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ON THE NUMERICAL SOLUTION OF SOME SEMILINEAR ELLIPTIC PROBLEMS*

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Abstract. We discuss a general framework for the numerical solution of a family of semilinear elliptic problems whose leading differential operator is the Laplacian. A problem is first transformed to one on a standard domain via a conformal mapping. The boundary value problem on the standard domain is then reduced to an equivalent integral operator equation. We employ the Galerkin method to solve the integral operator equation, using the eigenfunctions of the Laplacian on the standard domain. An error analysis of the method is given.

Key words. elliptic, nonlinear, integral equation, Galerkin method.

AMS subject classifications. 65R20, 65N99, 35J65.

1. Introduction. The purpose of the paper is to propose a general framework for the numerical solution of a semilinear elliptic boundary value problem of the form

$$\left\{ \begin{aligned} -\Delta U &= F(\cdot, U) \quad \text{in } \Omega, \\ U &= G \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a simply-connected open domain with a boundary $\partial\Omega$. For the domain Ω, assume there exists a standard open domain D, such that $\overline{\Omega}$ and $\overline{D}$ are conformally equivalent. The first step of the method is to reduce the problem (1.1) to an equivalent problem on the standard domain D,

$$\left\{ \begin{aligned} -\Delta u &= f(\cdot, u) \quad \text{in } D, \\ u &= g \quad \text{on } \partial D. \end{aligned} \right. \tag{1.2}$$

Then, (1.2) is converted to an equivalent integral equation. Finally, we use the Galerkin method to solve the integral equation, with the eigenfunctions of the Laplacian operator on the standard domain D as the basis functions.

An advantage of the above framework is its generality. For conformally equivalent domains, a boundary value problem (1.1) is reduced to a problem on a standard domain D, which is usually taken to assume a simple geometry, e.g., a disk or a square. On the standard domain, explicit forms of the eigenvalues and associated eigenfunctions of the Laplacian operator are usually available from the literature. The information on the eigenpairs can be used in the efficient implementation of the Galerkin method for the integral operator equation, as well as in the theoretical analysis of the proposed numerical method (convergence, convergence rate, etc.). In particular, we mention the possibility of using FFT algorithms for solving the resulting discrete systems for a problem on the standard domain.

To see how (1.1) is reduced to (1.2), let

$$\xi + i\,\eta = \phi(x + i\,y) = h(x, y) + i\,k(x, y) \tag{1.3}$$

be a conformal mapping from $\overline{D}$ to $\overline{\Omega}$, where (x, y) denotes a generic point in $\overline{D}$ and (ξ, η) is the corresponding point in $\overline{\Omega}$, and let $u(x, y) = U(\xi, \eta)$. It is then easy to verify that

$$-\Delta U(\xi, \eta) = F(\xi, \eta, U(\xi, \eta)), \quad (\xi, \eta) \in \Omega,$$

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is transformed to

$$-\Delta u(x,y) = f(x,y,u(x,y)), \quad (x,y) \in D,$$

where

$$f(x,y,u(x,y)) = |\phi'(x+i\,y)|^2 F(h(x,y),k(x,y),u(x,y)). \tag{1.4}$$

So without loss of generality, we only need consider the following problem on D,

$$\begin{cases} -\Delta u = f(\cdot,u) & \text{in } D, \\ \quad u = g & \text{on } \partial D, \end{cases} \tag{1.5}$$

where g is related to G by the relation $g(x,y) = G(\xi,\eta)$ for $(x,y) \in \partial D$. We assume $g \in H^{1/2}(\partial D)$. Notice that since ϕ is a conformal mapping, $g \in H^{1/2}(\partial D)$ if and only if $G \in H^{1/2}(\partial\Omega)$. The problem is solvable under mild assumptions on f; see the next section.

As an example of a related approach to solving (1.1), see [2]. They too convert the problem to an integral equation, but their approach is different from ours.

2. Some Existence Results. Concerning semilinear elliptic problems such as (1.5), one can find general solvability results in the context of applications of nonlinear operator theory, cf. Zeidler [15]. Here, we mention two such results.

A function $f(x,y,u) : D \times \mathbb{R} \to \mathbb{R}$ is called a Carathéodory function if, (1) for any $u \in \mathbb{R}$, $f(\cdot,u)$ is measurable, and (2) for a.e. $(x,y) \in D$, $f(x,y,\cdot)$ is continuous. Adapting a proof from [15, §27.4], we have following existence result.

