$.

Let $\mathcal{F}_0$ denote the C^*-algebra of all bounded sequences $\{A_n\}$ of matrices $A_n \in \mathbb{C}^{n\times n}$, provided with the supremum norm and elementwise operations. Further, let $\mathcal{N}$ be the two-sided closed ideal of $\mathcal{F}_0$ consisting of all sequences $\{A_n\} \in \mathcal{F}_0$ with $\lim_{n\to\infty} \|A_n\| = 0$.

For $\{A_n\} \in \mathcal{F}_0$, by $\Lambda_n(A_n)$ we denote the set of all singular values of A_n . We say that the singular values of a sequence $\{A_n\} \in \mathcal{F}_0$ have the k-splitting property if there is a sequence $\{\varepsilon_n\}$ of nonnegative numbers and a real number $d > 0$, such that $\lim_{n\to\infty} \varepsilon_n = 0$

and $\Lambda(A_n) \subset [0, \varepsilon_n] \cup [d, \infty)$ for all n, where, for all sufficiently large n, exactly k singular values lie in $[0, \varepsilon_n]$.

Case $\tau = \varphi$. With the help of Theorem 7.1 and relations (2.4), one can easily check that in this case the algebra $\mathcal{A}_0$ is a so called standard algebra (for a definition, see [5, p. 258]). Consequently, from [5, Theorems 6.1(b), 6.12] we get the following theorem.

THEOREM 8.1. *Let $\{A_n\} \in \mathcal{A}_0^\varphi$ and let $\{A_n\} + \mathcal{J}^\varphi$ be invertible in $\mathcal{A}_0^\varphi / \mathcal{J}^\varphi$. Then the operators $W_\omega^\varphi \{A_n\} : \mathbf{X}_\omega \longrightarrow \mathbf{X}_\omega$, $\omega = 1, 2, 3, 4$, are Fredholm and the singular values of $\{A_n\}$ have the k-splitting property with*

$$k = \sum_{\omega=1}^{4} \dim \ker W_\omega^\varphi \{A_n\} .$$

Case $\tau = \sigma$. In this case $\mathcal{A}_0$ is not longer a standard algebra (see Lemma 2.4). Hence, in the following we give another proof for the k-splitting property of the singular values (comp. [8, Section 5]), which applies in both cases.

For this aim, we continue with recalling some definitions and facts from a Fredholm theory for approximation sequences (comp. [5, 13]). Given a strongly monotonically increasing sequence $\eta : \mathbb{N} \longrightarrow \mathbb{N}$, let $\mathcal{F}_\eta$ refer to the C^*-algebra of all bounded sequences $\{A_n\}$ with $A_n \in \mathbb{C}^{\eta(n) \times \eta(n)}$, and write $\mathcal{N}_\eta$ for the ideal of all sequences $\{A_n\} \in \mathcal{F}_\eta$ which tend to zero in norm. Further, let $R_\eta : \mathcal{F}_0 \longrightarrow \mathcal{F}_\eta$, $\{A_n\} \mapsto \{A_{\eta(n)}\}$ denote the restriction mapping, which is a *-homomorphism from $\mathcal{F}_0$ onto $\mathcal{F}_\eta$ mapping $\mathcal{N}$ onto $\mathcal{N}_\eta$. For a C^*-subalgebra $\mathcal{B}$ of $\mathcal{F}_0$, let $\mathcal{B}_\eta = R_\eta(\mathcal{B})$ which is a C^*-algebra, too. A *-homomorphism $W : \mathcal{B} \longrightarrow \mathcal{C}$ from $\mathcal{B}$ into a C^*-algebra $\mathcal{C}$ is called fractal, if, for any strongly monotonically increasing sequence $\eta : \mathbb{N} \longrightarrow \mathbb{N}$, there is a *-homomorphism $W_\eta : \mathcal{B}_\eta \longrightarrow \mathcal{C}$ such that $W = W_\eta R_\eta$. The algebra $\mathcal{B}$ is called fractal, if the canonical homomorphism $\pi : \mathcal{B} \longrightarrow \mathcal{B}/(\mathcal{B} \cap \mathcal{N})$ is fractal.

LEMMA 8.2 ([5], Theorem 1.69). *Let $\mathcal{B}$ be a unital C^*-subalgebra of $\mathcal{F}_0$. Then $\mathcal{B}$ is fractal if and only if there exists a family $\{W_t\}_{t \in T_0}$ of unital and fractal *-homomorphisms $W_t : \mathcal{B} \longrightarrow \mathcal{C}_t$ from $\mathcal{B}$ into unital C^*-algebras $\mathcal{C}_t$ such that, for every sequence $\{B_n\} \in \mathcal{B}$, the following equivalence holds: The coset $\{B_n\} + \mathcal{B} \cap \mathcal{N}$ is invertible in $\mathcal{B}/(\mathcal{B} \cap \mathcal{N})$ if and only if $W_t\{B_n\}$ is invertible in $\mathcal{C}_t$ for all $t \in T_0$.*

COROLLARY 8.3. *The algebra $\mathcal{A}_0$ is fractal.*

Proof. Due to Theorem 7.1 and Lemma 8.2 we have only to show that the unital *-homomorphisms (see Cor. 2.8) $W_\omega : \mathcal{A}_0 \longrightarrow \mathcal{L}(\mathbf{X}_\omega)$, $\omega = 1, 2, 3, 4$, are fractal. But, this is evident since the images $W_\omega\{A_n\}$, $\{A_n\} \in \mathcal{A}_0$, are strong limits which are uniquely defined by each subsequence of $\{A_n\}$. □

Let $\mathcal{B}$ be a unital C^*-algebra. An element $k \in \mathcal{B}$ is said to be of central rank one if, for any $b \in \mathcal{B}$, there is an element $r(b)$ belonging to the center of $\mathcal{B}$, such that $k\, b\, k = r(b)\, k$. An element of $\mathcal{B}$ is called of finite central rank if it is the sum of a finite number of elements of central rank one, and it is called centrally compact if it lies in the closure of the set of all elements of finite central rank. Let $\mathcal{J}(\mathcal{B})$ denote the set of all centrally compact elements of $\mathcal{B}$.

LEMMA 8.4 ([8], Theorem 5.6). *Let $\mathcal{B}$ be a unital and fractal C^*-subalgebra of $\mathcal{F}_0$ which contains the ideal $\mathcal{N}$. Then, $\mathcal{K}(\mathcal{B}) = \mathcal{J}(\mathcal{B})$.*

LEMMA 8.5 ([13], Theorem 3). *Let $\mathcal{B}$ be a unital C^*-algebra and $\pi : \mathcal{B} \longrightarrow \mathcal{L}(H)$ an irreducible representation of $\mathcal{B}$. Then $\pi(\mathcal{J}(\mathcal{B})) \subset \mathcal{K}(H)$.*

Since every *-homomorphism between C^*-algebras, which preserves spectra, also preserves norms, we can conclude from Theorem 7.1 that the mapping

$$\mathrm{smb} : \mathcal{A}_0 \longrightarrow \mathcal{L}(\mathbf{L}^2_\nu) \times \mathcal{L}(\mathbf{L}^2_\nu) \times \mathcal{L}(\ell^2) \times \mathcal{L}(\ell^2) ,$$

$$\{A_n\} \mapsto (W_1\{A_n\}, W_2\{A_n\}, W_3\{A_n\}, W_4\{A_n\})$$

is a *-homomorphism with kernel $\mathcal{N}$. Since $\mathcal{K}(\mathbf{X}_\omega) \subset W_\omega(\mathcal{A}_0)$ for all $\omega \in T$, we can easily check that every $W_\omega : \mathcal{A}_0 \longrightarrow \mathcal{L}(\mathbb{X}_\omega)$, $\omega \in T$, is an irreducible representation of $\mathcal{A}_0$. Hence, the mapping

$$\mathrm{smb} : \mathcal{A}_0 \longrightarrow \mathcal{L}(\mathbf{L}^2_\nu) \times \mathcal{L}(\mathbf{L}^2_\nu) \times \mathcal{L}(\ell^2) \times \mathcal{L}(\ell^2)$$

is an irreducible representation of $\mathcal{A}_0$, too. Lemma 8.5 implies

$$\mathrm{smb}(\mathcal{J}(\mathcal{A}_0)) \subset \mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\ell^2) \times \mathcal{K}(\ell^2).$$

Recalling the definiton of the ideal $\mathcal{J}$ and the fact that every compact operator can be approximated as closely as desired by an operator of finite dimensional range, we find that $\mathcal{J} \subset \mathcal{K}(\mathcal{A}_0)$. Thus, due to Lemma 8.4, $\mathcal{J} \subset \mathcal{J}(\mathcal{A}_0)$. Obviously,

$$\mathrm{smb}(\mathcal{J}) = \mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\ell^2) \times \mathcal{K}(\ell^2).$$

Thus, we have proved the following.

LEMMA 8.6. *The homomorphism* smb *maps* $\mathcal{J}(\mathcal{A}_0)$ *onto* $\mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\mathbf{L}^2_\nu) \times \mathcal{K}(\ell^2) \times \mathcal{K}(\ell^2)$.

We say that a sequence $\{B_n\} \in \mathcal{F}_0$ is a Fredholm sequence if it is invertible modulo $\mathcal{J}(\mathcal{F}_0)$. Due to [13, Theorem 2] (or [5, Theorem 6.35]) the Fredholmness of a sequence from $\mathcal{F}_0$ is equivalent to the fact that the singular values of this sequence have the k-splitting property. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{F}_0$ is called Fredholm inverse closed if $\mathcal{J}(\mathcal{B}) = \mathcal{B} \cap \mathcal{J}(\mathcal{F}_0)$.

LEMMA 8.7 ([8], Theorem 5.8). *Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{F}_0$ and let $\{J_n\} \in \mathcal{J}(\mathcal{F}_0) \cap \mathcal{B}$. Then, for every irreducible representation $\pi : \mathcal{B} \longrightarrow \mathcal{L}(H)$ of $\mathcal{B}$, the operator $\pi\{J_n\}$ is compact.*

Let $\mathcal{B}$ be a unital and fractal C^*-subalgebra of $\mathcal{F}_0$ which contains the ideal $\mathcal{N}$. A central rank one sequence of $\mathcal{B}$ is said to be of essential rank one if it does not belong to the ideal $\mathcal{N}$. For every essential rank one sequence $\{K_n\}$, let $\mathcal{J}\{K_n\}$ refer to the smallest closed ideal of $\mathcal{B}$ which contains the sequence $\{K_n\}$ and the ideal $\mathcal{N}$.

In [13, Cor. 2] (see also [5, Cor. 6.43]) there is shown that, if $\{K_n\}$ and $\{J_n\}$ are sequences of essential rank one in $\mathcal{B}$, then eighter

$$\mathcal{J}\{K_n\} = \mathcal{J}\{J_n\} \quad \text{or} \quad \mathcal{J}\{K_n\} \cap \mathcal{J}\{J_n\} = \mathcal{N}. \tag{8.1}$$

Calling $\{K_n\}$ and $\{J_n\}$ equivalent in the first case we get a splitting of the sequences of essential rank one into equivalence classes, the collection of which we denote by $\mathcal{S}$. Moreover, with every $s \in \mathcal{S}$ there is associated a unique (up to unitary equivalence) irreducible representation $W^s : \mathcal{B} \longrightarrow \mathcal{L}(H_s)$ such that $W^s(\mathcal{J}\{K_n\}) = \mathcal{K}(H_s)$ and that the kernel of the mapping $W^s : \mathcal{J}\{K_n\} \longrightarrow \mathcal{K}(H_s)$ is $\mathcal{N}$ (see [13, Theorem 4] or [5, Theorem 6.39]).

From [13, Theorem 10] (or [5, Theorem 6.54]) and [5, Theorem 5.41] we infer the following.

THEOREM 8.8. *Let $\mathcal{B}$ be a unital, fractal and Fredholm inverse closed C^*-subalgebra of $\mathcal{F}_0$ which contains the ideal $\mathcal{N}$.*

(a) If $\{B_n\} \in \mathcal{B}$ is a Fredholm sequence, then the operators $W^s\{B_n\}$ are Fredholm operators for all $s \in \mathcal{S}$, there are only finitely many $s \in \mathcal{S}$ for which $W^s\{B_n\}$ is not invertible, and the singular values of $\{B_n\}$ have the k-splitting property with

$$k = \sum_{s \in \mathcal{S}} \dim \ker W^s\{B_n\}.$$

(b) *Let the family $\{W^s\}_{s\in\mathcal{S}}$ be sufficient for the stability of sequences in $\mathcal{B}$, i.e., the invertibility of all operators $W^s\{B_n\}$ implies the stability of $\{B_n\}$. Then the sequence $\{B_n\}\in\mathcal{B}$ is Fredholm, if all operators $W^s\{B_n\}$ are Fredholm and if there are only finitely many among them which are not invertible.*

Now we are ready to prove the following.

THEOREM 8.9. *The singular values of a sequence $\{A_n\}\in\mathcal{A}_0$ have the k-splitting property if all operators $W_\omega\{A_n\}$, $\omega=1,2,3,4$, are Fredholm. Moreover,*

$$k=\sum_{\omega=1}^{4}\dim\ker W_\omega\{A_n\}\,.$$

Proof. Due to Corollary 8.3, Theorem 8.8, and Theorem 7.1 we have to show that the algebra $\mathcal{A}_0$ is Fredholm inverse closed and that we can identify $\mathcal{S}$ with $\{1,2,3,4\}$ and W^ω with W_ω.

