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CONVERGENCE OF V-CYCLE AND F-CYCLE MULTIGRID METHODS FOR THE BIHARMONIC PROBLEM USING THE MORLEY ELEMENT*

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Abstract. Multigrid V-cycle and F-cycle algorithms for the biharmonic problem using the Morley element are studied in this paper. We show that the contraction numbers can be uniformly improved by increasing the number of smoothing steps.

Key words. multigrid, nonconforming, V-cycle, F-cycle, biharmonic problem, Morley element.

AMS subject classifications. 65N55, 65N30.

1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider the following variational problem for the biharmonic equation with homogeneous Dirichlet boundary conditions: Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \phi(v) \quad \forall v \in H_0^2(\Omega), \tag{1.1}$$

where

$$a(u,v) = \int_\Omega D^2u : D^2v\, dx := \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx,$$

and $\phi \in H^{-2}(\Omega) = [H_0^2(\Omega)]'$.

The elliptic regularity of the biharmonic equation (cf. [17], [18]) implies that there exists $\alpha \in (\frac{1}{2}, 1]$ such that the solution u of (1.1) belongs to $H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ whenever $\phi \in H^{-2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|\phi\|_{H^{-2+\alpha}(\Omega)}, \tag{1.2}$$

where C_Ω depends only on the shape of Ω.

The problem (1.1) can be solved numerically using the Bogner-Fox-Schmit element (cf. [4]), the Argyris element (cf. [1]), the Hsieh-Clough-Tocher element (cf. [16]), the Morley element (cf. [24]) and the incomplete biquadratic element (cf. [27]). In this paper we will concentrate on the Morley element, which is the simplest among all the finite element methods for the biharmonic problem. But of course, the analysis of the Morley element is more complicated.

Multigrid methods for the Morley element, which is nonconforming, have been studied in [8], [9], [10], [20], [26], [25] and [28]. It was shown in [10] that the W-cycle multigrid method converges uniformly if the number of smoothing steps is large enough and that the symmetric variable V-cycle algorithm is an optimal preconditioner, without assuming full elliptic regularity.

In [11] and [13], an additive theory was developed to study the asymptotic behavior, with respect to the number of smoothing steps, of the contraction numbers of V-cycle and F-cycle methods for conforming and nonconforming finite elements for second order problems, without assuming full elliptic regularity. In this paper we will apply this theory to the biharmonic

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problem. We use the Morley element to illustrate the theory, which can also be applied to other elements (cf. [33]).

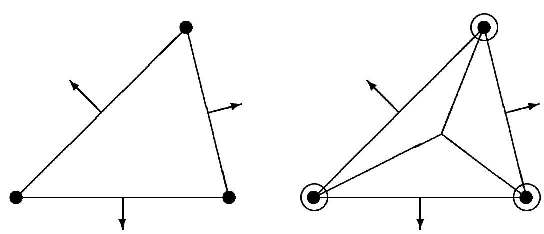
Let $\gamma_{k,m}$ be the contraction number of the k-th level symmetric V-cycle algorithm with m pre-smoothing and m post-smoothing steps. Our main result states that, there exists a constant C, independent of k and m, such that

$$\gamma_{k,m} \leq \frac{C}{m^{\alpha/2}} \quad \text{for } m \geq m_0,$$

where the positive integer m_0 is also independent of k. A similar result also holds for the F-cycle algorithm.

The rest of the paper is organized as follows. We discuss the Morley element and its relation with the Hsieh-Clough-Tocher element in Section 2. The relation is important for the analysis of the multigrid methods. We describe multigrid V-cycle and F-cycle algorithms in Section 3. In Section 4 we discuss mesh dependent norms and their properties. Some known results concerning the additive theory are summarized in Section 5. Convergence analysis is then carried out in Section 6. Numerical results are presented in Section 7.

2. The Morley element and the Hsieh-Clough-Tocher element. The Morley finite element is defined on a triangle. Its shape functions are quadratic polynomials on the triangle. Its nodal variables include the evaluations of the shape functions at the vertices of the triangles and the evaluations of the normal derivatives at the midpoints of the edges of the triangles (cf. Figure 2.1(a)).



(a) Morley (b) H-C-T

FIG. 2.1. *The Morley element and the H-C-T element*

The Hsieh-Clough-Tocher macro element is also defined on a triangle. The shape functions are those C^1 functions on the triangle whose restriction to each smaller triangle formed by connecting the centroid and two vertices of the triangle is a cubic polynomial. The nodal variables include the evaluations of the shape functions at the vertices of the triangle, the evaluations of the gradients at the vertices and of the normal derivatives at the midpoints of the edges of the triangle (cf. Figure 2.1(b)).

Since the shape functions and the nodal variables of the Morley element are also shape functions and nodal variables of the Hsieh-Clough-Tocher element, we call the Hsieh-Clough-Tocher element a "relative" of the Morley element (cf. [10]).

Let $\{\mathcal{T}_k\}_{k\geq 1}$ be a family of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. We denote the mesh size of $\mathcal{T}_k$ by $h_k = \max\{\text{diam}T : T \in \mathcal{T}_k\}$. Note that

$$h_{k-1} = 2h_k. \tag{2.1}$$

Let V_k be the Morley finite element space associated with $\mathcal{T}_k$. Then $v \in V_k$ if and only if it has the following three properties:

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1. $v_T = v|_T$ is a quadratic polynomial for all $T \in \mathcal{T}_k$,
2. v is continuous at the vertices of $\mathcal{T}_k$ and vanishes at the vertices along $\partial\Omega$,
3. The normal derivative $\partial v/\partial n$ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along $\partial\Omega$.

Note that the Morley finite element spaces are nonconforming (i.e., $V_k \not\subset H_0^2(\Omega)$) and nonnested (i.e., $V_{k-1} \not\subset V_k$).

Let $\tilde{V}_k$ be the Hsieh-Clough-Tocher macro element space associated with $\mathcal{T}_k$. Then a function $\tilde{v} \in \tilde{V}_k$ is C^1 on $\overline{\Omega}$, its restriction to each $T \in \mathcal{T}_k$ is a piecewise cubic polynomial function, and its nodal values along $\partial\Omega$ are zero. Note that $\tilde{V}_k \subset H_0^2(\Omega)$ (i.e., conforming).

We now discuss the relation between the Morley space and the Hsieh-Clough-Tocher space. We can define an operator $E_k : V_k \longrightarrow \tilde{V}_k$. For each $v \in V_k$, the function $E_k v \in \tilde{V}_k$ is defined as follows. For any internal vertex p and internal midpoint m,

$$\begin{aligned} (E_k v)(p) &= v(p), \\ \frac{\partial(E_k v)}{\partial n}(m) &= \frac{\partial v}{\partial n}(m), \\ (\partial^\beta(E_k v))(p) &= \text{ average of } (\partial^\beta v_i)(p), \end{aligned}$$

where $\beta = (0,1)$ or $(1,0)$, and $v_i = v|_{T_i}$ for T_i with p as a vertex.

We can also define an operator $F_k : \tilde{V}_k \longrightarrow V_k$ as follows. For each $\tilde{v} \in \tilde{V}_k$, $F_k\tilde{v}$ is the function in V_k satisfying

$$(F_k\tilde{v})(p) = \tilde{v}(p) \quad \text{and} \quad \frac{\partial(F_k\tilde{v})}{\partial n}(m) = \frac{\partial\tilde{v}}{\partial n}(m)$$

for every internal vertex p and midpoint m of $\mathcal{T}_k$.