THEOREM 2.1. *Consider the problem*

$$\begin{cases} -\Delta u + f_1(\cdot,u) = f_2 & \text{in } D, \\ \quad u = 0 & \text{on } \partial D, \end{cases} \tag{2.1}$$

where $f_1(x,y,u)$ is a Carathéodory function, $f_2 \in H^{-1}(D)$. Further assume

$$\inf_{(x,y)\in D,\, u\in\mathbb{R}} f_1(x,y,u)\,u > -\infty, \tag{2.2}$$

and

$$|f_1(x,y,u)| \le c\,(a(x,y) + |u|^r) \quad \textit{for some } r \in (0,\infty),\ a \in L^2(D). \tag{2.3}$$

Consider the weak formulation of (2.1): find $u \in H_0^1(D)$, such that

$$\int_D (\nabla u \cdot \nabla v + f_1(x,y,u)\,v)\,dx\,dy = \int_D f_2 v\,dx\,dy, \quad \forall\, v \in H_0^1(D). \tag{2.4}$$

Then this has a solution.

As for the uniqueness of a solution, let c_0 be the best constant in the *Poincaré-Friedrichs inequality*

$$\|u\|_{L^2(D)} \le c_0 \|\nabla u\|_{L^2(D)}, \quad \forall\, u \in H_0^1(D). \tag{2.5}$$

We have

$$c_0 = \frac{1}{\sqrt{\lambda_1}},$$

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where $\lambda_1 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

If either f_1 is monotone with respect to u, i.e.,

$$(f_1(x,y,u_1) - f_1(x,y,u_2))\,(u_1 - u_2) \ge 0, \quad \forall\,(x,y) \in D,\ u_1, u_2 \in \mathbb{R},$$

or f_1 is differentiable with respect to u and

$$\frac{\partial f_1(x,y,u)}{\partial u} > -\lambda_1, \quad \forall\,(x,y) \in D,\ u \in \mathbb{R},$$

then the solution u of (2.4) is unique.

3. Galerkin Approximations. Let $G(x,y;\xi,\eta)$ be the Green's function for the problem

$$\begin{cases} -\Delta u = f & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Let $u_g(x,y)$ be the harmonic function assuming the Dirichlet data $g(x,y)$ on ∂D. Then a solution u of the problem (1.5) satisfies

$$u(x,y) = u_g(x,y) + \int_D G(x,y;\xi,\eta)\, f(\xi,\eta,u(\xi,\eta))\, d\xi\, d\eta, \quad (x,y) \in \overline{D}. \tag{3.1}$$

As in Kumar and Sloan [11], we introduce $v(x,y) = f(x,y,u(x,y))$. The function v is a solution of

$$v(x,y) = f\left(x, y, u_g(x,y) + \int_D G(x,y;\xi,\eta)\, v(\xi,\eta)\, d\xi\, d\eta\right), \quad (x,y) \in \overline{D}. \tag{3.2}$$

For a further discussion of this approach, see [9].

We use Galerkin's method to solve (3.2). To do this, we consider the eigenvalue problem for the Laplacian operator:

$$\begin{cases} -\Delta \phi = \lambda \phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{3.3}$$

Let $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ be the sequence of the eigenvalues, and $\phi_1, \phi_2, \cdots$ be corresponding eigenfunctions. The existence of the eigenpair sequence is guaranteed, and the eigenfunctions can be chosen to form an orthogonal basis of $L^2(D)$, cf. [12]. Then we have

$$\begin{cases} -\Delta \phi_j = \lambda_j \phi_j & \text{in } D, \\ \phi_j = 0 & \text{on } \partial D, \end{cases}$$

or,

$$\int_D G(x,y;\xi,\eta)\, \phi_j(\xi,\eta)\, d\xi\, d\eta = \frac{1}{\lambda_j}\, \phi_j(x,y), \quad (x,y) \in \overline{D},\ j = 1, 2, \cdots. \tag{3.4}$$