Let $\mathbf{c}$ denote the set off all convergent sequences of complex numbers. Since the center of $\mathcal{A}_0$ is equal to $\Big\{\{\gamma_n I_n\}:\{\gamma_n\}\in\mathbf{c}\Big\}$ (I_n denotes the idetity matrix of order n) every central rank one sequence in $\mathcal{A}_0$ is also a central rank one sequence in $\mathcal{F}_0$, i.e. $\mathcal{J}(\mathcal{A}_0)\subset\mathcal{J}(\mathcal{F}_0)$. Hence for the Fredholm inverse closedness of $\mathcal{A}_0$, it remains to show that $\mathcal{A}_0\cap\mathcal{J}(\mathcal{F}_0)\subset\mathcal{J}(\mathcal{A}_0)$. For this, let $\{K_n\}\in\mathcal{A}_0\cap\mathcal{J}(\mathcal{F}_0)$. By Lemma 8.7 we get

$$\mathrm{smb}\{K_n\}\in\mathcal{K}(\mathbf{L}^2_\nu)\times\mathcal{K}(\mathbf{L}^2_\nu)\times\mathcal{K}(\ell^2)\times\mathcal{K}(\ell^2)\,,$$

and Lemma 8.6 implies the existence of a sequence $\{J_n\}\in\mathcal{J}(\mathcal{A}_0)$ such that

$$\mathrm{smb}\{K_n\}=\mathrm{smb}\{J_n\}\,.$$

Hence, $\{K_n-J_n\}\in\mathcal{N}$ and $\{K_n\}\in\mathcal{J}(\mathcal{A}_0)$.

Now, we show that, for each essential rank one sequence $\{K_n\}$, there exists an $\omega_0\in\{1,2,3,4\}$ such that

$$\mathcal{J}\{K_n\}=J_{\omega_0}:=\left\{\left\{(E_n^{(\omega_0)})^{-1}L_n^{(\omega_0)}TE_n^{(\omega_0)}+C_n\right\}:T\in\mathcal{K}(\mathbf{X}_{\omega_0}),\ \{C_n\}\in\mathcal{N}\right\}. \tag{8.2}$$

For some $\omega\in\{1,2,3,4\}$, let $K_n=(E_n^{(\omega)})^{-1}L_1^{(\omega)}E_n^{(\omega)}$. Then $\{K_n\}\in\mathcal{J}_\omega$ and, consequently, $\mathcal{J}\{K_n\}\subset\mathcal{J}_\omega$. This implies $W_\omega(\mathcal{J}\{K_n\})\subset W_\omega(\mathcal{J}_\omega)=\mathcal{K}(\mathbf{X}_\omega)$. Hence, $W_\omega(\mathcal{J}\{K_n\})=W_\omega(\mathcal{J}_\omega)$ and $\mathcal{J}\{K_n\}=\mathcal{J}_\omega$. On the other hand, for an arbitrary essential rank one sequence $\{K_n\}\in\mathcal{A}_0$, we get, using $\mathrm{smb}(\mathcal{J}(\mathcal{A}_0))=\mathrm{smb}(\mathcal{J})$, $\mathcal{J}\{K_n\}\subset\mathcal{J}$. This implies, due to (8.1), the existence of an $\omega_0\in\{1,2,3,4\}$ such that (8.2) holds. □

9. Appendix: Proof of Lemma 3.4 in case $\tau=\sigma$. At first we collect some known results needed in the sequel.

LEMMA 9.1 ([14], Lemma 4.13). *If $w\in\mathbf{C}^{0,\eta}$ with $\eta>\frac{1}{2}[1+\max\{\alpha,\beta,0\}]$, then the commutator $wS-SwI$ belongs to $\mathcal{K}(\mathbf{L}^2_\nu,\mathbf{C}^{0,\lambda})$ for some $\lambda>0$.*

LEMMA 9.2 ([11], Prop. 9.7, Theorem 9.9). *Assume that $a,b\in\mathbf{C}^{0,\eta}$ are real valued functions, where $\eta\in(0,1)$ and $[a(x)]^2+[b(x)]^2>0$ for all $x\in[-1,1]$. Furthermore, assume taht the integers $\lambda_\pm$ satisfy the relations*

$$\alpha_0:=\lambda_++g(1)\in(-1,1)\quad\text{and}\quad\beta_0:=\lambda_--g(-1)\in(-1,1)\,,$$

where $g:[-1,1]\longrightarrow\mathbb{R}$ is a continuous function such that

$$a(x)+\mathrm{i}\,b(x)=\sqrt{[a(x)]^2+[b(x)]^2}\,e^{\mathrm{i}\pi g(x)}\,,\quad x\in[-1,1]\,.$$

Then there exists a positive function $w \in \mathbf{C}^{0,\eta}$ *such that, for each polynomial* p *of degree* n *, the function* $a\,v^{\alpha_0,\beta_0} w\,p + \mathrm{i}\,S\,b\,v^{\alpha_0,\beta_0} w\,p$ *is a polynomial of degree* $n-\kappa$ *, where* $\kappa = -\lambda_+ - \lambda_-$ *and where, by definition, a polynomial of negative degree is identically zero.*

Suppose $\gamma, \delta \geq 0$. By $\mathbf{C}_{\gamma,\delta}$ we denote the Banach space of all continuous functions $f : (-1,1) \longrightarrow \mathbb{C}$, for which $v^{\gamma,\delta} f$ is continuous over $[-1,1]$. Moreover, by $\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}$ we refer to the Banach space of all functions f such that $v^{\alpha,\beta} f$ belongs to $\mathbf{L}^p(-1,1)$. The norms in $\mathbf{C}_{\gamma,\delta}$ and $\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}$ are defined by

$$\|f\|_{\gamma,\delta,\infty} := \left\|v^{\gamma,\delta} f\right\|_\infty , \quad \|f\|_{\widetilde{\mathbf{L}}^p_{v^{\alpha,\beta}}} := \left\|v^{\alpha,\beta} f\right\|_{\mathbf{L}^p(-1,1)} .$$

We introduce the operator $T_{\gamma,\delta}$ by

$$(T_{\gamma,\delta} u)(x) := \int_{-1}^{1} \left[1 - \frac{v^{\gamma,\delta}(y)}{v^{\gamma,\delta}(x)}\right] \frac{u(y)}{(y-x)}\, dy , \quad -1 < x < 1 .$$

LEMMA 9.3 ([6], Corollary 4.4). *If* $p > 2$,

$$\gamma, \delta \in \left(-\frac{1}{4}, -\frac{1}{p}\right) \cup \left(\frac{1}{p}, 1 - \frac{1}{2p}\right) , \quad 0 < \chi < \min\left\{\frac{1}{4} - \frac{1}{2p}, \frac{1}{4} + \gamma, \frac{1}{4} + \delta\right\} ,$$

then the operator $T_{\gamma,\delta} : \widetilde{\mathbf{L}}^p_{v^{\gamma - \frac{1}{2p}, \delta - \frac{1}{2p}}} \longrightarrow \mathbf{C}_{\gamma + \frac{1}{4} - \chi, \delta + \frac{1}{4} - \chi}$ *is compact.*

Of course, the assertion of this lemma remains true if one of the numbers γ or δ is equal to zero.

LEMMA 9.4 ([6], (2.9)). *The sequence* $\{W_n\}$ *converges weakly to* 0 *in the space* $\widetilde{\mathbf{L}}^p_\psi$ *with* $\psi = v^{\frac{1}{4} + \frac{\alpha}{2} - \frac{1}{2p}, \frac{1}{4} + \frac{\beta}{2} - \frac{1}{2p}}$.

Proof of Lemma 3.4 in case $\tau = \sigma$:

Since (1.5) holds, we can choose integers $\lambda_\pm$ such that $\alpha_0 - \lambda_+$ and $\lambda_- - \beta_0$ are in $(-1,0)$. Moreover, by $g(x)$ we denote a linear function such that $g(1) = \alpha_0 - \lambda_+$ and $g(-1) = \lambda_- - \beta_0$. Then, $\widehat{a}(x) := -\cot[\pi g(x)]$ is a continuous function on $[-1,1]$ and $\widehat{a}(x) - i = \sqrt{[\widehat{a}(x)]^2 + 1}\, e^{\mathrm{i}\pi g(x)}$. Due to Lemma 9.2 there exist a positive function $\omega \in \bigcap_{\eta \in (0,1)} \mathbf{C}^{0,\eta}$ such that $(\widehat{a}I + iS)\mu\omega u_n$ is a polynomial of degree less then $n-k$ for each $u_n \in \operatorname{im} L_n$, where $k = -\lambda_+ - \lambda_-$. Now we use the decomposition

$$\mu^{-1} S \mu I = \mathrm{i}\widehat{a} I - \mathrm{i}(\mu\omega)^{-1}(\widehat{a} I + \mathrm{i}S)\mu\omega I + (\mu\omega)^{-1}(\omega S - S\omega)\mu I . \tag{9.1}$$

The uniform boundedness of $\{M_n \widehat{a} L_n\}$ follows from Lemma 3.2. Taking into account (2.2), Lemma 2.1, and the boundedness of $S : \mathbf{L}^2_{v^{\alpha-2\gamma,\beta-2\delta}} \longrightarrow \mathbf{L}^2_{v^{\alpha-2\gamma,\beta-2\delta}}$ we get, for $u_n \in \operatorname{im} L_n$ and $q_n = (\widehat{a}I + \mathrm{i}S)\mu\omega u_n$,

$$\begin{aligned} \|M_n(\mu\omega)^{-1} q_n\|_\nu^2 &\leq 2 Q_n^\sigma |\vartheta^{-1}\varphi(\mu\omega)^{-1} q_n|^2 \\ &\leq \text{const} \int_{-1}^{1} |q_n|^2 \vartheta^{-2}\varphi^2\mu^{-2}\sigma\, dx \\ &= \text{const}\, \|q_n\|_{\nu\mu^{-2}}^2 \leq \text{const}\, \|\mu\omega u_n\|_{\nu\mu^{-2}}^2 \leq \text{const}\, \|u_n\|_\nu^2 , \end{aligned} \tag{9.2}$$

which proves the uniform boundedness of the second term in (9.1) corresponding to the collocation method. To handle the third term we set $H_\omega := \omega S - S\omega$. Due to (1.4), we have $\frac{1}{2}[1 + \max\{\alpha - 2\gamma, \beta - 2\delta, 0\}] < 1$. Thus, in view of Lemma 9.1, we have

$H_\omega \in \mathcal{K}(\mathbf{L}^2_{\nu\mu^{-2}}, \mathbf{C}^{0,\lambda})$, for some $\lambda > 0$, which implies $\mu^{-1}H_\omega\mu \in \mathcal{K}(\mathbf{L}^2_\nu)$. Moreover, choosing a $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\frac{1+\alpha}{2} - \gamma, \frac{1+\alpha}{2}, \frac{1+\beta}{2} - \delta, \frac{1+\beta}{2}\right\}$$

and applying Corollary 2.3, we get $\{(M_n - L_n)\omega^{-1}\mu^{-1}H_\omega\mu L_n\} \in \mathcal{N}$ and, consequently,

$$\{M_n\omega^{-1}\mu^{-1}H_\omega\mu L_n\} \in \mathcal{J}\,. \tag{9.3}$$

Using decomposition (9.1) together with Lemma 3.2 and Corollary 2.3, we infer that for each fixed $m = 0, 1, 2, \ldots$,

$$M_n\widehat{a}L_n\widetilde{u}_m \longrightarrow \widehat{a}\widetilde{u}_m\,,$$

$$M_n(\mu\omega)^{-1}(\widehat{a}I + S)\mu\omega L_n\widetilde{u}_m \longrightarrow (\mu\omega)^{-1}(\widehat{a}I + S)\mu\omega\widetilde{u}_m\,,$$

$$M_n(\mu\omega)^{-1}H_\omega\mu L_n\widetilde{u}_m \longrightarrow (\mu\omega)^{-1}H_\omega\mu\widetilde{u}_m\,.$$

Thus, $\{A_n\}$ converges strongly to A.