The operators E_k and F_k satisfy the following two properties (cf. [10]):

$$F_k \circ E_k = Id_k, \tag{2.2}$$

$$\|F_k\tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \text{and} \quad \|F_k\tilde{v}\|_{a_k} \lesssim |\tilde{v}|_{H^2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k, \tag{2.3}$$

where Id_k is the identity operator on V_k,

$$\|v\|_{a_k} = a_k(v,v)^{1/2} \qquad \forall v \in H_0^2(\Omega) + V_k, \tag{2.4}$$

and the bilinear form $a_k(\cdot,\cdot)$ on $H_0^2(\Omega) + V_k$ is defined by

$$a_k(u,v) := \sum_{T\in\mathcal{T}^k} \int_T D^2u : D^2v\,dx.$$

Note that the constructions of E_k and F_k and the properties (2.2) and (2.3) rely on the fact that the Morley element and the Hsieh-Clough-Tocher element are relatives. These operators are important for the multigrid analysis (cf. Lemmas 4.2 and 4.3).

We now define the Morley element method and the modified Morley element method for (1.1).

If $\phi(v) = \int_\Omega fv\,dx$ for a function $f \in L_2(\Omega)$, then the Morley finite element method for (1.1) is: Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega fv\,dx \quad \forall v \in V_k. \tag{2.5}$$

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In the case where $\phi \in H^{-2}(\Omega)$, the modified Morley finite element method for (1.1) is: Find $u'_k \in V_k$ such that

$$a_k(u'_k, v) = \phi(E_k v) \quad \forall v \in V_k. \tag{2.6}$$

The properties of these methods are discussed in [10]. The results of this paper can be applied to both of these methods.

3. V-cycle and F-cycle Multigrid methods. In this section we describe the V-cycle and F-cycle multigrid methods for the Morley finite element.

Let the discrete inner product $(\cdot,\cdot)_k$ on V_k be defined by

$$(v_1, v_2)_k := h_k^2 \left[\sum_{p \in \mathcal{V}_k} n(p) v_1(p) v_2(p) + \sum_{e \in \mathcal{E}_k} \left(\int_e \frac{\partial v_1}{\partial n}\, ds \right) \left(\int_e \frac{\partial v_2}{\partial n}\, ds \right) \right], \tag{3.1}$$

where $\mathcal{V}_k$ is the set of internal vertices of $\mathcal{T}_k$, $\mathcal{E}_k$ is the set of internal edges of $\mathcal{T}_k$ and $n(p) = \frac{1}{6}\times$ (the number of triangles sharing the node p as a vertex). We can represent the bilinear form $a_k(\cdot,\cdot)$ by the operator $A_k : V_k \longrightarrow V_k$ defined by

$$(A_k v_1, v_2)_k = a_k(v_1, v_2) \quad \forall v_1, v_2 \in V_k. \tag{3.2}$$

Then equations (2.5) and (2.6) can both be rewritten as

$$A_k u_k = f_k, \tag{3.3}$$

where $f_k \in V_k$ is defined by $(f_k, v)_k = \int_\Omega f v\, dx$ for all $v \in V_k$ for the standard Morley method, and $(f_k, v)_k = \phi(E_k v)$ for all $v \in V_k$ for the modified Morley method.

In order to describe multigrid methods, we need to define the intergrid transfer operators. We first define $I_{k-1}^k : V_{k-1} \longrightarrow V_k$, the coarse-to-fine intergrid transfer operator. The properties of this operator can be found in [9].

Let $v \in V_{k-1}$. We define $I_{k-1}^k v \in V_k$ by an averaging technique as follows:

1. If p is an internal vertex of $\mathcal{T}_k$, then

$$(I_{k-1}^k v)(p) = \frac{1}{|S_{p,k-1}|} \sum_{T \in S_{p,k-1}} v_T(p), \tag{3.4}$$

where $S_{p,k-1} := \{T \in \mathcal{T}_{k-1} : p \in \partial T\}$.

2. If e is an internal edge of $\mathcal{T}_k$, which means that $e \subset \partial T_1 \cap \partial T_2$ for some $T_1, T_2 \in \mathcal{T}_k$, then

$$\int_e \frac{\partial (I_{k-1}^k v)}{\partial n}\, ds = \frac{1}{2} \left(\int_e \frac{\partial v_{T_1}}{\partial n}\, ds + \int_e \frac{\partial v_{T_2}}{\partial n}\, ds \right). \tag{3.5}$$

We can then define the fine-to-coarse operator $I_k^{k-1} : V_k \longrightarrow V_{k-1}$ and the nonconforming Ritz "projection" operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ as follows:

$$\begin{aligned} (I_{k-1}^k v, w)_k &= (v, I_k^{k-1} w)_{k-1} \qquad \forall v \in V_{k-1}, w \in V_k, \\ a_k(I_{k-1}^k v, w) &= a_{k-1}(v, P_k^{k-1} w) \quad \forall v \in V_{k-1}, w \in V_k. \end{aligned}$$

Symmetric V-cycle Multigrid Method (cf. [5], [8], [14], [22], [19], [30] and [32]). The symmetric V-cycle multigrid algorithm is an iterative solver for equations of the form (3.3).

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Given $g \in V_k$ and an initial guess $z_0 \in V_k$, the output $MG_{\mathcal{V}}(k, g, z_0, m)$ of the algorithm is an approximate solution for the equation

$$A_k z = g, \tag{3.6}$$

where m is the number of pre-smoothing and post-smoothing steps.

For $k = 1$, we define

$$MG_{\mathcal{V}}(1, g, z_0, m) = A_1^{-1} g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{V}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}),$$

   where Λ_k is a constant dominating the spectral radius of A_k.
2. (Coarse Grid Correction) Compute z_{m+1} by

$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m).$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{V}}(k, g, z_0, m)$ to be z_{2m+1}.

In this algorithm we use Richardson relaxation as the smoother for simplicity. Other smoothers can also be used (cf. [2], [6] and [12]).

F-cycle Multigrid Method (cf. [23], [32], and [30]). The k-th level F-cycle algorithm (associated with the symmetric V-cycle algorithm) produces an approximate solution $MG_{\mathcal{F}}(k, g, z_0, m)$ for (3.6). For $k = 1$, we define

$$MG_{\mathcal{F}}(1, g, z_0, m) = A_1^{-1} g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{F}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

2. (Coarse Grid Correction) Compute $z_{m+\frac{1}{2}}$ and z_{m+1} by

$$\begin{aligned} z_{m+\frac{1}{2}} &= MG_{\mathcal{F}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m), \\ z_{m+1} &= z_m + I_k^{k-1} MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), z_{m+\frac{1}{2}}, m). \end{aligned}$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{F}}(k, g, z_0, m)$ to be z_{2m+1}.

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4. Mesh Dependent Norms. One of the main tools for the convergence analysis of multigrid methods is the mesh-dependent norm $|\!|\!| \cdot |\!|\!|_{s,k}$ (cf. [3]). For each $v \in V_k$ we define

$$\|\!|v|\!|\!|_{s,k} = \sqrt{(A_k^{s/2} v, v)_k}. \tag{4.1}$$

It is easy to see that

$$|\!|\!|v|\!|\!|_{0,k}^2 = (v,v)_k \qquad \forall v \in V_k, \tag{4.2}$$

$$|\!|\!|v|\!|\!|_{2,k} = \|v\|_{a_k} \lesssim h_k^{-2}\|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \tag{4.3}$$

To avoid the proliferation of constants we use two notations $\lesssim$ and $\approx$. The statement $A \lesssim B$ means that A is bounded by B multiplied by a constant which is independent of mesh sizes, mesh levels and all arguments in A and B, and $A \approx B$ means $A \lesssim B$ as well as $B \lesssim A$.

We can also easily see that (cf. [3] and [10])

$$|\!|\!|v|\!|\!|_{s,k} \lesssim h_k^{t-s}|\!|\!|v|\!|\!|_{t,k} \quad \text{where } 0 \le t \le s \le 4. \tag{4.4}$$

The following smoothing properties of R_k can also be easily verified (cf. [19] and [14]) :

$$|\!|\!|R_k v|\!|\!|_{s,k} \le |\!|\!|v|\!|\!|_{s,k} \qquad \forall v \in V_k,\, s \in \mathbb{R}, \tag{4.5}$$

$$|\!|\!|R_k^m v|\!|\!|_{s,k} \lesssim h_k^{(t-s)} m^{(t-s)/4}|\!|\!|v|\!|\!|_{t,k} \quad \forall v \in V_k,\ 0 < t < s < 4, \tag{4.6}$$

where $R_k = Id_k - \Lambda_k^{-1}A_k$.