Let $X_n = span\{\phi_1, \cdots, \phi_n\}$. The Galerkin method for (3.2) is to find $v_n = \sum_{j=1}^{n} \alpha_j \phi_j \in X_n$, such that

$$(v_n, \phi_i) = \left(f(x, y, u_g(x,y) + \int_D G(x,y;\xi,\eta)\, v_n(\xi,\eta)\, d\xi\, d\eta), \phi_i\right), \ 1 \le i \le n, \tag{3.5}$$

i.e., upon using the orthogonality of $\{\phi_i\}_i$,

$$\alpha_i(\phi_i,\phi_i) = \int_D \phi_i(x,y)\, f\Big(x,y,u_g(x,y) + \sum_{j=1}^n \frac{\alpha_j}{\lambda_j}\,\phi_j(x,y)\Big)\,dx\,dy,\ 1\le i\le n. \tag{3.6}$$

The advantage of applying the Galerkin method to solve the auxiliary problem (3.2) is that the integral

$$\int_D G(x,y;\xi,\eta)\,v_n(\xi,\eta)\,d\xi\,d\eta = \sum_{j=1}^n \frac{\alpha_j}{\lambda_j}\,\phi_j(x,y)$$

is available and has been computed exactly. If we apply the Galerkin method directly to the problem (3.1), in each iteration for solving the resulting nonlinear algebraic system, we will have to evaluate double integrals

$$\int_D\int_D G(x,y;\xi,\eta)\,f(\xi,\eta,u_n^{(k)}(\xi,\eta))\,\phi_i(x,y)\,d\xi\,d\eta\,dx\,dy,\quad 1\le i\le n,$$

where the superscript (k) refers to the number of iterations. Another advantage of the method is that the left side of the system (3.6) is diagonal; this may bring in some convenience in solving (3.6) numerically.

After we compute the Galerkin solution v_n, we can generate an approximation of u by, e.g., (4.8) of the next section.

REMARK 3.1. *Assume the standard domain D is the unit disk. The eigenvalues are*

$$\lambda_{m,n} = (j_{m,n})^2,\quad m=0,1,2,\cdots,\ n=1,2,\cdots. \tag{3.7}$$

A set of corresponding eigenfunctions is given by

$$u_{0,n} = J_0(j_{0,n}r),\quad n=1,2,\cdots,$$

and

$$u_{m,n}^{(1)} = J_m(j_{m,n}r)\,\cos m\theta,\ u_{m,n}^{(2)} = J_m(j_{m,n}r)\,\sin m\theta,\quad m=1,2,\cdots,\ n=1,2,\cdots,$$

where J_m is the m^{th} Bessel function, $j_{m,n}$ is the n^{th} zero of J_m (cf. [12]). Asymptotically ([1]),

$$j_{m,n} \sim \left(n+\frac{m}{2}-\frac14\right)\pi\quad \text{as } n\to\infty.$$

We have

$$\begin{aligned}
(u_{m,n}^{(1)},u_{k,l}^{(2)}) &= 0,\\
(u_{m,n}^{(1)},u_{k,l}^{(1)}) &= \frac{\pi}{2}\,[J_m'(j_{m,n})]^2\,\delta_{m,k}\delta_{n,l},\\
(u_{m,n}^{(2)},u_{k,l}^{(2)}) &= \frac{\pi}{2}\,[J_m'(j_{m,n})]^2\,\delta_{m,k}\delta_{n,l}.
\end{aligned}$$

REMARK 3.2. *When the standard domain is a square, $D=(0,\pi)\times(0,\pi)$, the eigenvalues and associated orthonormal eigenfunctions are*

$$\lambda_{m,n} = m^2+n^2,\ \ u_{m,n}(x,y) = \frac{2}{\pi}\,\sin(mx)\,\sin(ny),\quad m,n=1,2,\cdots.$$

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4. Error Analysis. In this section, we perform a convergence analysis for the Galerkin approximations defined in the last section. We will use $\|\cdot\|$ for $\|\cdot\|_{L^2(D)}$. Introduce the Nemyckii operator

$$(\mathcal{F}(u))(x,y) = f(x,y,u(x,y)), \tag{4.1}$$

and the linear integral operator

$$(\mathcal{G}(v))(x,y) = \int_D G(x,y;\xi,\eta)\, v(\xi,\eta)\, d\xi\, d\eta. \tag{4.2}$$

The operator $\mathcal{G}$ is a linear continuous mapping from $L^2(D)$ to $H_0^1(D)$,

$$\|\mathcal{G}(v)\|_{H^1(D)} \le c\,\|v\|, \quad \forall\, v \in L^2(D),$$

and $\mathcal{G}$ is compact on $L^2(D)$ to $L^2(D)$.