With the help of (3.2) and Lemma 2.1 we obtain, for $u_n = \vartheta p_n \in \operatorname{im} L_n$,

$$\|M_n a u_n\|^2_\nu \le 2\|a\|^2_\infty Q^\sigma_n|\vartheta^{-1}\varphi\vartheta p_n|^2 \le \text{const}\,\|a\|^2_\infty\|u_n\|^2_\nu\,. \tag{9.4}$$

To prove the strong convergence of $\{A_n^*\}$, at first we consider sequences of the form $\{M_n b_0 b\mu^{-1}S\mu L_n\}$, where $b_0 \in \mathbf{PC}$ and b is a differentiable function with $b' \in \mathbf{C}^{0,1}[-1,1]$ and $b(\pm1) = b'(\pm1) = 0$. We use the decomposition

$$\begin{aligned} b\mu^{-1}S\mu I &= b\rho^{-1}S\rho I + \mu^{-1}(bS - SbI)\mu I + \mu^{-1}(Sb\mu\rho^{-1}I - b\mu\rho^{-1}S)\rho I \\ &=: b\rho^{-1}S\rho I + K_1 + K_2\,. \end{aligned} \tag{9.5}$$

In the same way as for (9.3) one can show that $\{M_n K_j L_n\} \in \mathcal{J}$, $j = 1, 2$. Due to Lemma 3.2 and Lemma 3.3 the inclusion $\{M_n b_0 b\mu^{-1}S\mu L_n\} \in \mathcal{F}$ follows. Using this fact and the estimate (see (9.4))

$$\begin{aligned} \|M_n(b - \widetilde{b})\mu^{-1}S\mu L_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} &= \|M_n(b - \widetilde{b})L_n M_n\mu^{-1}S\mu L_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \\ &\le \text{const}\,\|b - \widetilde{b}\|_\infty \end{aligned} \tag{9.6}$$

we get

$$\{M_n b\mu^{-1}S\mu L_n\} \in \mathcal{F} \quad \text{for all} \quad b \in \mathbf{PC} \quad \text{with} \quad b(\pm1) = 0\,. \tag{9.7}$$

Now, for fixed m, we take the function $\varphi^{-1}\widetilde{u}_m$. This function belongs to $\mathbf{L}^2_\nu$ and fulfills the conditions of Corollary 2.3 such that $M_n\varphi^{-1}\widetilde{u}_m \longrightarrow \varphi^{-1}\widetilde{u}_m$. Because of $(M_n\varphi^{-1}L_n)^* = (2L_n - L_{n-1})M_n\varphi^{-1}\frac{1}{2}(L_n + L_{n-1})$ (see (3.7), which is also true for $a = \varphi^{-1}$) we get

$$\begin{aligned} &\lim_{n\to\infty}(M_n\mu^{-1}S\mu L_n)^*\widetilde{u}_m \\ &\quad= \lim_{n\to\infty}(M_n\varphi^{-1}L_n M_n\varphi\mu^{-1}S\mu L_n)^*\widetilde{u}_m \\ &\quad= \lim_{n\to\infty}(M_n\varphi\mu^{-1}S\mu L_n)^*(2L_n - L_{n-1})M_n\varphi^{-1}\frac{1}{2}(L_n + L_{n-1})\widetilde{u}_m \\ &\quad= W_1\{M_n\varphi\mu^{-1}S\mu L_n\}^*\varphi^{-1}\widetilde{u}_m \end{aligned}$$

in $\mathbf{L}^2_\nu$.

To prove the strong convergence of $\{W_n M_n \mu^{-1} S \mu W_n\}$ we write

$$\mu^{-1} S \mu I = \rho^{-1} S \rho I + \mu^{-1} K \mu I \tag{9.8}$$

with $K := S - \rho^{-1} \mu S \mu^{-1} \rho I$. Moreover, for $p \geq 2$, we set

$$\psi := v^{\frac{1}{4}+\frac{\alpha}{2}-\frac{1}{2p},\frac{1}{4}+\frac{\beta}{2}-\frac{1}{2p}}, \quad \widetilde{\psi} := \mu^{-1} \psi v^{\frac{1}{4}+\frac{\alpha}{2}-\gamma-\frac{1}{2p},\frac{1}{4}+\frac{\beta}{2}-\delta-\frac{1}{2p}} .$$

By assumption (1.4) we have $-\frac{1}{4} < \frac{1}{4} + \frac{\alpha}{2} - \gamma < \frac{3}{4}$ and $-\frac{1}{4} < \frac{1}{4} + \frac{\beta}{2} - \delta < \frac{3}{4}$. Thus, together with (1.5) we can apply Lemma 9.3 for sufficiently large p and sufficiently small $\chi > 0$ to conclude the compacness of

$$K_\mu := \mu^{-1} K \mu I : \widetilde{\mathbf{L}}^p_\psi \stackrel{\mu I}{\longrightarrow} \widetilde{\mathbf{L}}^p_{\widetilde{\psi}} \stackrel{K}{\longrightarrow} \mathbf{C}_{\frac{1+\alpha}{2}-\gamma-\chi,\frac{1-\beta}{2}-\delta-\chi} \stackrel{\mu^{-1} I}{\longrightarrow} \mathbf{C}_{\frac{1+\alpha}{2}-\chi,\frac{1-\beta}{2}-\chi} .$$

Using the decomposition (9.8) together with Lemma 3.3, it remains to prove that $W_n M_n K_\mu W_n \widetilde{u}_m$ converges to zero in $\mathbf{L}^2_\nu$ for each fixed $m = 0, 1, 2, \ldots$. As a consequence of Corollary 2.3 and the compactness of the operator K_μ we get

$$\lim_{n\to\infty} \|(M_n - I) K_\mu\|_{\widetilde{\mathbf{L}}^p_\psi \to \mathbf{L}^2_\nu} = 0$$

for some $p > 2$. Together with the uniform boundedness of $W_n : \widetilde{\mathbf{L}}^p_\psi \longrightarrow \widetilde{\mathbf{L}}^p_\psi$ (see Lemma 9.4) this leads to

$$\lim_{n\to\infty} \|W_n (M_n - I) K_\mu W_n\|_{\widetilde{\mathbf{L}}^p_\psi \to \mathbf{L}^2_\nu} = 0$$

Again Lemma 9.4 and the compactness of the operator K_μ imply, for some $p > 2$,

$$\lim_{n\to\infty} \|W_n K_\mu W_n u\|_\nu = 0 , \quad u \in \widetilde{\mathbf{L}}^p_\psi .$$

It remains to remark that $\widetilde{u}_m \in \widetilde{\mathbf{L}}^p_\psi$ for all $p \geq 1$ ..

For fixed m , the function $\varphi^{-\frac{1}{3}} T_m$ belongs to $\mathbf{L}^2_\sigma$ and fulfills the conditions of Lemma 2.2 such that

$$L^\sigma_n \varphi^{-\frac{1}{3}} T_m \longrightarrow \varphi^{-\frac{1}{3}} T_m \quad \text{in} \quad \mathbf{L}^2_\sigma . \tag{9.9}$$

Using (3.10) we get, for all $u, v \in \mathbf{L}^2_\nu$,

$$\begin{aligned} \langle W_n M_n a W_n u, v \rangle_\nu &= \left\langle J_\nu^{-1} L^\sigma_n a J_\nu L_n u, v \right\rangle_\nu = \left\langle L^\sigma_n a J_\nu L_n u, J_\nu^{-*} L_n v \right\rangle_\sigma \\ &= \frac{\pi}{n} \sum_{j=1}^n a(x^\sigma_{jn}) (J_\nu L_n u)(x^\sigma_{jn}) \overline{(J_\nu^{-*} L_n v)(x^\sigma_{jn})} \\ &= \left\langle J_\nu L_n u, L^\sigma_n \overline{a} J_\nu^{-*} L_n v \right\rangle_\sigma = \left\langle u, J_\nu^* L^\sigma_n \overline{a} J_\nu^{-*} L_n v \right\rangle_\nu , \end{aligned}$$

i.e.

$$(W_n M_n a W_n)^* = J_\nu^* L^\sigma_n \overline{a} J_\nu^{-*} L_n .$$

Together with (9.7) and (9.9) we conclude, for all fixed m ,

$$\begin{aligned} (W_n M_n \mu^{-1} S \mu W_n)^* \widetilde{u}_m &= (W_n M_n \varphi^{-\frac{1}{3}} W_n W_n M_n \varphi^{\frac{1}{3}} \mu^{-1} S \mu W_n)^* \widetilde{u}_m \\ &= (W_n M_n \varphi^{\frac{1}{3}} \mu^{-1} S \mu W_n)^* J_\nu^* L^\sigma_n \varphi^{-\frac{1}{3}} J_\nu^{-*} L_n \widetilde{u}_m \\ &\longrightarrow W_2 \{M_n \varphi^{\frac{1}{3}} \mu^{-1} S \mu L_n\}^* J_\nu^* \varphi^{-\frac{1}{3}} J_\nu^{-*} \widetilde{u}_m \end{aligned}$$

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in $\mathbf{L}^2_\nu$.

To get the strong limits of the sequences $\{V_nA_nV_n^{-1}P_n\}$ and $\{(V_nA_nV_n^{-1}P_n)^*\}$ we consider the structure of the corresponding matrices more closely. Setting $B := \mu^{-1}S\mu I - \rho^{-1}S\rho I$ and $B_n = M_nBL_n$ and using (3.12) and (3.13) we compute, for $x \neq x^\sigma_{kn}$,

$$
\begin{aligned}
&(B\widetilde{\ell}^\sigma_{kn})(x)\\
&= \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right] \frac{\vartheta(y)T_n(y)\,dy}{\vartheta(x^\sigma_{kn})T_n'(x^\sigma_{kn})(y-x^\sigma_{kn})(y-x)}\\
&= \frac{1}{\vartheta(x^\sigma_{kn})T_n'(x^\sigma_{kn})}\frac{1}{x^\sigma_{kn}-x}\frac{1}{\pi \mathrm{i}}\int_{-1}^1 \left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\left[\frac{1}{y-x^\sigma_{kn}} - \frac{1}{y-x}\right]\vartheta(y)T_n(y)\,dy\\
&= \frac{1}{\vartheta(x^\sigma_{kn})T_n'(x^\sigma_{kn})}\frac{1}{x^\sigma_{kn}-x} *\\
&\quad * \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \frac{1}{\mu(x)}\left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x)}{\rho(x)}\right]\left[\frac{1}{y-x^\sigma_{kn}} - \frac{1}{y-x}\right]\varphi(y)T_n(y)\,dy\\
&= \frac{1}{\vartheta(x^\sigma_{kn})T_n'(x^\sigma_{kn})}\frac{1}{x^\sigma_{kn}-x}\left\{\frac{1}{\pi \mathrm{i}}\int_{-1}^1 \left[\frac{\mu(x^\sigma_{kn})}{\mu(x)} - \frac{\rho(x^\sigma_{kn})}{\rho(x)}\right]\frac{1}{\rho(x^\sigma_{kn})}\frac{\varphi(y)T_n(y)}{y-x^\sigma_{kn}}\,dy\right.\\
&\quad + \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \left(\left[\frac{\mu(y)}{\mu(x)} - \frac{\rho(y)}{\rho(x)}\right]\frac{1}{\rho(y)} - \left[\frac{\mu(x^\sigma_{kn})}{\mu(x)} - \frac{\rho(x^\sigma_{kn})}{\rho(x)}\right]\frac{1}{\rho(x^\sigma_{kn})}\right)\frac{\varphi(y)T_n(y)}{y-x^\sigma_{kn}}\,dy\\
&\quad \left. - \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \frac{1}{\mu(x)}\left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x)}{\rho(x)}\right]\frac{\varphi(y)T_n(y)}{y-x}\,dy\right\}\\
&= \frac{1}{\vartheta(x^\sigma_{kn})T_n'(x^\sigma_{kn})}\frac{1}{x^\sigma_{kn}-x}\left\{\frac{1}{\mathrm{i}}\left[\frac{\mu(x^\sigma_{kn})}{\mu(x)} - \frac{\rho(x^\sigma_{kn})}{\rho(x)}\right]\frac{1}{\rho(x^\sigma_{kn})}\varphi^2(x^\sigma_{kn})U_{n-1}(x^\sigma_{kn})\right.\\
&\quad + \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x^\sigma_{kn})}{\rho(x^\sigma_{kn})}\right]\frac{1}{\mu(x)}\frac{\varphi(y)T_n(y)}{y-x^\sigma_{kn}}\,dy\\
&\quad \left. - \frac{1}{\pi \mathrm{i}}\int_{-1}^1 \frac{1}{\mu(x)}\left[\frac{\mu(y)}{\rho(y)} - \frac{\mu(x)}{\rho(x)}\right]\frac{\varphi(y)T_n(y)}{y-x}\,dy\right\}\\
&= \frac{1}{x^\sigma_{kn}-x}\left\{\frac{\varphi(x^\sigma_{kn})}{n\mathrm{i}}\left[\frac{\mu(x^\sigma_{kn})}{\mu(x)} - \frac{\rho(x^\sigma_{kn})}{\rho(x)}\right] + \frac{(-1)^{k+1}}{\sqrt{2\pi}}\frac{1}{n\mathrm{i}}\frac{\mu(x^\sigma_{kn})}{\mu(x)}\varphi(x^\sigma_{kn})d^n_k\right.\\
&\qquad\qquad \left. - \frac{(-1)^{k+1}}{\sqrt{2\pi}}\frac{1}{n\mathrm{i}}\frac{\rho(x^\sigma_{kn})}{\rho(x)}\varphi(x)d^n(x)\right\},
\end{aligned}
$$

where

$$d^n(x) := \int_{-1}^1 \left[\frac{\mu(y)\rho(x)}{\rho(y)\mu(x)} - 1\right]\frac{\varphi(y)}{\varphi(x)}\frac{T_n(y)}{y-x}\,dy\,, \quad d^n_k := d^n(x^\sigma_{kn}).$$

Consequently, we get

$$V_nB_nV_n^{-1}P_n = \left(\frac{\omega_{(j+1)n}}{\omega_{(k+1)n}}(B\widetilde{\ell}^\sigma_{k+1,n})(x^\sigma_{j+1,n})\right)_{j,k=0}^{n-1}$$