The following lemmas relate the mesh-dependent norms with the Sobolev norms.

LEMMA 4.1. *The following relation holds:*

$$|\!|\!|v|\!|\!|_{0,k} \approx \|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \tag{4.7}$$

Proof. Let $\hat{T}$ be a triangle with $|\hat{T}| \approx 1$. Then, for all quadratic polynomials v on $\hat{T}$, we have

$$\|v\|_{L_2(\hat{T})}^2 \approx \sum_{i=1}^{3} v(p_i)^2 + \sum_{i=1}^{3}\left(\int_{e_i}\frac{\partial v}{\partial n}\,ds\right)^2. \tag{4.8}$$

Using a scaling argument on (4.8), and by definition (3.1) of the inner product $(\cdot,\cdot)_k$, we have

$$(v,v)_k \approx \|v\|_{L_2(\Omega)}^2 \quad \forall v \in V_k. \tag{4.9}$$

The lemma follows from (4.2) and (4.9). ☐

LEMMA 4.2. *Let $E_k : V_k \longrightarrow \tilde{V}_k$ be the operator defined in Section 2. Then, the following relation holds:*

$$|E_k v|_{H^1(\Omega)} \approx |\!|\!|v|\!|\!|_{1,k} \quad \forall v \in V_k. \tag{4.10}$$

Proof. It is known from [10] that the operator E_k is a bounded operator from $(V_k, \|\cdot\|_{L_2(\Omega)})$ to $(L_2(\Omega), \|\cdot\|_{L_2(\Omega)})$, and from $(V_k, \|\cdot\|_{a_k})$ to $(H_0^2(\Omega), |\cdot|_{H^2(\Omega)})$, i.e.,

$$\|E_k v\|_{L_2(\Omega)} \lesssim \|v\|_{L_2(\Omega)} \quad \forall v \in V_k, \tag{4.11}$$

or equivalently

$$\|E_k v\|_{L_2(\Omega)} \lesssim \|\!|v|\!\|_{0,k} \quad \forall v \in V_k \tag{4.12}$$

and

$$|E_k v|_{H^2(\Omega)} \lesssim \|v\|_{a_k} \quad \forall v \in V_k, \tag{4.13}$$

or equivalently

$$|E_k v|_{H^2(\Omega)} \lesssim \|\!|v|\!\|_{2,k} \quad \forall v \in V_k. \tag{4.14}$$

By interpolations of Sobolev spaces and Hilbert scales(cf. [5] and [29]), we have

$$|E_k v|_{H^1(\Omega)} \lesssim \|\!|v|\!\|_{1,k} \quad \forall v \in V_k. \tag{4.15}$$

Conversely, let $Q_k : L_2(\Omega) \longrightarrow \tilde{V}_k$ be the L_2 projection operator on $\tilde{V}_k$, i.e., for each $\zeta \in L_2(\Omega)$, the function $Q_k\zeta \in \tilde{V}_k$ satisfies

$$(Q_k\zeta, \tilde{v})_{L_2(\Omega)} = (\zeta, \tilde{v})_{L_2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k.$$

It is known that (cf. [7])

$$\|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in L_2(\Omega), \tag{4.16}$$

$$|Q_k\zeta|_{H^2(\Omega)} \lesssim |\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H_0^2(\Omega). \tag{4.17}$$

We define $J_k : L_2(\Omega) \longrightarrow V_k$ by $J_k = F_k Q_k$. Then from (2.3), (4.16) and (4.17) we have

$$\begin{aligned} \|\!|J_k\zeta|\!\|_{0,k} &\approx \|F_k Q_k\zeta\|_{L_2(\Omega)} \lesssim \|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} && \forall \zeta \in L_2(\Omega), \\ \|\!|J_k\zeta|\!\|_{2,k} &= \|F_k Q_k\zeta\|_{a_k} \lesssim |Q_k\zeta|_{H^2(\Omega)} \lesssim |\zeta|_{H^2(\Omega)} && \forall \zeta \in H_0^2(\Omega). \end{aligned}$$

By interpolations of Sobolev spaces and Hilbert scales, we have

$$\|\!|J_k\zeta|\!\|_{1,k} \lesssim |\zeta|_{H^1(\Omega)} \quad \forall \zeta \in H_0^1(\Omega). \tag{4.18}$$

For each $v \in V_k$, we have $E_k v \in \tilde{v}_k$. Then, by (2.2) and the definition of Q_k, we have

$$J_k E_k v = F_k Q_k E_k v = F_k E_k v = v. \tag{4.19}$$

From equations (4.18) and (4.19) we have

$$\|\!|v|\!\|_{1,k} = \|\!|J_k E_k v|\!\|_{1,k} \lesssim |E_k v|_{H^2(\Omega)} \quad \forall v \in V_k.$$

□

LEMMA 4.3. *The following relation holds:*

$$\sum_{T \in \mathcal{T}_k} |v|^2_{H^1(T)} \approx \|\!|v|\!\|^2_{1,k} \quad \forall v \in V_k. \tag{4.20}$$

Proof. It is known that (cf. (E) of [10])

$$\|v - E_k v\|_{L_2(\Omega)} \lesssim h_k^2 \|v\|_{a_k} \quad \forall v \in V_k. \tag{4.21}$$

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From (2.4), (4.4), Lemma 4.2, (4.21) and a standard inverse estimate (cf. [14], [15]) we have

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_k} |v|^2_{H^1(T)} &\leq \sum_{T\in\mathcal{T}_k} \left(|v-E_kv|_{H^1(T)} + |E_kv|_{H^1(T)}\right)^2 \\
&\lesssim \sum_{T\in\mathcal{T}_k} |v-E_kv|^2_{H^1(T)} + \sum_{T\in\mathcal{T}_k} |E_kv|^2_{H^1(T)} \\
&\lesssim h_k^{-2}\|v-E_kv\|^2_{L_2(\Omega)} + |E_kv|^2_{H^1(\Omega)} \\
&\lesssim \sum_{T\in\mathcal{T}_k} h_k^2|v|^2_{H^2(T)} + |E_kv|^2_{H^1(\Omega)} \\
&\lesssim h_k^2 |\!|\!| v |\!|\!|^2_{2,k} + |E_kv|^2_{H^1(\Omega)} \lesssim |\!|\!| v |\!|\!|^2_{1,k}
\end{aligned}
$$

for all $v \in V_k$.

Conversely from Lemma 4.2, (4.21) and a standard inverse estimate we have

$$
\begin{aligned}
|\!|\!| v |\!|\!|^2_{1,k} \lesssim |E_kv|^2_{H^1(\Omega)} &= \sum_{T\in\mathcal{T}_k} |E_kv|^2_{H^1(T)} \\
&\leq \sum_{T\in\mathcal{T}_k} \left(|v-E_kv|_{H^1(T)} + |v|_{H^1(T)}\right)^2 \\
&\lesssim \sum_{T\in\mathcal{T}_k} |v-E_kv|^2_{H^1(T)} + \sum_{T\in\mathcal{T}_k} |v|^2_{H^1(T)} \\
&\lesssim h_k^{-2}\|v-E_kv\|^2_{L_2(\Omega)} + \sum_{T\in\mathcal{T}_k} |v|^2_{H^1(T)} \\
&\lesssim h_k^2\|v\|^2_{a_k} + \sum_{T\in\mathcal{T}_k} |v|^2_{H^1(T)} \\
&\lesssim \sum_{T\in\mathcal{T}_k} |v|^2_{H^1(T)}
\end{aligned}
$$

for all $v \in V_k$. □

5. Some known results for the additive theory. Let $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$ be the error propagation operator of the symmetric V-cycle algorithm applied to the equation (3.6), i.e.,

$$
\mathbb{E}_{k,m}(z-z_0) = z - MG_{\mathcal{V}}(k,g,z_0,m),
$$

where z is the exact solution of (3.6). The following relations (cf. [5] and [19]) are well-known:

$$
\mathbb{E}_{k,m} = R_k^m[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m} P_k^{k-1}]R_k^m \quad \text{for } k \geq 2, \tag{5.1}
$$

$$
\mathbb{E}_{1,m} = 0. \tag{5.2}
$$

From (5.1) and (5.2) the following additive expression for $\mathbb{E}_{k,m}$ can be derived that is the starting point of the additive theory (cf. [13]):

$$
\begin{aligned}
\mathbb{E}_{k,m} = \sum_{j=2}^{k} R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m \\
\times P_{j+1}^j R_{j+1}^m \cdots P_k^{k-1} R_k^m.
\end{aligned} \tag{5.3}
$$