We will need the following assumption.

> (**A1**) The operator $\mathcal{F}(u_g + \mathcal{G}(\cdot)) : L^2(D) \to L^2(D)$ is completely continuous.

The assumption (**A1**) is satisfied, if, e.g., the function f is Carathéodory, and satisfies the growth condition

$$|f(x,y,v)| \le c\,(a(x,y) + |v|^r), \quad \text{for some } a \in L^2(D), \text{ and some } r > 0. \tag{4.3}$$

Indeed, using the mapping property of $\mathcal{G}$, we then know that the mapping $v \mapsto \mathcal{F}(u_g + \mathcal{G}(v))$ is continuous and bounded from $L^2(D)$ to $L^2(D)$ with (cf. [15, §26.3])

$$\begin{aligned} \|\mathcal{F}(u_g + \mathcal{G}(v))\| &\le c\left(\|a\| + \|u_g + \mathcal{G}(v)\|^r_{L^{\max\{1,2r\}}(D)}\right) \\ &\le c\left(\|a\| + \|u_g\|^r_{H^1(D)} + \|v\|^r\right). \end{aligned} \tag{4.4}$$

Since $\mathcal{G}$ is compact as an operator from $L^2(D)$ to $L^2(D)$, it is easy to see that the map $\mathcal{F}(u_g + \mathcal{G}(\cdot))$ is compact from $L^2(D)$ to $L^2(D)$. Therefore, $\mathcal{F}(u_g + \mathcal{G}(\cdot)) : L^2(D) \to L^2(D)$ is completely continuous.

In the operator equation form, (3.1) and (3.2) can be rewritten as

$$u = u_g + \mathcal{G}\mathcal{F}(u) \tag{4.5}$$

and

$$v = \mathcal{F}(u_g + \mathcal{G}v). \tag{4.6}$$

We denote by u^* a solution of (4.5). Then $v^* = \mathcal{F}(u^*)$ is a solution of (4.6).

Let $\mathcal{P}_n$ be the $L^2(D)$ orthogonal projection of $L^2(D)$ onto

$$X_n = \text{span}\{\phi_1, \cdots, \phi_n\},$$

where the sequence $\{\phi_j\}_{j=1}^{\infty}$ has been normalized. Then the Galerkin solution $v_n \in X_n$ of (3.5) satisfies the operator equation

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$$v_n = \mathcal{P}_n \mathcal{F}(u_g + \mathcal{G} v_n). \tag{4.7}$$

We define

$$u_n = \mathcal{P}_n u_g + \mathcal{G} v_n, \tag{4.8}$$

and the iterated solution

$$\tilde{v}_n = \mathcal{F}(u_g + \mathcal{G} v_n). \tag{4.9}$$

Easily,

$$\mathcal{P}_n \tilde{v}_n = v_n.$$

Since $\mathcal{F}(u_g + \mathcal{G}(\cdot)) : L^2(D) \to L^2(D)$ is completely continuous, the following convergence result can be found from Krasnoselskii [10, Chap. 3, §3].

THEOREM 4.1. *Assume (**A1**) and*

*(**A2**) 1 is not an eigenvalue of $\mathcal{L} = \mathcal{F}'(u^*)\mathcal{G}$.*

Then $(I - \mathcal{L})^{-1} : L^2(D) \to L^2(D)$ is bounded, and for sufficiently large n, (4.7) has a solution v_n such that $v_n \to v^*$ as $n \to \infty$.

REMARK 4.2. *The assumption (**A2**) is equivalent to the statement that the equation*

$$\mathcal{F}'(u^*)\, \mathcal{G} w = w$$

does not admit a nontrivial solution, or letting $z = \mathcal{G} w$, that the equation

$$\mathcal{G}\mathcal{F}'(u^*) z = z \tag{4.10}$$

does not have a nonzero solution. Notice that the equation (4.10) is a linear boundary value problem

$$\begin{cases} -\Delta z = \dfrac{\partial f(\cdot, u^*(\cdot))}{\partial u}\, z & \text{in } D, \\ z = 0 \quad \text{on } \partial D. \end{cases} \tag{4.11}$$

The problem (4.11) does not have a nonzero solution, if, e.g.,

$$\sup_{(x,y) \in D} \frac{\partial f(x, y, u^*(x,y))}{\partial u} < \lambda_1.$$

From now on, we will assume the discrete solutions $\{v_n\}$ are the ones converging to v^*. The rest of the section is devoted to error estimations.