(9.10)

$$= \mathbf{B}_n + \mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1} - \mathbf{A}_n - \mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1}\mathbf{W}_n\mathbf{V}_n - \mathbf{V}_n\mathbf{A}_n^*\mathbf{W}_n\,,$$

with

$$\mathbf{B}_n := \left((B\widetilde{l}^\sigma_{(j+1)n})(x^\sigma_{(j+1)n})\delta_{j,k}\right)_{j,k=0}^{n-1}, \quad \mathbf{A}_n := \left(\frac{\varphi(x^\sigma_{k+1,n})}{n\mathrm{i}}\,\frac{1-\delta_{j,k}}{x^\sigma_{k+1,n}-x^\sigma_{j+1,n}}\right)_{j,k=0}^{n-1},$$

and

$$\mathbf{W}_n := \left(\frac{(-1)^{k+1}}{\sqrt{2\pi}}\delta_{j,k}\right)_{j,k=0}^{n-1}, \ \mathbf{V}_n := \left(d^n_{k+1}\delta_{j,k}\right)_{j,k=0}^{n-1}, \ \mathbf{D}_n := \left(\frac{\rho(x^\sigma_{j+1,n})}{\mu(x^\sigma_{j+1,n})}\delta_{j,k}\right)_{j,k=0}^{n-1},$$

where the diagonal elements in $\mathbf{A}_n$ are equal to zero by definition. We have to show that, for any fixed $m = 1, 2, \ldots$, the sequences

$$\{V_nA_nV_n^{-1}P_ne_{m-1}\} \quad \text{and} \quad \{(V_nA_nV_n^{-1}P_n)^*e_{m-1}\}$$

converge in ℓ^2 to $\mathbf{A}^\mu_+e_{m-1}$ and $(\mathbf{A}^\mu_+)^*e_{m-1}$, respectively.

a) At first we turn to the limits for the operators $\mathbf{A}_n$. We define

$$a^{(n)}_{jk} = \frac{\varphi(x^\sigma_{k+1,n})}{n\mathrm{i}}\,\frac{1-\delta_{j,k}}{x^\sigma_{k+1,n}-x^\sigma_{j+1,n}}, \quad 0 \le j,k \le n-1\,.$$

We observe that, for fixed j and k with $k \ne j$ and for $n \longrightarrow \infty$,

$$a^{(n)}_{jk} = \frac{1}{n\mathrm{i}}\,\frac{\sin\frac{2k+1}{2n}\pi}{2\sin\frac{k+j+1}{2n}\pi\sin\frac{j-k}{2n}\pi} \longrightarrow \frac{1}{\pi\mathrm{i}}\,\frac{2k+1}{(k+j+1)(j-k)} \tag{9.11}$$

and, for fixed k and $j = 0, 1, \ldots, n-1$, $j \ne k$, and $n > 2k$,

$$|a^{(n)}_{jk}| \le \text{const}\,\frac{2k+1}{|j-k|(k+j+1)}\,. \tag{9.12}$$

The same estimate holds true for fixed j and $k = 0, 1, \ldots, n-1$, $k \ne j$, and $n > 2k$. Using (9.11) and (9.12) together with Remark 3.1, we see that $\mathbf{A}_ne_{m-1} \longrightarrow \mathbf{A}e_{m-1}$ and $\mathbf{A}_n^*e_{m-1} \longrightarrow \mathbf{A}^*e_{m-1}$ for any fixed $m = 1, 2, \ldots$

b) In this item we consider the convergence of the operators $\mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1}$. We introduce the function $\chi(x) := \rho(x)[\mu(x)]^{-1} = (1-x)^{\chi_+}(1+x)^{\chi_-}$ with

$$\chi_+ := \frac{1}{4} + \frac{\alpha}{2} - \gamma\,, \quad \chi_+ := \frac{1}{4} + \frac{\beta}{2} - \delta \tag{9.13}$$

and define

$$\widetilde{a}^{(n)}_{jk} = \frac{\chi(x^\sigma_{j+1,n})}{\chi(x^\sigma_{k+1,n})}\,\frac{\varphi(\chi(x^\sigma_{k+1,n}))}{n\mathrm{i}}\,\frac{1-\delta_{j,k}}{x^\sigma_{j+1,n}-x^\sigma_{j+1,n}}, \quad 0 \le j,k \le n-1\,.$$

Then, condition (1.4) is equivalent to

$$-\frac{1}{4} < \chi_\pm < \frac{3}{4}\,. \tag{9.14}$$

We observe that, for fixed j and k with $k \neq j$ and for $n \longrightarrow \infty$,

$$\begin{aligned} \widetilde{a}_{jk}^{(n)} &= \left(\frac{\sin\frac{2j+1}{4n}\pi}{\sin\frac{2k+1}{4n}\pi}\right)^{2\chi_+}\left(\frac{\cos\frac{2j+1}{4n}\pi}{\cos\frac{2k+1}{4n}\pi}\right)^{2\chi_-}\frac{1}{n\mathrm{i}}\,\frac{\sin\frac{2k+1}{2n}\pi}{2\sin\frac{k+j+1}{2n}\pi\sin\frac{j-k}{2n}\pi} \\ &\longrightarrow \left(\frac{2j+1}{2k+1}\right)^{2\chi_+}\frac{1}{\pi\mathrm{i}}\,\frac{2k+1}{(k+j+1)(j-k)} =: \widetilde{a}_{jk}\,. \end{aligned} \tag{9.15}$$

For $0 \le j,k \le \frac{n}{2}$ and $j \neq k$, we have the estimate

$$|\widetilde{a}_{jk}^{(n)}| \le \text{conts}\left(\frac{2j+1}{2k+1}\right)^{2\chi_+}\left(\frac{1-\frac{2j+1}{4n}\pi}{1-\frac{2k+1}{4n}\pi}\right)^{2\chi_-}\frac{2k+1}{|j-k|(k+j+1)}\,.$$

For fixed $k, n > 3k$ and $n > j > \frac{n}{2}$ we get, if $\chi_- \ge 0$,

$$|\widetilde{a}_{jk}^{(n)}| \le \text{const}\left(\frac{n}{k}\right)^{2\chi_+}\frac{1}{n}\,\frac{k}{n} = \text{const}\,\frac{k^{1-2\chi_+}}{n^{2(1-\chi_+)}}$$

and, if $\chi_- < 0$,

$$|\widetilde{a}_{jk}^{(n)} \le \text{const}\left(\frac{n}{k}\right)^{2\chi_+}\left(\frac{2n-2j-1}{2n}\right)^{2\chi_-}\frac{1}{n}\,\frac{k}{n} \le \text{const}\,\frac{(n-j)^{2\chi_-}k^{1-2\chi_+}}{n^{2(1-\chi_++\chi_-)}}\,.$$

Thus, for fixed k and $j = 0,1,\ldots,n-1$, $j \neq k$, and $n > 3k$, we have

$$|\widetilde{a}_{jk}^{(n)}| \le \text{const}\begin{cases} \dfrac{1}{j^{\frac{1}{2}+\varepsilon}} & \text{if} \quad j \le \dfrac{n}{2}\,, \\ \dfrac{1}{n^\varepsilon}\,\dfrac{1}{(n-j)^{\frac{1}{2}+\varepsilon}} & \text{if} \quad j > \dfrac{n}{2} \end{cases} \tag{9.16}$$

with some $\varepsilon > 0$. For fixed j, $n > 3j$, and $n > k > \frac{n}{2}$ we get, if $\chi_- \le 0$,

$$|\widetilde{a}_{jk}^{(n)}| \le \text{const}\left(\frac{j}{n}\right)^{2\chi_+}\frac{1}{n} \le \text{const}\,\frac{j^{2\chi_+}}{n^{(1+2\chi_+)}}\,,$$

and, if $\chi_- > 0$,

$$\begin{aligned} |\widetilde{a}_{jk}^{(n)}| &\le \text{const}\left(\frac{j}{n}\right)^{2\chi_+}\left(\frac{2n}{2n-2k-1}\right)^{2\chi_-}\frac{1}{n}\,\frac{2n-2k-1}{2n} \\ &\le \text{const}\,\frac{(n-k)^{1-2\chi_-}j^{2\chi_+}}{n^{2(1+\chi_+-\chi_-)}}\,, \end{aligned}$$

Thus, we obtain, for fixed j and $k = 0,1,\ldots,n-1$, $k \neq j$, and $n > 3j$,

$$|\widetilde{a}_{jk}^{(n)}| \le \begin{cases} \text{const} & \dfrac{1}{k^{\frac{1}{2}+\varepsilon}} \quad \text{if} \quad k \le \dfrac{n}{2}\,, \\ & \dfrac{1}{n^\varepsilon}\,\dfrac{1}{(n-k)^{\frac{1}{2}+\varepsilon}} \quad \text{if} \quad k > \dfrac{n}{2} \end{cases} \tag{9.17}$$

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with some $\varepsilon > 0$. Using (9.15), (9.16), and (9.17) together with Remark 3.1 we conclude

$$\lim_{n\to\infty}\left(\sum_{j=0,j\neq k}^{\frac{n}{2}}|\widetilde{a}_{jk}^{(n)}-\widetilde{a}_{jk}|^2+\sum_{j=\frac{n}{2}+1,j\neq k}^{n-1}|\widetilde{a}_{jk}^{(n)}|^2+\sum_{j=\frac{n}{2}+1,j\neq k}^{\infty}|\widetilde{a}_{jk}|^2\right)=0$$

and

$$\lim_{n\to\infty}\left(\sum_{k=0,j\neq k}^{\frac{n}{2}}|\widetilde{a}_{jk}^{(n)}-\widetilde{a}_{jk}|^2+\sum_{k=\frac{n}{2}+1,j\neq k}^{n-1}|\widetilde{a}_{jk}^{(n)}|^2+\sum_{k=\frac{n}{2}+1,j\neq k}^{\infty}|\widetilde{a}_{jk}|^2\right)=0\,,$$

which imply the ℓ^2-convergences

$$\mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1}e_k\to\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1}e_k \text{ and } (\mathbf{D}_n\mathbf{A}_n\mathbf{D}_n^{-1})^*e_j\to(\mathbf{D}_+\mathbf{A}\mathbf{D}_+^{-1})^*e_j\,, \tag{9.18}$$

where $\mathbf{A}$ and $\mathbf{D}_+$ are defined in (3.19).

c) Next we compute the limits $b_k^+ := \lim_{n\to\infty} b_k^n$, where we have set $b_k^n := (B\ell_{kn}^\sigma)(x_{kn}^\sigma)$. In particular, we shall show that, for some $\varepsilon > 0$,

$$|b_k^n|\le\frac{\text{const}}{\min\{k,n+k-1\}^\varepsilon}\,,\quad k=1,2,\dots,n\,. \tag{9.19}$$

At first we consider the case $n \ge 2k-1$. Defining

$$\zeta(x):=[\rho(x)]^{-1}\mu(x)=[\chi(x)]^{-1}=:(1-x)^{\zeta_+}(1+x)^{\zeta_-}$$

and using (3.13) we get

$$\begin{aligned}
b_k^n &= \frac{1}{\pi \mathrm{i}}\int_{-1}^1\left[\frac{\mu(y)}{\mu(x_{kn}^\sigma)}-\frac{\rho(y)}{\rho(x_{kn}^\sigma)}\right]\frac{\vartheta(y)T_n(y)}{\vartheta(x_{kn}^\sigma)T_n'(x_{kn}^\sigma)(y-x_{kn}^\sigma)^2}\,dy\\
&=\frac{(-1)^{k+1}}{\sqrt{2\pi}}\frac{1}{n\mathrm{i}}\int_{-1}^1\frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{\zeta(x_{kn}^\sigma)}\frac{\varphi(y)T_n(y)}{(y-x_{kn}^\sigma)^2}\,dy\\
&=\left(\int_{-1}^{-\frac12}+\int_{-\frac12}^{\widetilde{x}_{2k,n}^\sigma}+\int_{\widetilde{x}_{2k,n}^\sigma}^{\frac12(1+x_{kn}^\sigma)}+\int_{\frac12(1+x_{kn}^\sigma)}^1\right)F(y,x_{kn}^\sigma)\,dy\\
&=: I_{1,k}^n+I_{2,k}^n+I_{3,k}^n+I_{4,k}^n\,,
\end{aligned}$$

where $\widetilde{x}_{2k,n}^\sigma=\max\left\{-\frac12,\cos\frac{2k-1}{n}\pi\right\}$ and

$$F(y,x):=\frac{(-1)^{k+1}}{\pi}\frac{1}{n\mathrm{i}}\frac{\varphi(y)}{\zeta(x)}\frac{\zeta(y)-\zeta(x)}{(y-x)^2}\cos s\,,\quad y=\cos\frac{s}{n}\,.$$

We observe $x_{k,n}^\sigma \ge 0$ for $n \ge 2k-1$. For $-1 < y < -\frac12$, we have $2 > |y - x_{k,n}^\sigma| > \frac12$ and $2 > 1-y > \frac32$. Thus,

$$\begin{aligned}
|I_{1,k}^n| &\le \frac{\text{const}}{n}\frac{1}{(1-x_{k,n}^\sigma)^{\zeta_+}}\int_{-1}^{-\frac12}\left[(1+y)^{\zeta_-}+(1-x_{kn}^\sigma)^{\zeta_+}\right](1+y)^{\frac12}\,dy\\
&\le\frac{\text{const}}{n}\left[1+\left(\frac{n}{k}\right)^{2\zeta_+}\right]\le\frac{\text{const}}{\sqrt{n}}\le\frac{\text{const}}{\sqrt{k}}\,,
\end{aligned} \tag{9.20}$$

since $-\frac{3}{4} < \zeta_\pm < \frac{1}{4}$ (recall (9.14) and $\zeta_\pm = -\chi_\pm$). From (9.20) we conclude $\lim_{n\to\infty} I_{1,k}^n = 0$ and