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Let $\tilde{\mathbb{E}}_{k,m} : \tilde{V}_k \longrightarrow \tilde{V}_k$ be the error propagation operator of the symmetric F-cycle algorithm applied to the equation (3.6), i.e.,

$$\tilde{\mathbb{E}}_{k,m}(z - z_0) = z - MG_{\mathcal{F}}(k, g, z_0, m),$$

where z is the exact solution of (3.6). The following relations are also well-known (cf. [30]):

$$\tilde{\mathbb{E}}_{1,m} = 0, \tag{5.4}$$

$$\tilde{\mathbb{E}}_{k,m} = R_k^m[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m}\tilde{\mathbb{E}}_{k-1,m} P_k^{k-1}]R_k^m, \ k \geq 2. \tag{5.5}$$

An additive theory for the convergence analysis of V-cycle and F-cycle multigrid algorithms is developed in [13] based on the expressions (5.3)–(5.5). It is shown there that, to complete the convergence analysis, we only need to verify the following assumptions.

Assumptions on V_k:

$$\begin{aligned} (v, v)_k &\approx \|v\|_{L_2(\Omega)}^2 \qquad \forall v \in V_k, \\ \|v\|_{a_k} &\lesssim h_k^{-2}\|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \end{aligned}$$

Assumptions on I_{k-1}^k *and* P_k^{k-1}:

$$\begin{aligned} |||I_{k-1}^k v|||_{2,k}^2 &\leq (1+\theta^2)|||v|||_{2,k-1}^2 + C_1\theta^{-2}h_k^{2\alpha}|||v|||_{2+\alpha,k-1}^2 \\ &\qquad \forall v \in V_{k-1}, \theta \in (0,1), \\ |||I_{k-1}^k v|||_{2-\alpha,k}^2 &\leq (1+\theta^2)|||v|||_{2-\alpha,k-1}^2 + C_2\theta^{-2}h_k^{2\alpha}|||v|||_{2,k-1}^2 \\ &\qquad \forall v \in V_{k-1}, \theta \in (0,1), \\ |||P_k^{k-1} v|||_{2-\alpha,k-1}^2 &\leq (1+\theta^2)|||v|||_{2-\alpha,k}^2 + C_3\theta^{-2}h_k^{2\alpha}|||v|||_{2,k}^2 \\ &\qquad \forall v \in V_k, \theta \in (0,1). \end{aligned}$$

Assumptions on $I_{k-1}^k P_k^{k-1}$ *and* $P_k^{k-1} I_{k-1}^k$:

$$\begin{aligned} |||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{2-\alpha,k} &\lesssim h_k^{2\alpha}|||v|||_{2+\alpha,k} \quad \forall v \in V_k, \\ |||(Id_{k-1} - P_k^{k-1} I_{k-1}^k)v|||_{2-\alpha,k-1} &\lesssim h_k^{\alpha}|||v|||_{2,k-1} \quad \forall v \in V_k. \end{aligned}$$

It is also shown in [13] that these assumptions can be verified for a specific nonconforming multigrid method by the use of the following framework.

First, we should establish a relation between the nonconforming finite element space V_k and its conforming relative $\tilde{V}_k$. In addition to (2.2) and (2.3), the two spaces are also assumed to satisfy the following properties, which have been established in [10].

Let $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ be related by

$$\begin{aligned} a(\zeta, E_k v) &= a_k(\zeta_k, v) \qquad \forall v \in V_k, \\ a(\zeta, E_{k-1} v) &= a_{k-1}(\zeta_{k-1}, v) \quad \forall v \in V_{k-1}. \end{aligned}$$

Then the following estimates hold:

$$\|\zeta - \zeta_k\|_{a_k} \lesssim h_k^{\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{5.6}$$

$$|||\Pi_k\zeta - \zeta_k|||_{2-\alpha,k} \lesssim h_k^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{5.7}$$

$$|||\zeta_{k-1} - P_k^{k-1}\zeta_k|||_{2-\alpha,k-1} \lesssim h_k^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}, \tag{5.8}$$

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where α is the index of elliptic regularity in (1.2) and $\Pi_k : H_0^2(\Omega) \longrightarrow V_k$ is the Morley interpolation operator defined as follows. For each $\zeta \in H_0^2(\Omega)$, the function $\Pi_k\zeta \in V_k$ satisfies

$$(\Pi_k v)(p) = v(p) \quad \text{and} \quad \int_e \frac{\partial(\Pi_k v)}{\partial n}\,ds = \int_e \frac{\partial v}{\partial n}\,ds, \tag{5.9}$$

where p and e range over the internal vertices and edges of $\mathcal{T}_k$.

Secondly, we need the following estimates concerning I_{k-1}^k and Π_k, which have also been established in [10] :

$$\|I_{k-1}^k v\|_{s,k} \lesssim \|v\|_{s,k-1} \qquad \forall v \in V_{k-1},\ 0 \le s \le 2, \tag{5.10}$$

$$\|\zeta - \Pi_k\zeta\|_{L_2(\Omega)} \lesssim h_k^2|\zeta|_{H^2(\Omega)} \qquad \forall \zeta \in H_0^2(\Omega), \tag{5.11}$$

$$\|\zeta - \Pi_k\zeta\|_{a_k} \lesssim h_k^\alpha\|\zeta\|_{H^{2+\alpha}} \qquad \forall \zeta \in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega), \tag{5.12}$$

$$\|\Pi_k\zeta - I_{k-1}^k\Pi_{k-1}\zeta\|_{2-\alpha,k} \lesssim h_k^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega). \tag{5.13}$$

Finally, the following new estimates are required for relating mesh-dependent norms between two consecutive levels. First of all, we have

$$\|I_{k-1}^k v\|_{0,k}^2 \le (1+\theta^2)\|v\|_{0,k-1}^2 + C_0\theta^{-2}h_k^{2\alpha}\|v\|_{\alpha,k-1}^2 \qquad \forall v \in V_k,\ \theta \in (0,1), \tag{5.14}$$

where the positive constant C_0 is mesh-independent. Moreover, the operator Π_{k-1} can be extended to map $H_0^2(\Omega) + V_k$ to V_{k-1} (see the next section for details) and we have

$$\|\Pi_{k-1}v\|_{a_k} \lesssim \|v\|_{a_k} \qquad \forall v \in H_0^2(\Omega) + V_k, \tag{5.15}$$

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^2\|v\|_{a_k} \qquad \forall v \in V_k, \tag{5.16}$$

$$\|\Pi_{k-1}v\|_{0,k-1}^2 \le (1+\theta^2)\|v\|_{0,k}^2 + C_0'\theta^{-2}h_k^{2\alpha}\|v\|_{\alpha,k}^2 \qquad \forall v \in V_k,\ \theta \in (0,1), \tag{5.17}$$

where the positive constant C_0' is mesh-independent.

The theory in [13] can be applied to V-cycle and F-cycle multigrid methods using the Morley element once we have (2.2), (2.3), (5.6)–(5.8) and (5.10)–(5.17).

We will establish the new estimates (5.14)–(5.17) in the next section and complete the convergence analysis.

6. Convergence Analysis. We first extend the definition of Π_k to a larger space. In fact, the definition (5.9) can be extended to $H_0^2(\Omega) + V_k + V_{k+1}$.