PROPOSITION 4.3.

$$\|v^* - v_n\| \leq (1 + \delta_n)\, \|v^* - \mathcal{P}_n v^*\|,$$

where

$$\delta_n = \frac{\|v^* - \tilde{v}_n\|}{\|v^* - \mathcal{P}_n v^*\|}.$$

Proof.

$$\begin{aligned}\|v^* - v_n\| &\le \|v^* - \mathcal{P}_n v^*\| + \|\mathcal{P}_n v^* - v_n\| \\ &= \|v^* - \mathcal{P}_n v^*\| + \|\mathcal{P}_n (v^* - \tilde{v}_n)\| \\ &\le \|v^* - \mathcal{P}_n v^*\| + \|v^* - \tilde{v}_n\| \\ &= (1+\delta_n)\,\|v^* - \mathcal{P}_n v^*\|.\end{aligned}$$

∎

The significance of Proposition 4.3 is that the discrete solution v_n is almost as accurate as the $L^2(D)$ orthogonal projection of the exact solution to X_n, as long as $\delta_n \to 0$, which is true most of the time. Indeed, it is shown in [5] that if (**A2**) is satisfied, then

$$\delta_n \le c\,\max\{b_n, r_n\}, \tag{4.12}$$

where

$$r_n = \frac{\|\mathcal{F}(u_g + \mathcal{G}v_n) - \mathcal{F}(u_g + \mathcal{G}v^*) - \mathcal{L}(v_n - v^*)\|}{\|v_n - v^*\|}, \tag{4.13}$$

$$b_n = \|\mathcal{L}(I - \mathcal{P}_n)\| = \|(I - \mathcal{P}_n)\,\mathcal{L}^*\|. \tag{4.14}$$

For the behavior of δ_n, we have the following result.

PROPOSITION 4.4. *Assume (**A1**) and (**A2**). Assume further that $\mathcal{F}(u_g + \mathcal{G}v)$ is differentiable at v^*. Then, δ_n is uniformly bounded. Furthermore, if $\mathcal{F}'(u_g + \mathcal{G}v)$ is Lipschitz continuous in some neighborhood V of v^*,*

$$\|\mathcal{F}'(u_g + \mathcal{G}v_1) - \mathcal{F}'(u_g + \mathcal{G}v_2)\| \le q\,\|v_1 - v_2\|, \quad \forall\, v_1, v_2 \in V, \text{ for some } q \ge 0,$$

then we have the estimate:

$$\delta_n \le c\,\min\left\{\|v_n - v^*\|, 1/\lambda_{n+1}\right\}.$$

Proof. From the convergence $v_n \to v^*$ and the differentiability of $\mathcal{F}(u_g + \mathcal{G}v)$ at v^*, it is easy to see that r_n is uniformly bounded. Obviously, b_n is uniformly bounded. Thus, δ_n is uniformly bounded.

Now we further assume that $\mathcal{F}'(u_g + \mathcal{G}v)$ is Lipschitz continuous in some neighborhood V of v^*. For sufficiently large n, $v_n \in V$, and so

$$r_n \le c\,\|v_n - v^*\|. \tag{4.15}$$

For the term b_n, we have

$$b_n \le \|(I - \mathcal{P}_n)\,\mathcal{G}\|\,\|\mathcal{F}'(u^*)\|.$$

With $v = \sum_{k=1}^{\infty} c_k \phi_k$, we have

$$\mathcal{G}(v) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\, c_k \phi_k,$$

$$(I - \mathcal{P}_n)\mathcal{G}(v) = \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k}\, c_k \phi_k.$$

Hence,

$$\|(I-\mathcal{P}_n)\mathcal{G}(v)\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^2}\, c_k^2 \le \frac{1}{\lambda_{n+1}^2}\, \|(I-\mathcal{P}_n)v\|^2 \le \frac{\|v\|^2}{\lambda_{n+1}^2}.$$