(9.21) $$b_k^+ = \lim_{n\to\infty} (I_{2,k}^n + I_{3,k}^n + I_{4,k}^n) = \lim_{n\to\infty} \int_0^\infty G(s, x_{kn}^\sigma)\, ds\,,$$

where

$$G(s,x) := \begin{cases} \frac{1}{n} F\left(\cos\frac{s}{n}, x\right) \sin\frac{s}{n} & \text{if} \quad 0 < s < \frac{2\pi}{3} n\,, \\ 0 & \text{if} \quad \frac{2\pi}{3} n < s\,. \end{cases}$$

Now, we consider the case $\frac{1}{2}(1+x_{kn}^\sigma) < y = \cos\frac{s}{n} < 1$, which is equivalent to $0 < s < s_k^n$, where $\frac{2k-1}{4}\pi < s_k^n < \frac{2k-1}{2}\pi$. Thus $y - x_{kn}^\sigma > \frac{1}{2}(1 - x_{kn}^\sigma)$ and

(9.22) $$\begin{aligned} |G(s, x_{kn}^\sigma)| &\le \frac{\text{const}}{n} \left[\left(\frac{1-y}{1-x_{kn}^\sigma} \right)^{\zeta_+} + 1 \right] \frac{(1-y)^{\frac{1}{2}}}{(1-x_{kn}^\sigma)^2} \frac{1}{n} \sin\frac{s}{n} \,|\cos s| \\ &\le \frac{\text{const}}{n} \left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1 \right] \frac{s}{n} \left(\frac{n}{k}\right)^4 \frac{s}{n^2} \le \frac{\text{const}}{k^4} \left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1 \right] s^2\,. \end{aligned}$$

Consequently,

(9.23) $$|I_{4,k}^n| \le \frac{\text{const}}{k^4} \int_0^{\frac{2k-1}{2}\pi} \left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1 \right] s^2\, ds \le \frac{\text{const}}{k}\,.$$

For the case $\widetilde{x}_{2k,n}^\sigma < y = \cos\frac{s}{n} < \frac{1}{2}(1+x_{kn}^\sigma)$ we have $s_k^n < s < \min\{(2k-1)\pi, \frac{2\pi}{3}n\}$ and

$$\begin{aligned} |F(y, x_{kn}^\sigma)| &\le \text{const}\, \frac{\varphi(y)}{n} \frac{|\zeta'(\zeta_1)|}{\zeta_(x_{kn}^\sigma)} \left| \frac{\cos s - \cos\frac{2k-1}{2}\pi}{\cos\frac{s}{n} - \cos\frac{2k-1}{2n}\pi} \right| \\ &= \text{const}\, \frac{\varphi(y)}{n} \frac{|\zeta'(\zeta_1)|}{\zeta_(x_{kn}^\sigma)} \frac{\left| \int_0^1 \sin\left[\frac{2k-1}{2}\pi + \lambda\left(s - \frac{2k-1}{2}\pi \right) \right] d\lambda \right|}{\frac{1}{n} \int_0^1 \sin\frac{1}{n}\left[\frac{2k-1}{2}\pi + \lambda\left(s - \frac{2k-1}{2}\pi \right) \right] d\lambda} \\ &\le \text{const}\, \frac{\varphi(y)}{n} \frac{|\zeta'(\zeta_1)|}{\zeta_(x_{kn}^\sigma)} \frac{\left| \frac{1}{s - \frac{2k-1}{2}\pi} \int_{\frac{2k-1}{2}\pi}^s \sin u\, du \right|}{\frac{1}{n} \int_0^{\frac{1}{2}} \sin\frac{1}{n}\left[\frac{2k-1}{2}\pi + \lambda\left(s - \frac{2k-1}{2}\pi \right) \right] d\lambda} \\ &\le \text{const}\, \frac{\varphi(y)}{n} \frac{|\zeta'(\zeta_1)|}{\zeta_(x_{kn}^\sigma)} \frac{\min\left\{ 1, \left| s - \frac{2k-1}{2}\pi \right|^{-1} \right\}}{\frac{k}{n^2}} \end{aligned}$$

for some $\zeta_1 \in \left(\widetilde{x}_{2k,n}^\sigma, [1 + x_{kn}^\sigma]/2 \right)$. Since in this case

$$1 - y > 1 - \frac{1}{2}(1 + x_{kn}^\sigma) = \frac{1}{2}(1 - x_{kn}^\sigma)$$

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and

$$1-y<1-\cos\frac{2k-1}{n}\pi=2\sin^2\frac{2k-1}{2n}\pi$$
$$=2\left(1+\cos\frac{2k-1}{2n}\pi\right)\left(1-\cos\frac{2k-1}{2n}\pi\right)4(1-x^\sigma_{kn})\,,$$

we get $1-y\sim 1-x^\sigma_{kn}\sim 1-\zeta_1$,

(9.24)
$$|G(s,x^\sigma_{kn})|\le \text{const}\,\frac{1}{n}\,\frac{s}{n}\,\frac{n^2}{k^2}\,\frac{n^2}{k}\,\min\left\{1,\left|s-\frac{2k-1}{2}\pi\right|^{-1}\right\}\frac{s}{n^2}$$
$$=\text{const}\,\frac{s^2}{k^3}\min\left\{1,\left|s-\frac{2k-1}{2}\pi\right|^{-1}\right\}\,,$$

and

(9.25)
$$|I^n_{3,k}|\le \text{const}\,\frac{1}{k^3}\int_0^{(2k-1)\pi}s^2\min\left\{1,\left|s-\frac{2k-1}{2}\pi\right|^{-1}\right\}$$
$$\le \text{const}\,\frac{1}{k}\int_0^{(2k-1)\pi}\min\left\{1,\left|s-\frac{2k-1}{2}\pi\right|^{-1}\right\}=\text{const}\,\frac{1+\log k}{k}\,.$$

In the last case $-\frac{1}{2}<y<\widetilde{x}^\sigma_{2k,n}$, i.e. $(2k-1)\pi<s<\frac{2\pi}{3}n$, we obtain the relations

$$1-y>1-\cos\frac{2k-1}{n}\pi=2\sin^2\frac{2k-1}{2n}\pi$$
$$=2\left(1+\cos\frac{2k-1}{2n}\pi\right)\left(1-\cos\frac{2k-1}{2n}\pi\right)\ge 2(1-x^\sigma_{kn})\,,$$

and

(9.26)
$$1-y>x^\sigma_{kn}-y=(1-y)-(1-x^\sigma_{kn})>\frac{1}{2}(1-y)\,.$$

Consequently, we get

$$|F(y,x^\sigma_{kn})|\le\frac{\text{const}}{n}\left[\left(\frac{1-y}{1-x^\sigma_{kn}}\right)^{\zeta_+}+1\right]\frac{(1-y)^{\frac{1}{2}}}{(1-y)^2}$$

and

(9.27)
$$|G(s,x^\sigma_{kn})|\le\frac{\text{const}}{n}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{n^3}{s^3}\,\frac{s}{n^2}=\text{const}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{1}{s^2}\,.$$

Since $2(1-\zeta_+)>1$, we obtain the estimate

(9.28)
$$|I^n_{2,k}|\le\text{const}\int_{(2k-1)\pi}^{\infty}\left[\left(\frac{s}{k}\right)^{2\zeta_+}+1\right]\frac{1}{s^2}\,ds\le\frac{\text{const}}{k}\,.$$

From the estimates (9.22), (9.24), and (9.27) we conclude that the function

$$f(s):=C\begin{cases}\max\{s^{2\zeta_++2},s^2\} & \text{if}\quad 0<s<(2k-1)\pi\,,\\ (s^{2\zeta_+}+1)s^{-2} & \text{if}\quad (2k-1)\pi<s<\infty\,,\end{cases}$$

with the constant C depending only on $\zeta_\pm$ and k, is an integrable majorant for the functions $G(s,x_{kn}^\sigma)$, $n > \frac{3}{2}(2k-1)$, in (9.21). Thus, we can change the order between the limit and the integration, and we obtain

$$\begin{aligned}
b_k^+ &= \int_0^\infty \lim_{n\to\infty} G(s,x_{kn}^\sigma)\,ds \\
&= \frac{(-1)^{k+1}}{\pi \mathrm{i}} \int_0^\infty \lim_{n\to\infty} \left\{ \frac{1}{n^2}\,\sin\frac{s}{n} \left[\frac{\left(2\sin^2\frac{s}{2n}\right)^{\zeta_+}\left(2\cos^2\frac{s}{2n}\right)^{\zeta_-}}{\left(2\sin^2\frac{2k-1}{4n}\pi\right)^{\zeta_+}\left(2\cos^2\frac{2k-1}{4n}\pi\right)^{\zeta_-}} - 1 \right] \right. \\
&\qquad\qquad \left. \frac{1}{4\sin^2\frac{(2k-1)\pi-2s}{4n}\sin^2\frac{(2k-1)\pi+2s}{4n}}\cos s\,\sin\frac{s}{n} \right\} ds \\
&= \frac{(-1)^{k+1}}{\pi \mathrm{i}} \int_0^\infty \lim_{n\to\infty} \frac{s^2}{n^4} \left[\left(\frac{2s}{(2k-1)\pi} \right)^{2\zeta_+} - 1 \right] \frac{(4n)^4\cos s}{4\left([(2k-1)\pi]^2-[2s]^2\right)^2}\,ds \\
&= \frac{64(-1)^{k+1}}{\pi \mathrm{i}} \int_0^\infty \frac{\left(\frac{2s}{(2k-1)\pi}\right)^{2\zeta_+} - 1}{\left([(2k-1)\pi]^2-[2s]^2\right)^2}\, s^2\cos s\,ds\,.
\end{aligned}$$

Hence, formula (3.21) is shown.

Due to the estimates (9.20), (9.23), (9.25), and (9.28) we have $|b_k^n| \le \mathrm{const}\,k^{-\varepsilon}$ for some $\varepsilon > 0$ and for $1 \le k \le \frac{n+1}{2}$. Let us consider the case $\frac{n+1}{2} < k \le n$, $j = n+1-k$. It follows $1 \le j \le \frac{n+1}{2}$ and, in view of $x_{n+1-j,n}^\sigma = -x_{jn}^\sigma$, $\varphi(-y) = \varphi(y)$, and $T_n(-y) = (-1)^n T_n(y)$,

$$b_k^n = \frac{(-1)^j}{\sqrt{2\pi}}\,\frac{1}{n\mathrm{i}} \int_{-1}^1 \frac{\widetilde{\zeta}(y)-\widetilde{\zeta}(x_{jn}^\sigma)}{\widetilde{\zeta}(x_{jn}^\sigma)}\,\frac{\varphi(y)T_n(y)}{(y-x_{jn}^\sigma)^2}\,dy\,,$$

where $\widetilde{\zeta}(y) = \zeta(-y)$. Hence, we get $|b_k^n| \le \mathrm{const}\,j^{-\varepsilon} = \mathrm{const}\,(n+1-k)^{-\varepsilon}$ for $\frac{n+1}{2} \le k \le n$, and (9.19) is proved.

d) Now we compute the limits $d_k^+ = \lim_{n\to\infty} d_k^n$ with

$$d_k^n = \frac{1}{\zeta(x_{kn}^\sigma)} \int_{-1}^1 \frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{y-x_{kn}^\sigma}\,\frac{\varphi(y)}{\varphi(x_{kn}^\sigma)}\,T_n(y)\,dy\,.$$

In particular, we shall show that, for some $\varepsilon > 0$,

$$|d_k^n| \le \frac{\mathrm{const}}{\min\{k,n+1-k\}^\varepsilon}\,, \qquad k = 1,2,\ldots,n\,. \tag{9.29}$$

At first, let $n \ge 2k-1$ and consider the polynomials

$$S_n(x) := \left[\frac{1}{n+1}\,T_{n+1}(x) - \frac{1}{n-1}\,T_{n-1}(x) \right],$$

for which we have the relations (see (3.2) and (3.13))

$$T_n(x) = \frac{1}{2}\left[U_n(x) - U_{n-2}(x)\right] = \frac{1}{2}\,S_n'(x)\,. \tag{9.30}$$

We obtain, for $n \geq 2$,

$$S_n(x_{1n}^\sigma) = \frac{1}{n+1}\cos\frac{(n+1)\pi}{2n} - \frac{1}{n-1}\cos\frac{(n-1)\pi}{2n}$$
$$= -\frac{1}{n+1}\sin\frac{\pi}{2n} - \frac{1}{n-1}\sin\frac{\pi}{2n} < 0\,,$$

and

$$S_n(x_{2n}^\sigma) = \frac{1}{n+1}\,\cos\frac{3(n+1)\pi}{2n} - \frac{1}{n-1}\,\cos\frac{3(n-1)\pi}{2n}$$
$$= \frac{1}{n+1}\,\sin\frac{3\pi}{2n} + \frac{1}{n-1}\,\sin\frac{3\pi}{2n} > 0\,.$$

Moreover, since $S_n'(x) = 2T_n(x)$ and $T_n(x) < 0$ for $x \in (x_{2n}^\sigma, x_{1n}^\sigma)$, we get that $S_n(x)$ decreases monotonously on the interval $(x_{2n}^\sigma, x_{1n}^\sigma)$. Consequently, the polynomial $S_n(x)$ has exactly one root in the interval $(x_{2n}^\sigma, x_{1n}^\sigma)$. We denote this root by x_n^+. Obviously, x_n^+ has the form

$$x_n^+ = \cos\frac{s_n^*}{n} \quad \text{with} \quad \frac{\pi}{2} < s_n^* < \frac{3\pi}{2}\,. \tag{9.31}$$