Let $v \in H_0^2(\Omega) + V_k + V_{k+1}$. First of all, the value $v(p)$ is well-defined for $p \in \mathcal{V}_k$. Secondly, the integral $\int_e \frac{\partial v}{\partial n}\,ds$ is also well-defined for $e \in \mathcal{E}_k$. In particular, if $v \in V_{k+1}$, then

$$\int_e \frac{\partial v}{\partial n}\,ds = \int_{e_1} \frac{\partial v}{\partial n}\,ds + \int_{e_2} \frac{\partial v}{\partial n}\,ds \tag{6.1}$$

where $e_1, e_2 \in \mathcal{E}_{k+1}$ with $e = e_1 \cup e_2$ (cf. Figure 6.1). Therefore the linear operator Π_k is well-defined from the larger space $H_0^2(\Omega) + V_k + V_{k+1}$ into V_k. In particular, $\Pi_k : H_0^2(\Omega) + V_k \longrightarrow V_k$ and $\Pi_{k-1} : H_0^2(\Omega) + V_k \longrightarrow V_{k-1}$ are both well-defined.

Before we prove the estimates (5.14)–(5.17), we give two lemmas.

LEMMA 6.1. *The following estimate holds:*

$$\|v - \Pi_k v\|_{L_2(\Omega)} + h_k^2\|\Pi_k v\|_{a_k} \lesssim h_k^2\|v\|_{a_k} \qquad \forall v \in H_0^2(\Omega) + V_k. \tag{6.2}$$

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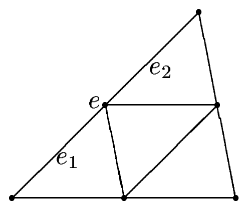

FIG. 6.1. *An edge* $e \in \mathcal{E}_{k-1}$.

Proof. Let $T \in \mathcal{T}_k$, $\zeta \in H^2(T)$ and the quadratic polynomial $\Pi_T\zeta$ on T be the Morley nodal interpolant of ζ, i.e.,

$$(\Pi_T\zeta)(p_i) = \zeta(p_i) \text{ and } \int_{e_i} \frac{\partial(\Pi_T\zeta)}{\partial n}\, ds = \int_{e_i} \frac{\partial\zeta}{\partial n}\, ds, \tag{6.3}$$

for $i = 1$, 2 and 3, where p_1, p_2 and p_3 are the vertices of T, and e_1, e_2 and e_3 are the edges of T. It is well-known that (cf. [14] and [15])

$$\|\zeta - \Pi_T\zeta\|_{L_2(T)} + h_k^2|\Pi_T\zeta|_{H^2(T)} \lesssim h_k^2|\zeta|_{H^2(T)}. \tag{6.4}$$

Let $v \in H_0^2(\Omega) + V_k$ and $T \in \mathcal{T}_k$. Then $v_T \in H^2(T)$ and $\Pi_k v = \Pi_T v_T$ on T. Therefore

$$\|v - \Pi_k v\|_{L_2(T)} + h_k^2|\Pi_k v|_{H^2(T)} \lesssim h_k^2|v|_{H^2(T)}. \tag{6.5}$$

The estimate (6.2) holds because (6.5) is valid for all $T \in \mathcal{T}_k$. □

LEMMA 6.2. *The following equality holds:*

$$\Pi_{k-1}\Pi_k v = \Pi_{k-1}v \quad \forall v \in H_0^2(\Omega) + V_k. \tag{6.6}$$

Proof. Let $v \in H_0^2(\Omega) + V_k$ be arbitrary. The functions $\Pi_{k-1}\Pi_k v$ and $\Pi_{k-1}v$ are both in V_{k-1}. Moreover, we have

$$(\Pi_{k-1}\Pi_k v)(p) = (\Pi_{k-1}v)(p)$$

for all $p \in \mathcal{V}_{k-1}$, and

$$\begin{aligned}
\int_e \frac{\partial(\Pi_{k-1}\Pi_k v)}{\partial n}\, ds &= \int_{e_1} \frac{\partial(\Pi_k v)}{\partial n}\, ds + \int_{e_2} \frac{\partial(\Pi_k v)}{\partial n}\, ds \\
&= \int_{e_1} \frac{\partial v}{\partial n}\, ds + \int_{e_2} \frac{\partial v}{\partial n}\, ds \\
&= \int_e \frac{\partial v}{\partial n}\, ds = \int_e \frac{\partial(\Pi_{k-1}v)}{\partial n}\, ds
\end{aligned}$$

for all $e \in \mathcal{E}_{k-1}$, where e_1, $e_2 \in \mathcal{E}_k$ with $e = e_1 \cup e_2$ (cf. Figure 6.1). Therefore $\Pi_{k-1}\Pi_k v = \Pi_{k-1}v$. □

LEMMA 6.3. *The estimate* (5.16) *holds. That is,*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^2\|v\|_{a_k} \quad \forall v \in V_k.$$

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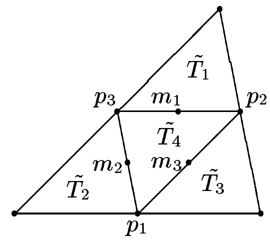

FIG. 6.2. *A reference triangle $\tilde{T}$ divided into 4 triangles $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ and $\tilde{T}_4$*

Proof. Let $T \in \mathcal{T}_{k-1}$ be divided into 4 triangles T_1, T_2, T_3 and T_4 in $\mathcal{T}_k$ and $\tilde{T} = T/h_{k-1}$. Then $|\tilde{T}| \approx 1$ (cf. Figure 6.2).

For each $v \in V_k$, define $\tilde{v}(\tilde{x}) = v(h_{k-1}\tilde{x})$ for $\tilde{x} \in \tilde{T}$. Note that $\tilde{x} \in \tilde{T}$ if and only if $h_{k-1}\tilde{x} \in T$. If $w = \Pi_{k-1}v$, then we define $\tilde{\Pi}_{k-1}\tilde{v}$ to be $\tilde{w}$.

Let $V(\tilde{T})$ be the Morley finite element space associated with $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ and $\tilde{T}_4$. Note that $V(\tilde{T})$ is the space of functions $\tilde{v} \in L_2(\tilde{T})$ such that $\tilde{v}|_{\tilde{T}_i}$ is a quadratic polynomial on $\tilde{T}_i$ for $i = 1,\ 2,\ 3$ and 4, $\tilde{v}$ is continuous at $p_1,\ p_2$ and p_3, and $\partial\tilde{v}/\partial n$ is continuous at $m_1,\ m_2$ and m_3. We can see that $V(\tilde{T})$ is a finite dimensional linear space and

$$\|\tilde{v}\|_* = \left[\sum_{i=1}^{4} |\tilde{v}|^2_{H^2(\tilde{T}_i)}\right]^{1/2}$$

defines a norm on the quotient space $V(\tilde{T})/P_1(\tilde{T})$, where $P_1(\tilde{T})$ is the space of linear functions on $\tilde{T}$. On the other hand,

$$v \longrightarrow \|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})}$$

defines a semi-norm on $V(\tilde{T})/P_1(\tilde{T})$. Therefore

$$\|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})} \lesssim \left[\sum_{i=1}^{4} |\tilde{v}|^2_{H^2(\tilde{T}_i)}\right]^{1/2}. \tag{6.7}$$

A scaling argument on (6.7) yields

$$\|\Pi_{k-1}v - v\|_{L_2(T)} \lesssim h_k^2 \left[\sum_{i=1}^{4} |v|^2_{H^2(T_i)}\right]^{1/2}. \tag{6.8}$$

The estimate (5.16) follows. ◻

LEMMA 6.4. *The estimate (5.15) holds. That is,*

$$\|\Pi_{k-1}v\|_{a_k} \lesssim \|v\|_{a_k} \quad \forall v \in H_0^2(\Omega) + V_k.$$

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Proof. From (6.2), (6.6), Lemma 6.3 and an inverse estimate we have that, for all $v \in H_0^2(\Omega) + V_k$,

$$\begin{aligned}
\|\Pi_{k-1} v\|_{a_k} &= \|\Pi_{k-1}\Pi_k v\|_{a_k} \\
&\leq \|\Pi_{k-1}\Pi_k v - \Pi_k v\|_{a_k} + \|\Pi_k v\|_{a_k} \\
&\lesssim h_k^{-2}\|\Pi_{k-1}\Pi_k v - \Pi_k v\|_{L_2(\Omega)} + \|\Pi_k v\|_{a_k} \\
&\lesssim \|\Pi_k v\|_{a_k} \lesssim \|v\|_{a_k}.
\end{aligned}$$

□

Before we prove the estimates (5.14) and (5.17), we state an elementary inequality:

$$(a+b)^2 \leq (1+\theta^2)a^2 + (1+\theta^{-2})b^2 \quad \forall a, b \in \mathbb{R},\ \theta \in (0,1). \tag{6.9}$$

In the rest of the section, we use C for a mesh-independent constant. The values of C at different appearances are not necessarily identical.