Thus,

$$\|(I-\mathcal{P}_n)\mathcal{G}\| \le \frac{1}{\lambda_{n+1}},$$

and

$$b_n \le \frac{\|\mathcal{F}'(u^*)\|}{\lambda_{n+1}}. \tag{4.16}$$

The estimate for δ_n follows from (4.15) and (4.16). □

We have seen that under the assumptions stated in the first part of Proposition 4.4, the Galerkin solution is as accurate as the orthogonal projection. Actually, we can show that

$$\frac{\|v_n - \mathcal{P}_n v^*\|}{\|v^* - \mathcal{P}_n v^*\|} \to 0 \quad \text{as } n \to \infty, \tag{4.17}$$

which is a type of superconvergence result. Write

$$v^* = \sum_{k=1}^{\infty} c_k \phi_k .$$

Then

$$\|v^* - \mathcal{P}_n v^*\|^2 = \sum_{k=n+1}^{\infty} c_k^2 .$$

Also write

$$v_n = \sum_{k=1}^{n} c_k^{(n)} \phi_k .$$

Then

$$\mathcal{P}_n v^* - v_n = \sum_{k=1}^{n} [c_k - c_k^{(n)}]\, \phi_k ,$$

$$v^* - v_n = \sum_{k=1}^{n} [c_k - c_k^{(n)}]\, \phi_k + \sum_{k=n+1}^{\infty} c_k \phi_k .$$

Using Proposition 4.3, we obtain

$$\|v^* - v_n\|^2 = \sum_{k=1}^{n} [c_k - c_k^{(n)}]^2 + \|v^* - \mathcal{P}_n v^*\|^2 \le (1+\delta_n)^2 \|v^* - \mathcal{P}_n v^*\|^2 .$$

Hence,

$$\|v_n - \mathcal{P}_n v^*\|^2 = \sum_{k=1}^{n} [c_k - c_k^{(n)}]^2 \le \delta_n\, (2+\delta_n)\, \|v^* - \mathcal{P}_n v^*\|^2 . \tag{4.18}$$

Since $\delta_n \to 0$, we have (4.17).

Finally we consider an error estimate for $u^* - u_n$. We have

$$\begin{aligned} u^* - u_n &= u_g - \mathcal{P}_n u_g + \mathcal{G}\mathcal{F}(u^*) - \mathcal{G} v_n \\ &= u_g - \mathcal{P}_n u_g + \mathcal{G}(v^* - v_n) \\ &= u_g - \mathcal{P}_n u_g + \sum_{k=1}^{n} \frac{1}{\lambda_k} [c_k - c_k^{(n)}]\, \phi_k + \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k} c_k \phi_k \\ &= \sum_{k=1}^{n} \frac{1}{\lambda_k} [c_k - c_k^{(n)}]\, \phi_k + u^* - \mathcal{P}_n u^*, \end{aligned}$$

and

$$\begin{aligned} \|u^* - u_n\|^2 &= \sum_{k=1}^{n} \frac{1}{\lambda_k^2} [c_k - c_k^{(n)}]^2 + \|u^* - \mathcal{P}_n u^*\|^2 \\ &\le \frac{1}{\lambda_1^2} \sum_{k=1}^{n} [c_k - c_k^{(n)}]^2 + \|u^* - \mathcal{P}_n u^*\|^2 \\ &\le \frac{1}{\lambda_1^2}\, \delta_n \,(2 + \delta_n)\, \|v^* - \mathcal{P}_n v^*\|^2 + \|u^* - \mathcal{P}_n u^*\|^2. \end{aligned}$$

In conclusion, we have proved the following theorem.

THEOREM 4.5. *We make the assumptions stated in Proposition 4.4. For the Galerkin solution v_n, we have the error estimate*

$$\|v^* - v_n\| \le (1 + \delta_n)\, \|v^* - \mathcal{P}_n v^*\|, \quad \delta_n \to 0 \text{ as } n \to \infty.$$

For the approximation u_n of the problem (3.1), defined by the relation (4.8), we have the error estimate

$$\|u^* - u_n\|^2 \le \frac{1}{\lambda_1^2}\, \delta_n \,(2 + \delta_n)\, \|v^* - \mathcal{P}_n v^*\|^2 + \|u^* - \mathcal{P}_n u^*\|^2.$$