Now, we take an arbitrary $s \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and compute, for sufficiently large n,

$$\begin{aligned}
S_n(\cos\frac{s}{n}) &= \frac{1}{n+1}\,\cos\frac{n+1}{n}s - \frac{1}{n-1}\,\cos\frac{n-1}{n}s \\
&= \frac{1}{n+1}\left[\cos s\,\cos\frac{s}{n} - \sin s\,\sin\frac{s}{n}\right] - \frac{1}{n-1}\left[\cos s\,\cos\frac{s}{n} - \sin s\,\sin\frac{s}{n}\right] \\
&= \left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos s\,\cos\frac{s}{n} - \left[\frac{1}{n+1} + \frac{1}{n-1}\right]\sin s\,\sin\frac{s}{n} \\
&= -\frac{2\cos s}{n^2-1}\left[1 - O\left(\frac{1}{n^2}\right)\right] - \frac{2n\sin s}{n^2-1}\left[\frac{s}{n} + O\left(\frac{1}{n^3}\right)\right] \\
&= -\frac{2[\cos s + s\sin s]}{n^2-1} + O\left(\frac{1}{n^4}\right)\,,
\end{aligned}$$

which means that there exist constants $c, d \in \mathbb{R}$ and $N \in \mathbb{N}$ such that, for each $s \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and for any $n > N$,

$$-2[\cos s + s\sin s] + \frac{c}{n} < (n^2-1)S_n(\cos\frac{s}{n}) < -2[\cos s + s\,\sin s] + \frac{d}{n}\,.$$

Since $S_n\left(\cos\frac{s_n^*}{n}\right) = 0$ the inequalities

$$\frac{c}{n} \leq 2[\cos s_n^* + s_n^*\sin s_n^*] \leq \frac{d}{n}$$

are fulfilled. Whence we conclude that s_n^* tends to s^*, where $s^* \in (\frac{\pi}{2}, \frac{3\pi}{2})$ is the solution of the equation $\cos s + s\sin s = 0$. Define $x_n^- = -x_n^+$ and taking into acount

$S_n(-x) = (-1)^{n+1}S_n(x)$, we obtain $S_n(x_n^\pm) = 0$. In view of (9.30) we get, by applying partial integration two times,

$$\begin{aligned}
d_k^n &= \frac{1}{\zeta(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)} \int_{-1}^{1} \frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{y-x_{kn}^\sigma}\, \varphi(y) T_n(y)\, dy \\
&= \frac{1}{\zeta(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)} \left(\int_{-1}^{x_n^-} + \int_{x_n^+}^{1} \right) \frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{y-x_{kn}^\sigma}\, \varphi(y) T_n(y)\, dy \\
&\quad + \frac{1}{2\zeta(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)} \int_{x_n^-}^{x_n^+} \left[\varphi(y) \frac{\zeta(y)-\zeta(x_{kn}^\sigma)-\zeta'(y)(y-x_{kn}^\sigma)}{(y-x_{kn}^\sigma)^2} \right. \\
&\qquad\qquad \left. -\varphi'(y)\, \frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{y-x_{kn}^\sigma} \right] T_n(y)\, dy \\
&= \frac{1}{\zeta(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)} \left(\int_{-1}^{x_n^-} + \int_{x_n^+}^{1} \right) \frac{\zeta(y)-\zeta(x_{kn}^\sigma)}{y-x_{kn}^\sigma}\, \varphi(y) T_n(y)\, dy \\
&\quad \pm \frac{\widetilde{S}_n(x_n^\pm)}{4\zeta(x_{kn}^\sigma)\varphi(x_{kn}^\sigma)} \left[\varphi(x_n^\pm) \frac{\zeta(x_n^\pm)-\zeta(x_{kn}^\sigma)-zeta'(x_n^\pm)(x_n^\pm - x_{kn}^\sigma)}{(x_n^\pm - x_{kn}^\sigma)^2} \right. \\
&\qquad\qquad \left. -\varphi'(x_n^\pm)\, \frac{\zeta(x_n^\pm)-\zeta(x_{kn}^\sigma)}{x_n^\pm - x_{kn}^\sigma} \right] + \int_{x_n^-}^{x_n^+} \widetilde{F}(y, x_{kn}^\sigma)\, dy \\
&=: d_{k,-}^{n,1} + d_{k,+}^{n,1} + d_{k,+}^{n,2} - d_{k,-}^{n,2} + \widetilde{d}_k^n\,,
\end{aligned}$$

where

$$\widetilde{S}_n(y) = \frac{1}{(n+1)(n+2)} T_{n+2}(y) + \frac{1}{(n-1)(n-2)} T_{n-2}(y) - \frac{1}{n^2-1} T_n(y)$$

and

$$\begin{aligned}
\widetilde{F}(y,x) = \frac{\widetilde{S}_n(y)}{4\varphi(x)\zeta(x)} &\left[2\varphi(y)\, \frac{\zeta(y)-\zeta(x)-\zeta'(y)(y-x)+\frac{1}{2}\zeta''(y)(y-x)^2}{(y-x)^3} \right. \\
&\left. -2\varphi'(y)\, \frac{\zeta(y)-\zeta(x)-\zeta'(y)(y-x)}{(y-x)^2} + \varphi''(y)\, \frac{\zeta(y)-\zeta(x)}{y-x} \right].
\end{aligned}$$

For $n \ge 5$, the term $d_{k,-}^{n,1}$ can be estimated (remark that in this case $x_n^- < -\frac{1}{2}$) by

$$\begin{aligned}
|d_{k,-}^{n,1}| &\le \text{const} \int_{-1}^{x_n^-} \left[\frac{(1+y)^{\zeta^-}}{(1-x_{kn}^\sigma)^{\zeta_+}} + 1 \right] \frac{(1+y)^{1/2}}{(1-x_{kn}^\sigma)^{1/2}}\, dy \\
&= \text{const} \left[\frac{(1+x_n^-)^{3/2+\zeta_-}}{(1-x_{kn}^\sigma)^{1/2+\zeta_+}} + \frac{(1+x_n^-)^{3/2}}{(1-x_{kn}^\sigma)^{1/2}} \right] \\
&\le \text{const} \left[\left(\frac{1}{n}\right)^{3+2\zeta_-} \left(\frac{n}{k}\right)^{1+2\zeta_+} + \frac{1}{n^3}\, \frac{n}{k} \right]
\end{aligned}$$

$$\leq \text{const}\left[\frac{1}{n^{2(1+\zeta_- - \zeta_+)}k^{1+2\zeta_+}} + \frac{1}{n^2}\right]$$

such that $\lim_{n\to\infty} d_{k,-}^{n,1} = 0$ and

$$|d_{k,-}^{n,1}| \leq \frac{C}{k^{3/2}}\,. \tag{9.32}$$

To consider $d_{k,+}^{n,1}$ we use the substitution $y = \cos\frac{s}{n}$ and get

$$d_{k,+}^{n,1} = \int_{x_n^+}^1 H(y, x_{kn}^\sigma)\,dy = \left(\int_{x_n^+}^{\max\{x_n^+,\frac{1}{2}(1+x_{kn}^\sigma)\}} + \int_{\max\{x_n^+,\frac{1}{2}(1+x_{kn}^\sigma)\}}^1\right) H(y, x_{kn}^\sigma)\,dy$$

$$= \left(\int_{\min\{s_n^*,s_k^n\}}^{s_n^*} + \int_0^{\min\{s_n^*,s_k^n\}}\right)\widetilde{H}(s, x_{kn}^\sigma)\,ds =: J_{1,k}^n + J_{2,k}^n$$

with $s_k^n = n\arccos\frac{1}{2}(1+x_{kn}^\sigma)$,

$$H(y,x) = \sqrt{\frac{2}{\pi}}\frac{\varphi(y)}{\varphi(x)\zeta(x)}\frac{\zeta(y)-\zeta(x)}{y-x}\cos s\,, \quad y = \cos\frac{s}{n}\,,$$

and

$$\widetilde{H}(s,x) = \begin{cases} \frac{1}{n}H\left(\cos\frac{s}{n}, x\right)\sin\frac{s}{n} & \text{if} \quad 0 < s < s_n^*\,, \\ 0 & \text{if} \quad s_n^* \leq s \leq \frac{3\pi}{2}\,. \end{cases}$$

If $s_k^n < s < s_n^*$, i.e. $x_n^+ < y = \cos\frac{s}{n} < \frac{1}{2}(1+x_{kn}^\sigma)$, we have the estimate

$$|H(y, x_{kn}^\sigma)| \leq \text{const}\,\frac{\varphi(y)}{\varphi(x_{kn}^\sigma)}\,\frac{|\zeta'(\zeta_1)|}{\zeta(x_{kn}^\sigma)}$$

for some $\zeta_1 \in \left(x_n^+, \frac{1}{2}(1+x_{kn}^\sigma)\right)$. Since in this case $(1-y) \sim (1-x_{kn}^\sigma) \sim (1-\zeta_1)$, we get

$$|\widetilde{H}(s, x_{kn}^\sigma)| \leq \frac{\text{const}}{n}\,\frac{s}{k}\,\frac{n^2}{k^2}\,\frac{s}{n} = \frac{\text{const}\,s^2}{k^3} \quad \text{and} \quad |J_{1,k}^n| \leq \frac{\text{const}}{k^3}\int_0^{\frac{3}{2}\pi} s^2\,ds \leq \frac{\text{const}}{k^3}\,.$$

For $J_{2,k}^n$, we have $0 < s < s_k^n$, which equivalent to $\frac{1}{2}(1+x_{kn}^\sigma) < y = \cos\frac{s}{n} < 1$. Hence $y - x_{kn}^\sigma > \frac{1}{2}(1-x_{kn}^\sigma)$,

$$|H(y, x_{kn}^\sigma)| \leq \text{const}\left(\frac{1-y}{1-x_{kn}^\sigma}\right)^{1/2}\left[\left(\frac{1-y}{1-x_{kn}^\sigma}\right)^{\zeta_+} + 1\right]\frac{1}{1-x_{kn}^\sigma}\,,$$

and

$$|\widetilde{H}(s, x_{kn}^\sigma)| \leq \frac{\text{const}}{n}\,\frac{s}{k}\left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1\right]\frac{n^2}{k^2}\,\frac{s}{n} = \frac{\text{const}}{k^3}\left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1\right]s^2\,.$$

Thus,

$$|J_{2,k}^n| \leq \frac{\text{const}}{k^3}\int_0^{\frac{3}{2}\pi}\left[\left(\frac{s}{k}\right)^{2\zeta_+} + 1\right]s^2\,ds \leq \frac{\text{const}}{k^{3/2}}\,.$$

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Consequently,

$$|d_{k,+}^{n,1}| \le \frac{\text{const}}{k^{3/2}}\,, \tag{9.33}$$

and the functions $\widetilde{H}(s,x_{kn}^{\sigma})$ possess an integrable majorant,

$$\left|\widetilde{H}(s,x_{kn}^{\sigma})\right| \le C\left[s^{2+2\zeta_+}+s^2\right]\,, \quad 0<s<\frac{3\pi}{2}\,,$$

where the constant C depends only on ζ_+ and k . So, we can change the order between the limit and the integration and obtain

$$\begin{aligned}\lim_{n\to\infty} d_{k,+}^{n,1} &= \int_0^{s^*} \lim_{n\to\infty} \widetilde{H}(s,x_{kn}^{\sigma})\,ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^{s^*} \lim_{n\to\infty} \frac{2s}{2k-1}\left[\left(\frac{2s}{2k-1}\right)^{2\zeta_+}-1\right]\frac{s}{n^2}\,\frac{8n^2\cos s}{[(2k-1)\pi]^2-[2s]^2}\,ds \\ &= \sqrt{\frac{2}{\pi}}\,\frac{16}{2k-1}\int_0^{s^*}\frac{\left(\frac{2s}{2k-1}\right)^{2\zeta_+}-1}{[(2k-1)\pi]^2-[2s]^2}\,s^2\cos s\,ds\,.\end{aligned}$$

To estimate $d_{k,\pm}^{n,2}$ we remark that $\widetilde{S}_n(-x)=(-1)^n\widetilde{S}_n(x)$ and write

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\,\widetilde{S}_n(\cos t) &= \frac{\cos(n+2)t}{(n+1)(n+2)}-\frac{\cos(n-2)t}{(n-1)(n-2)}-\frac{2\cos nt}{n^2-1} \\ &= \left[\frac{1}{(n+1)(n+2)}+\frac{1}{(n-1)(n-2)}\right]\cos nt\,\cos 2t-\frac{2}{n^2-1}\cos nt \\ &\qquad + \left[\frac{1}{(n-1)(n-2)}-\frac{1}{(n+1)(n+2)}\right]\sin nt\,\sin 2t \qquad (9.34)\\ &= \left[\frac{(2n^2+4)\cos 2t}{(n^2-1)(n^2-4)}-\frac{2}{n^2-1}\right]\cos nt+\frac{6n}{(n^2-1)(n^2-4)}\sin nt\,\sin 2t \\ &= \frac{2n^2(\cos 2t-1)\cos nt+4\cos 2t\,\cos nt+8\cos nt+6n\sin 2t\,\sin nt}{(n^2-1)(n^2-4)} \\ &= \frac{8\cos nt+4\cos 2t\cos nt+6n\sin 2t\,\sin nt-4n^2\sin^2 t\,\cos nt}{(n^2-1)(n^2-4)}\,.\end{aligned}$$