LEMMA 6.5. *The estimate (5.14) holds. That is,*

$$|\!|\!| I_{k-1}^k v |\!|\!|_{0,k}^2 \leq (1+\theta^2)|\!|\!| v |\!|\!|_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} |\!|\!| v |\!|\!|_{\alpha,k-1}^2 \tag{6.10}$$

for all $v \in V_{k-1}$ and $\theta \in (0,1)$.

Proof. Let $v \in V_{k-1}$ be arbitrary and $w = I_{k-1}^k v$. Then by (3.1) we have

$$|\!|\!| I_{k-1}^k v |\!|\!|_{0,k}^2 = (w,w)_k = h_k^2 \left[\sum_{p \in \mathcal{V}_k} n(p) w(p)^2 + \sum_{e \in \mathcal{E}_k} \left(\int_e \frac{\partial w}{\partial n}\, ds \right)^2 \right] \tag{6.11}$$

and

$$|\!|\!| v |\!|\!|_{0,k-1}^2 = h_k^2 \left[\sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + \sum_{e \in \mathcal{E}_{k-1}} \left(\int_e \frac{\partial v}{\partial n}\, ds \right)^2 \right], \tag{6.12}$$

where $n(p) = |S_p|/6$ and S_p is the set of triangles sharing p as a common vertex. Note that $n(p)$ is independent of k.

If $p \in \mathcal{V}_{k-1}$, then the value $v(p)$ is well-defined, i.e., $v_T(p) = v|_T(p) = v(p)$ for all $T \in \mathcal{T}_{k-1}$ sharing p as a common vertex. From (3.4) in the definition of I_{k-1}^k we have $w(p) = v(p)$. If $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$. Then p is the midpoint of some $e \in \mathcal{E}_{k-1}$, which is the common edge of two triangles $T, T' \in \mathcal{T}_{k-1}$ (cf. Figure 6.3). After subdivision, p is the common vertex of 6 triangles in $\mathcal{T}_k$ and therefore $n(p) = 1$. Hence we can write

$$\sum_{p \in \mathcal{V}_k} n(p) w(p)^2 = \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} w(p)^2. \tag{6.13}$$

Suppose p_1 and p_2 are the endpoints of e (cf. Figure 6.3). We have

$$w(p)^2 = \left[\frac{1}{2}(v_T(p) + v_{T'}(p)) \right]^2 \leq \frac{1}{2}(v_T(p))^2 + \frac{1}{2}(v_{T'}(p))^2. \tag{6.14}$$

Then from (6.9) we can write

$$\begin{aligned}
\frac{1}{2}(v_T(p))^2 &= \frac{1}{2}\left[v(p_1) + (v_T(p) - v(p_1)) \right]^2 \\
&\leq \frac{1}{2}(1+\theta^2) v(p_1)^2 + C\theta^{-2}[v_T(p) - v(p_1)]^2.
\end{aligned}$$

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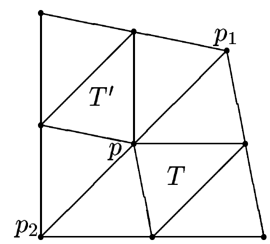

FIG. 6.3. *A vertex* $p \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$

Note that the Mean-Value Theorem and a standard inverse estimate imply that

$$[v_T(p) - v(p_1)]^2 \leq |p - p_1|^2 \|\nabla v\|^2_{L_\infty(T)} \leq C|v|^2_{H^1(T)}.$$

Therefore we have

$$\frac{1}{2}(v_T(p))^2 \leq \frac{1}{2}(1+\theta^2)v(p_1)^2 + C\theta^{-2}|v|^2_{H^1(T)}, \tag{6.15}$$

and similarly

$$\frac{1}{2}(v_{T'}(p))^2 \leq \frac{1}{2}(1+\theta^2)v(p_2)^2 + C\theta^{-2}|v|^2_{H^1(T')}. \tag{6.16}$$

Thus from (6.14), (6.15), and (6.16) we have

$$w(p)^2 \leq \frac{1}{2}(1+\theta^2)[v(p_1)^2 + v(p_2)^2] + C\theta^{-2}[|v|^2_{H^1(T)} + |v|^2_{H^1(T')}]. \tag{6.17}$$

Taking summation of (6.17) over $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ gives

$$\begin{aligned}\sum_{p\in\mathcal{V}_k\setminus\mathcal{V}_{k-1}} w(p)^2 &\leq \frac{1}{2}(1+\theta^2)\sum_{p\in\mathcal{V}_{k-1}} |S_p|v(p)^2 + C\theta^{-2}\sum_{T\in\mathcal{T}_{k-1}} |v|^2_{H^1(T)} \\ &= 3(1+\theta^2)\sum_{p\in\mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}\sum_{T\in\mathcal{T}_{k-1}} |v|^2_{H^1(T)}.\end{aligned}$$

Therefore it follows from (6.13) that

$$\sum_{p\in\mathcal{V}_k} n(p)w(p)^2 \leq 4(1+\theta^2)\sum_{p\in\mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}\sum_{T\in\mathcal{T}_{k-1}} |v|^2_{H^1(T)}. \tag{6.18}$$

By the definition of I^k_{k-1} (cf. (3.5)), we have

$$\sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial w}{\partial n}\,ds\right)^2 \leq C\sum_{T\in\mathcal{T}_k}\left(\int_{\partial T}\frac{\partial v_T}{\partial n}\,ds\right)^2. \tag{6.19}$$

From the Mean-Value Theorem and a standard inverse estimate we have

$$\left(\int_{\partial T}\frac{\partial v_T}{\partial n}\,ds\right)^2 \leq |\partial T|^2\|\nabla v\|^2_{L_\infty(T)} \leq C|v|^2_{H^1(T)} \tag{6.20}$$

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for all $T \in \mathcal{T}_k$. Therefore from (6.19) and (6.20) we have

$$\sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial w}{\partial n}\,ds\right)^2 \le C \sum_{T\in\mathcal{T}_{k-1}} |v|^2_{H^1(T)}. \tag{6.21}$$

By (2.1), (4.4), Lemma 4.3, (6.11), (6.12), (6.18) and (6.21), we have

$$\begin{aligned}
&\|I^k_{k-1}v\|^2_{0,k}\\
&\le h_k^2\left[4(1+\theta^2)\sum_{p\in\mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}\sum_{T\in\mathcal{T}_{k-1}}|v|^2_{H^1(T)} + C\sum_{T\in\mathcal{T}_{k-1}}|v|^2_{H^1(T)}\right]\\
&\le (1+\theta^2)h_{k-1}^2\sum_{p\in\mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}h_k^2\sum_{T\in\mathcal{T}_{k-1}}|v|^2_{H^1(T)}\\
&\le (1+\theta^2)\|v\|^2_{0,k-1} + C\theta^{-2}h_k^2\|v\|^2_{1,k-1}\\
&\le (1+\theta^2)\|v\|^2_{0,k-1} + C\theta^{-2}h_k^{2\alpha}\|v\|^2_{\alpha,k-1}.
\end{aligned}$$

□

LEMMA 6.6. *The estimate (5.17) holds. That is,*

$$\|\Pi_{k-1}v\|^2_{0,k-1} \le (1+\theta^2)\|v\|^2_{0,k} + C\theta^{-2}h_k^{2\alpha}\|v\|^2_{\alpha,k} \tag{6.22}$$

for all $v \in V_k$ and $\theta \in (0,1)$.