REMARK 4.6. *The above discussion on error estimations is based on the convergence result, Theorem 4.1. Alternatively, if we assume $\{v_n\}$ is a bounded sequence, then Proposition 4.4 still holds. This follows from the fact that r_n is uniformly bounded, a property implied by the boundedness of $\{v_n\}$, the inequality (4.4) and the differentiability of $\mathcal{F}(u_g + \mathcal{G}(\cdot))$ at v^*. Then by Proposition 4.3, we obtain the convergence $v_n \to v^*$. Thus all the error estimates derived above hold. We also notice that if the function f is Carathéodory and satisfies the inequality (4.3) with $r \le 1$, then r_n is uniformly bounded even without the boundedness assumption of $\{v_n\}$. Consequently, if $r \le 1$, then (4.6) has a unique solution v^* satisfying the condition (**A2**), and any discrete solution of (4.7) converges to v^*.*

5. Numerical Example. We conclude the paper with an illustrative example. Further results on the implementation of the proposed method and further numerical experiments are given in [6].

Let Ω be the bounded elliptical region

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1 \tag{5.1}$$

with $a > b > 0$. For the standard region D, we use the unit circle in the plane, $x^2 + y^2 < 1$. We construct the conformal mapping $\phi : D \to \Omega$ and its derivative by using the *SC Toolbox* (to be run from within *Matlab*) that is described in [8, Appendix].

Begin by introducing the *elliptic integral of the first kind*

$$F_k(z) \equiv \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{5.2}$$

with

$$0 < k < 1.$$

The function $w = F_k(z)$ maps conformally the half-space $\mathrm{imag}(z) > 0$, denoted by H, onto the rectangle

$$\begin{aligned} -K &< \mathrm{real}(w) < K \\ 0 &< \mathrm{imag}(w) < K' \end{aligned}$$

which we denote by R. Introduce

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{5.3}$$

$$K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\left(1-(k')^2\, t^2\right)}} \tag{5.4}$$

with

$$k' = \sqrt{1-k^2}$$

or

$$K'(k) = K(k').$$

The mapping $F_k(z)$ can be produced using *SC*.

For a conformal mapping of the closed unit disk onto the closure of the ellipse in (5.1), use

$$w = \phi(z) = c \sin\left[\frac{\pi}{2K} F_k\left(\frac{z}{\sqrt{k}}\right)\right], \qquad \mathrm{imag}(z) \geq 0 \tag{5.5}$$

with $c = \sqrt{a^2 - b^2}$, $z = x + iy$, $x^2 + y^2 \leq 1$. The definition extends to the lower half of the unit disk by using

$$\phi(z) = \overline{\phi(\overline{z})}, \qquad \mathrm{imag}(z) < 0.$$

The points $(\pm c, 0)$ are the foci of the ellipse in (5.1). The constant k can be produced as follows, using the construction of Szegö [14]. Introduce

$$\begin{aligned} \alpha &= \frac{a}{\sqrt{a^2-b^2}} > 1, \\ q &= \left(\alpha + \sqrt{\alpha^2 - 1}\right)^{-4} < 1. \end{aligned}$$

Then

$$k = 4\sqrt{q}\left[\frac{\left(1+q^2\right)\left(1+q^4\right)\left(1+q^6\right)\cdots}{\left(1+q\right)\left(1+q^3\right)\left(1+q^5\right)\cdots}\right]^4 < 1.$$

The derivative is given by

$$\begin{aligned}\phi'(z) &= \frac{\pi c}{2K\sqrt{k}}\cos\left[\frac{\pi}{2K}F_k\left(\frac{z}{\sqrt{k}}\right)\right]F_k'\left(\frac{z}{\sqrt{k}}\right),\\ F_k'(z) &\equiv \frac{1}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}.\end{aligned}$$

According to (1.4), we need only the absolute value of the derivative of ϕ, and therefore we do not need to choose a particular branch to the square root function. More precisely, we need to evaluate

$$\left|\phi'(z)\right|^2 = \frac{1}{k}\left(\frac{\pi c}{2K}\right)^2\left|\cos\left[\frac{\pi}{2K}F_k\left(\frac{z}{\sqrt{k}}\right)\right]\right|^2\left|\left(1-\frac{z^2}{k}\right)\left(1-kz^2\right)\right|^{-1} \tag{5.6}$$

for $|z| \leq 1$. Note that $F_k(\pm 1) = \pm K$, and therefore the apparent singularity in (5.6) at $z = \pm\sqrt{k}$ is actually removable.