For $n\ge 5$, the term $d_{k,-}^{n,2}$ can be estimated by (comp. (9.31))

$$\begin{aligned}|d_{k,-}^{n,2}| &\le \text{const}\,\frac{1+s_n^*+(s_n^*)^2}{n^4}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{s_n^*}{n}\left[\left(\frac{k}{n}\right)^{2\zeta_+}+\left(\frac{s_n^*}{n}\right)^{2\zeta_--2}\right]\right. \\ &\qquad\qquad \left.+\frac{n}{s_n^*}\left[\left(\frac{k}{n}\right)^{2\zeta_+}+\left(\frac{s_n^*}{n}\right)^{2\zeta_-}\right]\right\} \\ &\le \text{const}\left\{\frac{1}{n^4}\,\frac{1}{k}+\frac{1}{n^2}\,\frac{1}{k}+\frac{1}{n^{2(1+\zeta_--\zeta_+)}k^{1+2\zeta_+}}\right\}\,,\end{aligned}$$

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such that $\lim_{n\to\infty} d_{k,-}^{n,2} = 0$ and

$$|d_{k,-}^{n,2}| \le \frac{\text{const}}{k^{3/2}}\,. \tag{9.35}$$

For the term $d_{k,+}^{n,2}$, there are two possible cases $x_n^+ \ge \frac{1}{2}(1+x_{kn}^\sigma)$ and $x_{2k,n}^\sigma < x_n^+ < \frac{1}{2}(1+x_{kn}^\sigma)$. In the first case, we have $x_n^+ - x_{kn}^\sigma \ge \frac{1}{2}(1-x_{kn}^\sigma)$ and

$$|d_{k,+}^{n,2}| \le \frac{\text{const}}{n^4}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{1}{n}\left[\left(\frac{k}{n}\right)^{2\zeta_+}+\left(\frac{1}{n}\right)^{2\zeta_+}+\left(\frac{1}{n}\right)^{2\zeta_+-2}\frac{k^2}{n^2}\right]\frac{n^4}{k^4}\right.$$
$$\left.+n\left[\left(\frac{k}{n}\right)^{2\zeta_+}+\left(\frac{1}{n}\right)^{2\zeta_+}\right]\frac{n^2}{k^2}\right\}$$
$$=\text{const}\left\{\left(\frac{1}{k}\right)^5+\left(\frac{1}{k}\right)^{5+2\zeta_+}+\left(\frac{1}{k}\right)^3+\left(\frac{1}{k}\right)^{3+2\zeta_+}\right\}\le\frac{\text{const}}{k^{3/2}}\,.$$

In the second case, we have

$$|d_{k,+}^{n,2}| \le \frac{\text{const}}{n^4}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{1}{n}|\zeta''(\zeta_2|+n|\zeta'(\zeta_1)|\right\}\,,$$

for some $\zeta_1,\zeta_2 \in \left(x_n^+, \frac{1}{2}(1+x_{kn}^\sigma)\right)$. Since in this case $1-\zeta_{1,2} > \frac{1}{2}(1-x_{kn}^\sigma)$, we conclude

$$|d_{k,+}^{n,2}| \le \frac{\text{const}}{n^4}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{1}{n}\left(\frac{k}{n}\right)^{2\zeta_+-4}+n\left(\frac{1}{n}\right)^{2\zeta_+-2}\right\}=\text{const}\left\{\frac{1}{k^5}+\frac{1}{k^3}\right\}\,.$$

Consequently,

$$|d_{k,+}^{n,2}| \le \frac{\text{const}}{k^{3/2}} \tag{9.36}$$

and, taking into account (9.34),

$$\lim_{n\to\infty} d_{k,+}^{n,2}$$

$$=\frac{1}{4}\sqrt{\frac{2}{\pi}}\lim_{n\to\infty}\frac{8\cos s_n^*+4\cos s_n^*+12s_n^*\sin s_n^*-4(s_n^*)^2\cos s_n^*}{(n^2-1)(n^2-4)}\,\frac{8^{1/2+\zeta_+}}{2^{1/2+\zeta_-}}$$
$$*\left(\frac{n}{(2k-1)\pi}\right)^{1+2\zeta_+}\left\{\frac{s_n^*}{n}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left[\left(\frac{2s_n^*}{n}\right)^{2\zeta_+}-\left(\frac{(2k-1)\pi}{n}\right)^{2\zeta_+}\right]*\right.$$
$$*\frac{64n^4}{([(2k-1)\pi]^2-[2s_n^*]^2)^2}-\frac{s_n^*}{n}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left(\frac{2s_n^*}{n}\right)^{2\zeta_+}\left[\frac{\zeta_-}{2}-\frac{8\zeta_+n^2}{[2s_n^*]^2}\right]\frac{8n^2}{[(2k-1)\pi]^2-[2s_n^*]^2}$$
$$\left.+\frac{n}{s_n^*}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left[\left(\frac{2s_n^*}{n}\right)^{2\zeta_+}-\left(\frac{2k-1}{n}\right)^{2\zeta_+}\right]\frac{8n^2}{[(2k-1)\pi]^2-[2s_n^*]^2}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{12\cos s^* + 12 s^*\sin s^* - 4(s^*)^2\cos s^*}{[(2k-1)\pi]^{1+2\zeta_+}}\left\{32 s^*\frac{[2s^*]^{2\zeta_+} - [(2k-1)\pi]^{2\zeta_+}}{([(2k-1)\pi]^2 - [2s^*]^2)^2}\right.$$
$$\left. + \frac{4}{s^*}\,\frac{(1+2\zeta_+)[2s^*]^{2\zeta_+} - [(2k-1)\pi]^{2\zeta_+}}{[(2k-1)\pi]^2 - [2s^*]^2}\right\}.$$

Finally, we write

$$\widetilde{d}_k^n = \left(\int_{x_n^-}^{-\frac12} + \int_{-\frac12}^{\widetilde{x}_{2k,n}^\sigma} + \int_{\widetilde{x}_{2k,n}^\sigma}^{\widetilde{x}_n^+} + \int_{\widetilde{x}_n^+}^{x_n^+}\right)\widetilde{F}(y, x_{kn}^\sigma)\,dy =: \widetilde{I}_{1,k}^n + \widetilde{I}_{2,k}^n + \widetilde{I}_{3,k}^n + \widetilde{I}_{4,k}^n\,,$$

where $\widetilde{x}_{2k,n}^\sigma = \max\left\{-\frac12, \cos\frac{2k-1}{n}\pi\right\}$ and $\widetilde{x}_n^+ = \min\left\{x_n^+, \frac12(1+x_{kn}^\sigma)\right\}$. For $x_n^- < y < -\frac12$, we have $2 > |y - x_{kn}^\sigma| > \frac12$ and $2 > 1 - y > \frac32$. With the help of the relations $-x_n^- = x_n^+$ and $\widetilde{S}_n(-y) = (-1)^n\widetilde{S}_n(y)$ together with the substitution $y = \cos\frac{s}{n}$ we obtain

$$|\widetilde{I}_{1,k}^n| \le \frac{C}{(1-x_{kn}^\sigma)^{1/2+\zeta_+}}\int_{\frac12}^{x_n^+}\left\{(1-y)^{\frac12}\left[(1-y)^{\zeta_- - 2} + (1-x_{kn}^\sigma)^{\zeta_+}\right]\right.$$
$$+(1-y)^{-\frac12}\left[(1-y)^{\zeta_- - 1} + (1-x_{kn}^\sigma)^{\zeta_+}\right]$$
$$\left.+(1-y)^{-\frac32}\left[(1-y)^{\zeta_-} + (1-x_{kn}^\sigma)^{\zeta_+}\right]\right\}\widetilde{S}_n(y)\,dy$$

$$\le \text{const}\left(\frac{n}{k}\right)^{1+2\zeta_+}\int_{\frac{\pi}{2}}^{\frac{\pi}{3}n}\left\{\frac{s}{n}\left[\left(\frac{s}{n}\right)^{2\zeta_- - 4} + \left(\frac{k}{n}\right)^{2\zeta_+}\right]\right.$$
$$\left.+\frac{n}{s}\left[\left(\frac{s}{n}\right)^{2\zeta_- - 2} + \left(\frac{k}{n}\right)^{2\zeta_+}\right] + \frac{n^3}{s^3}\left[\left(\frac{s}{n}\right)^{2\zeta_-} + \left(\frac{k}{n}\right)^{2\zeta_+}\right]\right\}\frac{1+s+s^2}{n^4}\,\frac{s}{n^2}\,ds$$

$$\le \text{const}\int_{\frac{\pi}{2}}^{\frac{\pi}{3}n}\left\{\frac{s^4}{n^6 k} + \frac{s^2}{n^4 k} + \frac{1}{n^2 k} + \frac{s^{2\zeta_-}}{n^{2(1-\zeta_+ + \zeta_-)}k^{1+2\zeta^+}}\right\}ds$$

$$\le \text{const}\left\{\begin{array}{ll} \dfrac1n + \dfrac{1}{n^{1-2\zeta_+}k^{1+2\zeta_+}} & \text{if}\quad \zeta_- > -\dfrac12 \\ \dfrac1n + \dfrac{\log n}{n^{1-2\zeta_+}k^{1+2\zeta_+}} & \text{if}\quad \zeta_- = -\dfrac12 \\ \dfrac1n + \dfrac{1}{n^{2(1-\zeta_+ + \zeta_-)}k^{1+2\zeta_+}} & \text{if}\quad \zeta_- < -\dfrac12 \end{array}\right\} \le \frac{\text{const}}{n^\varepsilon}$$

for some $\varepsilon > 0$. Consequently, $\lim_{n\to\infty}\widetilde{I}_{1,k}^n = 0$,

$$|\widetilde{I}_{1,k}^n| \le \frac{\text{const}}{k^\varepsilon}\,, \tag{9.37}$$

and

$$\widetilde{d}_k^+ = \lim_{n\to\infty}\widetilde{d}_{k,+}^n = \lim_{n\to\infty}\widetilde{I}_{2,k}^n + \widetilde{I}_{3,k}^n + \widetilde{I}_{4,k}^n = \lim_{n\to\infty}\int_{\frac{\pi}{2}}^{\infty}\widetilde{G}(s, x_{kn}^\sigma)\,ds\,, \tag{9.38}$$

where

$$\widetilde{G}(s,x) = \begin{cases} 0 & \text{if} \quad \frac{\pi}{2} < s < s_n^*\,, \\ \frac{1}{n}\widetilde{F}(\cos\frac{s}{n},x)\sin\frac{s}{n} & \text{if} \quad s_n^* < s < \frac{2\pi}{3}n\,, \\ 0 & \text{if} \quad \frac{2\pi}{3}n < s\,. \end{cases}$$

Now, let $\frac{1}{2}(1+x_{kn}^\sigma) < y = \cos\frac{s}{n} < x_n^+$, which is equivalent to $s_n^* < s < s_k^n$, where $\frac{2k-1}{4}\pi < s_k^n < \frac{2k-1}{2}\pi$. Hence, $y - x_{kn}^\sigma > \frac{1}{2}(1-x_{kn}^\sigma)$ and

$$\begin{aligned} |\widetilde{G}(s,x_{kn}^\sigma)| \le \text{const}\Bigg\{ & \frac{s}{n}\left[\left(\frac{s}{n}\right)^{2\zeta_+} + \left(\frac{k}{n}\right)^{2\zeta_+} + \left(\frac{s}{n}\right)^{2\zeta_+-2}\frac{k^2}{n^2} + \left(\frac{s}{n}\right)^{2\zeta_+-4}\frac{k^4}{n^4}\right]\frac{n^6}{k^6} \\ & + \frac{n}{s}\left[\left(\frac{s}{n}\right)^{2\zeta_+} + \left(\frac{k}{n}\right)^{2\zeta_+} + \left(\frac{s}{n}\right)^{2\zeta_+-2}\frac{k^2}{n^2}\right]\frac{n^4}{k^4} \\ & + \frac{n^3}{s^3}\left[\left(\frac{s}{n}\right)^{2\zeta_+} + \left(\frac{k}{n}\right)^{2\zeta_+}\right]\frac{n^2}{k^2}\Bigg\}\left(\frac{n}{k}\right)^{1+2\zeta_+}\frac{s^3}{n^6} \\ \le \text{const}\left\{\frac{s^{4+2\zeta_+}}{k^{7+2\zeta_+}} + \frac{s^{2+2\zeta_+}}{k^{5+2\zeta_+}} + \frac{s^{2\zeta_+}}{k^{3+2\zeta_+}} + \frac{s^4}{k^7} + \frac{s^2}{k^5} + \frac{1}{k^3}\right\} & \\ \le \text{const}\left\{\frac{s^{2\zeta_+}}{k^{3+2\zeta_+}} + \frac{1}{k^3}\right\} & \end{aligned} \tag{9.39}$$

Consequently,

$$|\widetilde{I}_{4,k}^n| \le \text{const}\int_{\frac{\pi}{2}}^{\frac{2k-1}{2}\pi}\left\{\frac{s^{2\zeta_+}}{k^{3+2\zeta_+}} + \frac{1}{k^3}\right\}ds \le \frac{\text{const}}{k^{3/2}} \tag{9.40}$$