Proof. Let $v \in V_k$ be arbitrary. It is easy to see from (3.1) that $\|v\|^2_{0,k}$ can be expressed as follows:

$$\|v\|^2_{0,k} = h_k^2\left[\frac{1}{6}\sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{V}_T} v(p)^2 + \sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial v}{\partial n}\,ds\right)^2\right], \tag{6.23}$$

where $\mathcal{V}_T$ is the set of the vertices of the triangle T.

Let $w = \Pi_{k-1}v$. Then

$$\|w\|^2_{0,k-1} = h_{k-1}^2\left[\frac{1}{6}\sum_{T\in\mathcal{T}_{k-1}}\sum_{p\in\mathcal{V}_T} w(p)^2 + \sum_{e\in\mathcal{E}_{k-1}}\left(\int_e \frac{\partial w}{\partial n}\,ds\right)^2\right]. \tag{6.24}$$

By the definition (5.9) of Π_{k-1} and (6.1), we have $w(p) = v(p)$ for all $p \in \mathcal{V}_{k-1}$ and

$$\begin{aligned}
\left(\int_e \frac{\partial w}{\partial n}\,ds\right)^2 &= \left(\int_{e_1}\frac{\partial v}{\partial n}\,ds + \int_{e_2}\frac{\partial v}{\partial n}\,ds\right)^2\\
&\le 2\left(\int_{e_1}\frac{\partial v}{\partial n}\,ds\right)^2 + 2\left(\int_{e_2}\frac{\partial v}{\partial n}\,ds\right)^2
\end{aligned}$$

for all $e \in \mathcal{E}_{k-1}$, where $e_1,\ e_2 \in \mathcal{E}_k$ with $e = e_1 \cup e_2$ (cf. Figure 6.1). Therefore

$$\|\Pi_{k-1}v\|^2_{0,k-1} \le \frac{h_{k-1}^2}{6}\sum_{T\in\mathcal{T}_{k-1}}\sum_{p\in\mathcal{V}_T} v(p)^2 + Ch_k^2\sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial v}{\partial n}\,ds\right)^2. \tag{6.25}$$

Let $T \in \mathcal{T}_{k-1}$ be divided into four triangles T_1, T_2, T_3 and T_4 in $\mathcal{T}_k$, whose vertices are labeled as in Figure 6.4. Then we have

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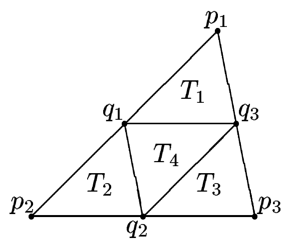

FIG. 6.4. *A Triangle $T \in \mathcal{T}_{k-1}$ divided into four triangles in $\mathcal{T}_k$*

.

$$
\begin{aligned}
4\sum_{p\in\mathcal{V}_T} v(p)^2 &= \sum_{i=1}^{3} v(p_i)^2 + 3\sum_{i=1}^{3}[v(q_i) + (v(p_i) - v(q_i))]^2 \\
&\le \sum_{i=1}^{3} v(p_i)^2 + 3\sum_{i=1}^{3}[(1+\theta^2)v(q_i)^2 + (1+\theta^{-2})(v(p_i) - v(q_i))^2] \\
&\le (1+\theta^2)\sum_{i=1}^{4}\sum_{p\in\mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2}\sum_{i=1}^{4}|v|^2_{H^1(T_i)}.
\end{aligned} \tag{6.26}
$$

From (2.1) and (6.26) we have

$$
h_{k-1}^2 \sum_{p\in\mathcal{V}_T} v(p)^2 \le (1+\theta^2)h_k^2 \sum_{i=1}^{4}\sum_{p\in\mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2}h_k^2\sum_{i=1}^{4}|v|^2_{H^1(T_i)}. \tag{6.27}
$$

Summing up over all $T \in \mathcal{T}_{k-1}$ gives

$$
\begin{aligned}
&h_{k-1}^2 \sum_{T\in\mathcal{T}_{k-1}}\sum_{p\in\mathcal{V}_T} v(p)^2 \\
&\le h_k^2(1+\theta^2)\sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2\sum_{T\in\mathcal{T}_k}|v|^2_{H^1(T)}.
\end{aligned} \tag{6.28}
$$

Using a similar argument as in (6.19)–(6.21) we have

$$
\sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial v}{\partial n}\,ds\right)^2 \le C\sum_{T\in\mathcal{T}_k}|v|^2_{H^1(T)}. \tag{6.29}
$$

Therefore from (4.4), Lemma 4.3, (6.23), (6.25), (6.28) and (6.29) we have

$$
\begin{aligned}
&|||\Pi_{k-1}v|||^2_{0,k-1} \le \frac{h_{k-1}^2}{6}\sum_{T\in\mathcal{T}_{k-1}}\sum_{p\in\mathcal{V}_T} v(p)^2 + Ch_k^2\sum_{e\in\mathcal{E}_k}\left(\int_e \frac{\partial v}{\partial n}\,ds\right)^2 \\
&\le \frac{h_k^2}{6}(1+\theta^2)\sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2\sum_{T\in\mathcal{T}_k}|v|^2_{H^1(T)} \\
&\le (1+\theta^2)|||v|||^2_{0,k} + C\theta^{-2}h_k^2|||v|||^2_{1,k} \\
&\le (1+\theta^2)|||v|||^2_{0,k} + C\theta^{-2}h_k^{2\alpha}|||v|||^2_{\alpha,k}.
\end{aligned}
$$

□

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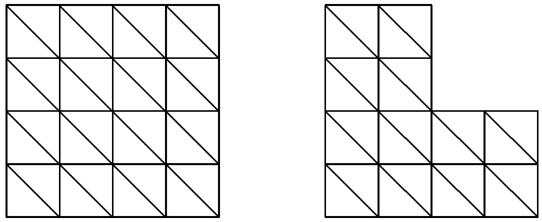

(a) Square domain (b) L-shaped domain

FIG. 7.1. *The triangulation $\mathcal{T}_k$ for $k = 2$.*

We have proved the required estimates (5.14)−(5.17) in Lemmas 6.3–6.6. The following theorems are then established by the additive theory (cf. [13]).

THEOREM 6.7. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{V}}(k, g, z_0, m)\|_{a_k} \leq Cm^{-\alpha/2}\|z - z_0\|_{a_k}, \tag{6.30}$$

where z is the exact solution of (3.6).

THEOREM 6.8. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{F}}(k, g, z_0, m)\|_{a_k} \leq Cm^{-\alpha/2}\|z - z_0\|_{a_k}, \tag{6.31}$$

where z is the exact solution of (3.6).

7. Numerical Experiments. In this section we present some experimental results to illustrate Theorem 6.7 and Theorem 6.8.

First, let Ω be the unit square $(0, 1) \times (0, 1)$ (cf. Figure 7.1(a)). Since the domain Ω is convex, we have full elliptic regularity, i.e., the index α in (1.2) is 1. Let $\gamma_{k,m}$ be the contraction number of the kth level V-cycle iteration with m pre-smoothing and m post-smoothing steps. According to Theorem 6.7, there is a constant C, independent of k and m, such that

$$m^{1/2}\gamma_{k,m} \leq C. \tag{7.1}$$

The numerical results in Table 7.1 are consistent with (7.1). In fact, they seem to indicate that C could be some number less than 10, and that (7.1) is valid for $m \geq 50$.