As a simple illustration of our numerical method from §3, we use the true solution

$$\overline{U}(x,y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1, \qquad (x,y) \in \Omega. \tag{5.7}$$

We solve the problem

$$\begin{aligned}-\Delta U &= e^{-U} + g(x,y), \qquad &(x,y) \in \Omega,\\ U(x,y) &= 0, &(x,y) \in \partial\Omega.\end{aligned} \tag{5.8}$$

The function g is to be so chosen that the $\overline{U}$ of (5.7) is the true solution of the problem.

The details of the implementation of our numerical method are given in the paper [6], along with further theoretical results on the rate of convergence. Here we simply give some numerical results for this one sample problem. From the solution u_n defined on the unit disk, we obtain the solution U_n defined on the elliptical region Ω. For our example, we double n repeatedly, where n is the number of distinct eigenvalues taken in increasing order from (3.7). Note that the multiplicity can vary, and thus the number of basis functions $d_n > n$, with $d_n \approx 2n$, roughly. In Table 5.1, we give two measures of error:

$$\begin{aligned}E_{n,1} &= \max_{1\leq \ell\leq m}\left|\overline{U}(x_\ell,y_\ell) - U_n(x_\ell,y_\ell)\right| \approx \left\|\overline{U} - U_n\right\|_\infty,\\ E_{n,2} &= \sqrt{\frac{1}{m}\sum_{j=1}^{m}\left|\overline{U}(x_\ell,y_\ell) - U_n(x_\ell,y_\ell)\right|^2} \approx \frac{1}{\sqrt{\pi ab}}\left\|\overline{U} - U_n\right\|_2.\end{aligned}$$

The points $\{(x_\ell,y_\ell) : 1 \leq \ell \leq m\}$, m large, are chosen to cover well the closed ellipse $\overline{\Omega}$. The numerical results are given for the case of $(a,b) = (3,2)$. For the construction of $F_k(z)$ in (5.2), $k \doteq 0.688$.

These results in Table 5.1 are converging slowly to zero; but the cost of setting up and solving the nonlinear system is also fairly small. We plan on exploring extrapolation methods for accelerating the convergence, using techniques described in the book of Sidi [13]. We are also looking into alternative polynomial bases for approximating the solution u using collocation.

TABLE 5.1
Errors in solving a problem over an ellipse

n	$E_{n,1}$	$E_{n,2}$
2	1.23E−1	7.56E−2
4	3.97E−2	2.46E−2
8	3.77E−2	2.02E−2
16	1.76E−2	8.93E−3
32	5.79E−3	2.53E−3
64	2.45E−3	1.08E−3
128	1.16E−3	4.82E−4

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, eds., *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1972.
[2] A. ARTHURS, J. CLEGG, AND A. NAGAR, *On the solution of the Liouville equation over a rectangle*, J. Appl. Math. Stochastic Anal., 9 (1996), pp. 57–67.
[3] K. ATKINSON, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.
[4] K. ATKINSON AND W. HAN, *Theoretical Numerical Analysis*, Springer-Verlag, New York, 2001.
[5] K. ATKINSON AND F. A. POTRA, *Projection and iterated projection methods for nonlinear integral equations*, SIAM J. Numer. Anal., 24 (1987), pp. 1352–1373.
[6] K. ATKINSON AND A. SOMMARIVA, *On the numerical solution of some semilinear elliptic problems – II*, Computing, to appear.
[7] H. BRÉZIS, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier, 18 (1968), pp. 115–175.
[8] T. DRISCOLL AND L. TREFETHEN, *Schwarz-Christoffel Mapping*, Cambridge University Press, 2002.
[9] J. FRANKEL, *A note on the integral formulation of Kumar and Sloan*, J. Comput. Appl. Math., 61 (1995), pp. 263–274.
[10] M. A. KRASNOSELSKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, London-New York, 1964.
[11] S. KUMAR AND I. SLOAN, *A new collocation-type method for Hammerstein integral equations*, Math. Comp., 48 (1987), pp. 585–593.
[12] J. R. KUTTLER AND V. G. SIGILLITO, *Eigenvalues of the Laplacian in two dimensions*, SIAM Rev., 26 (1984), pp. 163–193.
[13] A. SIDI, *Practical Extrapolation Methods*, Cambridge University Press, Cambridge, 2003.
[14] G. SZEGÖ, *Conformal mapping of the interior of an ellipse onto a circle*, Amer. Math. Monthly, 57 (1950), pp. 474–478.
[15] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications, II/B Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.