For the case $\widetilde{x}_{2k,n}^\sigma < y = \cos\frac{s}{n} < \widetilde{x}_n^+ \le \frac{1}{2}(1+x_{kn}^\sigma)$, we have $\max\{s_k^n, s_n^*\} < s < \min\{(2k-1)\pi, \frac{2\pi}{3}n\}$ and, for some $\zeta_1,\zeta_2,\zeta_3 \in \left(\widetilde{x}_{2k,n}^\sigma, \widetilde{x}_n^+\right)$, the estimate

$$|\widetilde{F}(y,x_{kn}^\sigma)| \le \frac{\widetilde{S}_n(y)}{\varphi(x_{kn}^\sigma)\zeta(x_{kn}^\sigma)}\Big(\varphi(y)|\zeta'''(\zeta_3)| + |\varphi'(y)||\zeta''(\zeta_2)| + |\varphi''(y)||\zeta'(\zeta_1)|\Big)\,.$$

Because of $(1-y) \sim (1-x_{kn}^\sigma) \sim (1-\zeta_{1,2,3})$ we get

$$\begin{aligned} |G(s,x_{kn}^\sigma)| &\le \text{const}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{k}{n}\left(\frac{k}{n}\right)^{2\zeta_+-6} + \frac{n}{k}\left(\frac{k}{n}\right)^{2\zeta_+-4} + \frac{n^3}{k^3}\left(\frac{k}{n}\right)^{2\zeta_+-3}\right\}\frac{k^3}{n^6} \\ &\le \frac{\text{const}}{k^2} \end{aligned} \tag{9.41}$$

and

$$|\widetilde{I}_{3,k}^n| \le \frac{\text{const}}{k^2}\int_{\pi/2}^{(2k-1)\pi} ds \le \frac{\text{const}}{k} \tag{9.42}$$

In last case $-\frac{1}{2} < y < \widetilde{x}^{\sigma}_{2k,n}$, i.e. $(2k-1)\pi < s < \frac{2\pi}{3}n$, we have the relation (comp. (9.26))

$$1 - y > x^{\sigma}_{kn} - y > \frac{1}{2}(1-y)\,.$$

Consequently,

$$\begin{aligned} |G(s,x^{\sigma}_{kn})| &\le \text{const}\left(\frac{n}{k}\right)^{1+2\zeta_+}\left\{\frac{s}{n}\left[\left(\frac{s}{n}\right)^{2\zeta_+}+\left(\frac{k}{n}\right)^{2\zeta_+}\right]\frac{n^6}{s^6}\right. \\ &\qquad \left.+\frac{n}{s}\left[\left(\frac{s}{n}\right)^{2\zeta_+}+\left(\frac{k}{n}\right)^{2\zeta_+}\right]\frac{n^4}{s^4}+\frac{n^3}{s^3}\left[\left(\frac{s}{n}\right)^{2\zeta_+}+\left(\frac{k}{n}\right)^{2\zeta_+}\right]\frac{n^2}{s^2}\right\}\frac{s^3}{n^6} \qquad (9.43) \\ &\le \text{const}\left\{\frac{1}{s^{2(1-\zeta_+)}k^{1+2\zeta_+}}+\frac{1}{s^2 k}\right\} \end{aligned}$$

and, since $2(1-\zeta_+) > 1$,

$$|\widetilde{I}^{n}_{2,k}| \le \text{const}\int_{(2k-1)\pi}^{\infty}\left\{\frac{1}{s^{2(1-\zeta_+)}k^{1+2\zeta_+}}+\frac{1}{s^2 k}\right\}ds \le \frac{\text{const}}{k^2} \qquad (9.44)$$

From the estimates (9.39), (9.41), and (9.43) we conclude that the function

$$\widetilde{f}(s) := C\left\{\begin{array}{ll} \max\{s^{2\zeta_+},1\} & \text{if} \quad \dfrac{\pi}{2} < s < (2k-1)\pi\,, \\ (s^{2\zeta_+}+1)s^{-2} & \text{if} \quad (2k-1)\pi < s < \infty\,, \end{array}\right.$$

with the constant C depending only on $\zeta_\pm$ and k , is an integrable majorant for the functions $G(s,x^{\sigma}_{kn})$, $n > \frac{3}{2}(2k-1)$, in (9.38). Thus, we can change the order between the limit and the integration and obtain

$$\begin{aligned} \widetilde{d}^{+}_{k} = \int_{s^*}^{\infty}\frac{1}{4}\sqrt{\frac{2}{\pi}}\lim_{n\to\infty}\frac{12\cos s+12s\sin s-4s^2\cos s}{(n^2-1)(n^2-4)}\,\frac{8^{1/2+\zeta_+}}{2^{1/2}+\zeta_-}\left(\frac{n}{(2k-1)\pi}\right)^{1+2\zeta_+} * \\ *\left\{\frac{2s}{n}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left[\left(\frac{2s}{n}\right)^{2\zeta_+}-\left(\frac{(2k-1)\pi}{n}\right)^{2\zeta_+}\right]\frac{512n^6}{([(2k-1)\pi]^2-[2s]^2)^3}\right. \\ -\frac{2s}{n}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left(\frac{2s}{n}\right)^{2\zeta_+}\left[\frac{\zeta_-}{2}-\frac{8\zeta_+n^2}{[2s]^2}\right]\frac{64n^4}{([(2k-1)\pi]^2-[2s]^2)^2} \\ +\frac{s}{n}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left(\frac{2s}{n}\right)^{2\zeta_+}\left[\frac{(\zeta_+^2-\zeta_+)64n^4}{[2s]^4}\,\frac{\zeta_-^2-\zeta_-}{4}-\frac{8\zeta_-\zeta_+n^2}{[2s]^2}\right]\frac{8n^2}{[(2k-1)\pi]^2-[2s]^2} \\ +\frac{2n}{s}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left[\left(\frac{2s}{n}\right)^{2\zeta_+}-\left(\frac{(2k-1)\pi}{n}\right)^{2\zeta_+}\right]\frac{64n^4}{([(2k-1)\pi]^2-[2s]^2)^2} \\ -\frac{2n}{s}\frac{2^{\zeta_-}}{8^{\zeta_+}}\left(\frac{2s}{n}\right)^{2\zeta_+}\left[\frac{\zeta_-}{2}-\frac{8\zeta_+n^2}{[2s]^2}\right]\frac{8n^2}{[(2k-1)\pi]^2-[2s]^2} \\ \left.-\left[\frac{n}{s}+\frac{n^3}{s^3}\right]\frac{2^{\zeta_-}}{8^{\zeta_+}}\left[\left(\frac{2s}{n}\right)^{2\zeta_+}-\left(\frac{(2k-1)\pi}{n}\right)^{2\zeta_+}\right]\frac{8n^2}{[(2k-1)\pi]^2-[2s]^2}\right\}\frac{s}{n^2}\,ds \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \int_{s^*}^{\infty} \left\{ 512s \frac{[2s]^{2\zeta_+} - [(2k-1)\pi]^{2\zeta_+}}{([(2k-1)\pi]^2 - [2s]^2)^3} + \frac{64}{s} \frac{(1+2\zeta_+)[2s]^{2\zeta_+} - [(2k-1)\pi]^{2\zeta_+}}{([(2k-1)\pi]^2 - [2s]^2)^2} \right.$$

$$\left. + \frac{4}{s^3} \frac{(4\zeta_+^2 - 1)[2s]^{2\zeta_+} - [(2k-1)\pi]^{2\zeta_+}}{[(2k-1)\pi]^2 - [2s]^2} \right\} \frac{12\cos s + 12 s \sin s - 4s^2 \cos s}{[(2k-1)\pi]^{1+2\zeta_+}} \, s \, ds$$

Formula (3.22) is proved.

Due to the estimates (9.32), (9.33), (9.35), (9.36), (9.37), (9.40), (9.42), and (9.44) we have $|d_k^n| \le \text{const}\, k^{-\varepsilon}$ for some $\varepsilon > 0$ and for $1 \le k \le \frac{n+1}{2}$. Additionally, let $\frac{n+1}{2} < k \le n$ and $j = n+1-k$. Then $1 \le j \le \frac{n+1}{2}$ and, in view of $x_{n+1-j,n}^\sigma = -x_{jn}^\sigma$, $\varphi(-y) = \varphi(y)$, and $T_n(-y) = (-1)^n T_n(y)$,

$$d_k^n = \frac{(-1)^{n+1}}{\widetilde{\zeta}(x_{jn}^\sigma)} \int_{-1}^{1} \frac{\widetilde{\zeta}(y) - \widetilde{\zeta}(x_{jn}^\sigma)}{y - x_{jn}^\sigma} \frac{\varphi(y)}{\varphi(x_{jn}^\sigma)} T_n(y) \, dy,$$

where $\widetilde{\zeta}(y) = \zeta(-y)$. Hence, we get $|d_n^k| \le \text{const}\, j^{-\varepsilon} = \text{const}\, (n+1-k)^{-\varepsilon}$ for $\frac{n+1}{2} \le k \le n$, and (9.29) is proved.

e) Using the estimates (9.12) and (9.17) together with (9.29) and Remark 3.1, we get, for each fixed $m = 1, 2, \dots$, the ℓ^2-convergences

$$\mathbf{V}_n \mathbf{A}_n^* \mathbf{W}_n e_{m-1} \longrightarrow \mathbf{V}_+ \mathbf{A}^* \mathbf{W} e_{m-1}$$

and

$$\mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1} \mathbf{W}_n \mathbf{V}_n e_{m-1} \longrightarrow \mathbf{D}_+ \mathbf{A} \mathbf{D}_+^{-1} \mathbf{W} \mathbf{V}_+ e_{m-1}$$

as well as the corresponding limit relations for the adjoint operators, where the operators V_+ and W are defined by (3.20). Together with items **a)**,**b)**, **c)**, and Lemma 3.3, we obtain the strongly convergence of the sequences $\{V_n A_n V_n^{-1} P_n\}$ and $\{(V_n A_n V_n^{-1} P_n)^*\}$.

The strong convergence of $\widetilde{V}_n A_n \widetilde{V}_n^{-1} P_n$ and $(\widetilde{V}_n A_n \widetilde{V}_n^{-1} P_n)^*$ follows from the previous considerations and the relations

$$a_{n-1-j,n-1-k}^{(n)} = \frac{\varphi(x_{n-k,n}^\sigma)}{ni} \frac{1 - \delta_{n-1-j,n-1-k}}{x_{n-k,n}^\sigma - x_{n-j,n}^\sigma}$$

$$= -\frac{\varphi(x_{k+1,n}^\sigma)}{ni} \frac{1 - \delta_{j,k}}{x_{k+1,n}^\sigma - x_{j+1,n}^\sigma} = -a_{jk}^{(n)}, \quad 0 \le j,k \le n-1,$$

$$\widetilde{a}_{n-1-j,n-1-k}^{(n)} = \frac{\chi(x_{n-j,n}^\sigma)}{\chi(x_{n-k,n}^\sigma)} \frac{\varphi(x_{n-k,n}^\sigma)}{ni} \frac{1 - \delta_{n-1-j,n-1-k}}{x_{n-k,n}^\sigma - x_{n-j,n}^\sigma}$$

$$= -\frac{\widetilde{\chi}(x_{j+1,n}^\sigma)}{\widetilde{\chi}(x_{k+1,n}^\sigma)} \frac{\varphi(x_{k+1,n}^\sigma)}{ni} \frac{1 - \delta_{j,k}}{x_{k+1,n}^\sigma - x_{j+1,n}^\sigma}, \quad 0 \le j,k \le n-1,$$

$$b_{n+1-k}^n = \frac{(-1)^{n-k}}{\sqrt{2\pi}} \frac{1}{n\mathrm{i}} \int_{-1}^{1} \frac{\zeta(y) - \zeta(x_{n+1-k,n}^\sigma)}{\zeta(x_{n+1-k,n}^\sigma)} \frac{\varphi(y) T_n(y)}{(y - x_{n+1-k,n}^\sigma)^2} \, dy$$

$$= -\frac{(-1)^{k+1}}{\sqrt{2\pi}} \frac{1}{n\mathrm{i}} \int_{-1}^{1} \frac{\widetilde{\zeta}(y) - \widetilde{\zeta}(x_{kn}^\sigma)}{\widetilde{\zeta}(x_{kn}^\sigma)} \frac{\varphi(y) T_n(y)}{(y - x_{kn}^\sigma)^2} \, dy, \quad 1 \le k \le n,$$

$$d^n_{n+1-k} = \frac{1}{\zeta(x^\sigma_{n+1-k,n})} \int_{-1}^{1} \frac{\zeta(y) - \zeta(x^\sigma_{n+1-k,n})}{y - x^\sigma_{n+1-k,n}} \, \frac{\varphi(y)}{\varphi(x^\sigma_{n+1-k,n})} \, T_n(y)\, dy$$

$$= \frac{(-1)^{n+1}}{\widetilde{\zeta}(x^\sigma_{kn})} \int_{-1}^{1} \frac{\widetilde{\zeta}(y) - \widetilde{\zeta}(x^\sigma_{kn})}{y - x^\sigma_{kn}} \, \frac{\varphi(y)}{\varphi(x^\sigma_{kn})} \, T_n(y)\, dy\,, \quad 1 \le k \le n\,,$$

where $\widetilde{\chi}(y) = \chi(-y)$, $\widetilde{\zeta}(y) = \zeta(-y)$. The numbers $a^{(n)}_{jk}$ $\widetilde{a}^{(n)}_{jk}$, b^n_k , and d^n_k are defined in items **a)**,**b)**, **c)**, and **d)**, respectively.

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