$m^{1/2}\gamma_{m,k}$	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.1347	1.0005	0.9075	0.8251	0.7454	0.6683	0.5948
k=4	1.5547	2.2969	2.1645	2.0767	2.0067	1.9455	1.8904
k=5	3.9972	3.5275	3.3043	3.1722	3.0803	3.0009	2.9491
k=6	5.3075	4.5923	4.2589	4.0665	3.9393	3.8459	3.7726
k=7	6.4352	5.4653	5.0207	4.7645	4.5984	4.4800	4.3898
k=8	7.3727	6.1637	5.6143	5.2982	5.0953	4.9528	4.8459

TABLE 7.1
V-cycle results on the unit square

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REMARK 7.1. Note that the condition number of the operator A_k (cf. (3.2) and (4.3)) is of order h_k^{-4} while the condition number for second order problems is of order h_k^{-2}. Therefore the effect of m smoothing steps for fourth order problems is equivalent to the effect of $\sqrt{m}$ smoothing steps for second order problems.

The key to the improvement of the performance of multigrid methods for fourth order problems is in the design of new smoothing operators. Besides [2], [6] and [12], there are also some new composite relaxation schemes that may also apply (cf. [21]).

Let $\tilde{\gamma}_{k,m}$ be the contraction number of the kth level F-cycle iteration with m pre-smoothing and m post-smoothing steps. According to Theorem 6.8, there is a constant C, independent of k and m, such that

$$m^{1/2}\tilde{\gamma}_{k,m} \leq C. \tag{7.2}$$

The numerical results in Table 7.2 are consistent with (7.2) and seem to indicate that $C = 2$ and (7.2) is valid as long as $m \geq 15$.

$m^{1/2}\tilde{\gamma}_{m,k}$	m=10	m=11	m=12	m=13	m=14	m=15	m=16
k=3	1.2132	1.1890	1.1706	1.1524	1.1364	1.1231	1.1071
k=4	1.4359	1.4037	1.3764	1.4097	1.3907	1.3838	1.4097
k=5	1.4310	1.3909	1.4163	1.4140	1.4030	1.4000	1.3933
k=6	1.4057	1.4041	1.3989	1.4017	1.3924	1.3908	1.3905
k=7	1.7918	1.3958	1.4035	1.3841	1.3756	1.3949	1.3759
k=8	5.4706	3.5578	2.4361	1.7541	1.3662	1.3775	1.3700

TABLE 7.2
F-cycle results on the unit square

In the case of the L-shaped domain (cf. Figure 7.1(b)), the index of elliptic regularity is $\alpha_* = 0.5444837368$. Numerical results for V-cycle and F-cycle algorithms are reported in Table 3 and Table 4, which are also consistent with (7.1) and (7.2).

$m^{\alpha_*/2}\gamma_{m,k}$	m=30	m=40	m=50	m=60	m=70	m=80	m=90
k=3	0.2237	0.1404	0.0879	0.0546	0.0337	0.0208	0.0127
k=4	0.9099	0.7905	0.7137	0.6597	0.6173	0.6833	0.5561
k=5	1.5698	1.3924	1.2888	1.2124	1.1605	1.1192	1.0877
k=6	2.1111	1.8776	1.7316	1.6282	1.5592	1.5073	1.4635
k=7	2.5752	2.2753	2.0991	1.9814	1.8938	1.8276	1.7715
k=8	2.9648	2.6232	2.4243	2.2913	2.1894	2.1138	2.0517

TABLE 7.3
V-cycle results on an L-shaped domain

REMARK 7.2. Even though the asymptotic convergence rate for both algorithms is $O(m^{-\alpha/2})$, the performance of the F-cycle algorithm is clearly superior, as demonstrated by the numerical results in Table 7.5 and Table 7.6. Similar results also hold for the L-shaped domain.

Compared with the W-cycle algorithm, the contraction numbers for F-cycle are larger for small numbers of smoothing steps. In Table 7.7, the contraction numbers of W-cycle

$m^{\alpha_*/2}\tilde{\gamma}_{m,k}$	m=11	m=12	m=13	m=14	m=15	m=16	m=17
k=3	0.5550	0.5266	0.5006	0.4762	0.4532	0.4313	0.4112
k=4	0.8529	0.8459	0.8126	0.8080	0.7965	0.7858	0.7743
k=5	0.8274	0.8092	0.7942	0.7701	0.7646	0.7303	0.7341
k=6	0.8134	0.7958	0.7830	0.7627	0.7515	0.7391	0.7192
k=7	0.8205	0.8038	0.7894	0.7759	0.7601	0.7381	0.7246
k=8	2.0406	1.4087	1.0198	0.7749	0.7449	0.7264	0.7140

TABLE 7.4
F-cycle results on an L-shaped domain

algorithm are given for $3 \leq m \leq 8$. In general F-cycle algorithm diverges for these m's. However, for larger m (say, for $m \geq 13$), the contraction numbers for both algorithms are almost the same, and sometimes the F-cycle algorithm is even better (cf. Table 7.6 and Table 7.8). Considering the fact that the cost for W-cycle is higher, we could say that the F-cycle algorithm is even more efficient then the W-cycle algorithm for m between 11 and 16.

It would be interesting to find a theoretical explanation for the superior performance of the F-cycle algorithm (see also [31]).

$\gamma_{m,k}$	m=34	m=35	m=36	m=37	m=38	m=39	m=40	m=41
k=3	0.1648	0.1609	0.1571	0.1535	0.1500	0.1467	0.1435	0.1404
k=4	0.3834	0.3756	0.3683	0.3613	0.3546	0.3483	0.3422	0.3365
k=5	0.5869	0.5746	0.5630	0.5521	0.5417	0.5318	0.5225	0.5135
k=6	0.7605	0.7438	0.7281	0.7133	0.6993	0.6860	0.6734	0.6614
k=7	0.9014	0.8807	0.8613	0.8430	0.8257	0.8093	0.7935	0.7791
k=8	1.0128	0.9887	0.9667	0.9448	0.9247	0.9057	0.8877	0.8707

TABLE 7.5
Contraction numbers for V-cycle algorithms on the unit square

$\tilde{\gamma}_{m,k}$	m=11	m=12	m=13	m=14	m=15	m=16
k=3	0.3580	0.3379	0.3196	0.3037	0.2900	0.2768
k=4	0.4232	0.3973	0.3910	0.3717	0.3573	0.3524
k=5	0.4194	0.4089	0.3922	0.3750	0.3615	0.3483
k=6	0.4234	0.4038	0.3888	0.3721	0.3591	0.3476
k=7	0.4208	0.4051	0.3839	0.3677	0.3602	0.3440
k=8	1.0727	0.7032	0.4865	0.3651	0.3557	0.3425

TABLE 7.6
Contraction numbers for F-cycle algorithms on the unit square

Acknowledgment. The author would like to thank his advisor, Professor Susanne C. Brenner, for her advice and encouragement.

$\gamma_{m,k,w}$	m=3	m=4	m=5	m=6	k=7	m=8
k=3	0.7260	0.6620	0.5922	0.5313	0.4883	0.4245
k=4	0.7628	0.7005	0.6294	0.5787	0.5323	0.4898
k=5	0.8499	0.7477	0.6505	0.5743	0.5348	0.4988
k=6	0.8926	0.7673	0.6660	0.5850	0.5445	0.4990
k=7	0.9349	0.8051	0.6544	0.5874	0.5384	0.5003
k=8	0.9334	0.8214	0.6747	0.5856	0.5362	0.5009

TABLE 7.7
Contraction numbers for W-cycle algorithms on the unit square

$\gamma_{m,k,w}$	m=11	m=12	m=13	m=14	m=15	m=16
k=4	0.4250	0.4113	0.3943	0.3790	0.3660	0.3473
k=5	0.4288	0.4078	0.3958	0.3774	0.3670	0.3558
k=6	0.4296	0.4137	0.3957	0.3803	0.3642	0.3553
k=7	0.4285	0.4117	0.3954	0.3817	0.3667	0.3546

TABLE 7.8
Contraction numbers for W-cycle algorithms for large m's.

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