**Electronic Transactions on Numerical Analysis.**
**Volume 17, pp. 218-233, 2004.**

**ISSN 1068-9613.**

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**

# A NOTE ON THE EFFICIENCY OF RESIDUAL-BASED A-POSTERIORI ERROR ESTIMATORS FOR SOME MIXED FINITE ELEMENT METHODS*

GABRIEL N. GATICA†

**Abstract.** In this paper we present a unified proof of the efficiency of residual-based a-posteriori error estimates for the dual-mixed variational formulations of linear boundary value problems in the plane. We consider the interior problem determined by a second order elliptic equation in divergence form with mixed boundary conditions, and the exterior transmission problem given by the same equation in a bounded domain, coupled with Laplace equation in the surrounding unbounded exterior region. The corresponding Galerkin scheme reduces to a mixed finite element method with Lagrange multipliers for the first problem, and to the coupling of the mixed finite element method with the boundary element method for the second one. Our analysis makes use of inverse inequalities in finite element subspaces and the localization technique based on triangle-bubble and edge-bubble functions.

**Key words.** mixed finite elements, boundary elements, residual-based estimates, efficiency.

**AMS subject classifications.** 65N15, 65N30, 65N50, 35J25.

**1. Introduction.** One of the main advantages in using dual-mixed variational formulations lies on the possibility of introducing further unknowns with a physical interest, such as stresses and fluxes, so that they can be approximated directly, thus avoiding any numerical postprocessing yielding additional sources of error. This fact has motivated the utilization of the mixed finite element method for the numerical solution of diverse problems in elasticity, heat conduction, and other areas (see, e.g. [5]). However, in order to guarantee a good convergence behaviour of these discrete solutions, one usually needs to apply a refinement algorithm based on a-posteriori error estimates. These are represented by global quantities $\boldsymbol{\eta}$ that are expressed in terms of local estimators $\eta_T$ defined on each element $T$ of a given triangulation. The estimator $\boldsymbol{\eta}$ is said to be efficient (resp. reliable) if there exists $C_1 > 0$ (resp. $C_2 > 0$) such that $C_1\,\boldsymbol{\eta} \;\leq\; \|error\| \;\leq\; C_2\,\boldsymbol{\eta}$.

The first results concerning a-posteriori error analysis of mixed formulations are given in [16], where an estimator of explicit residual type is obtained for the Stokes problem. Then, estimators based on residuals and on the solution of local problems, using Raviart-Thomas and Brezzi-Douglas-Marini spaces, are provided in [1] for elliptic partial differential equations of second order. The main novelty of the approach in [1] is the use of a Helmholtz decomposition to prove reliability and efficiency of error estimators for mixed finite elements. In connection with Raviart-Thomas spaces, we also refer to [4] where a non-natural norm is employed to derive residual error estimators. The drawback of the approach in [4] is the use of a saturation assumption. This hypothesis is avoided in [7], where a Helmholtz decomposition is also applied to obtain reliable and efficient residual-based error estimators for the Poisson problem in the usual norm of $H(\mathrm{div}\,;\Omega) \times L^2(\Omega)$. In addition, the analysis from [7] is extended in [8] to the linear elasticity problem with mixed boundary conditions. A comparison of four different kinds of error estimators for mixed finite element discretizations by Raviart-Thomas elements is presented in [14].

A-posteriori error estimators for the combination of the mixed finite element method with other techniques have also been developed in recent years. In particular, a similar approach to the one from [7] is utilized in [10] to derive a reliable residual-based a-posteriori error

*Received December 10, 2003. Accepted for publication July 19, 2004. Recommended by V. L. Druskin.

†**GI$^2$MA**, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. E-mail: `ggatica@ing-mat.udec.cl`. This research was partially supported by Conicyt-Chile through the FONDAP Program in Applied Mathematics, and by the Dirección de Investigación of the Universidad de Concepción through the Advanced Research Groups Program.

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



estimator for the coupling with the boundary element method of an exterior transmission problem in the plane (see also [15] and [13]). These results are obtained, independently, in [13], where the proof of reliability is also given in details. However, except for some remarks given in [10] on the tools that should be used, which refer mainly to the analysis in [6], [7], and [9], no explicit proof for the efficiency is available neither in [10] nor in [13]. On the other hand, the mixed finite element method with Lagrange multipliers from [2] is considered in [12] to obtain a reliable residual-based a-posteriori error estimator for the Poisson problem with mixed boundary conditions in a bounded inner region of the plane. It is important to remark that the estimators from [12] and [10] (or [13]), although related to different problems, have several terms in common and other with the same structure.

According to the above, the purpose of this note is to present a unified and detailed proof for the efficiency of the residual-based a-posteriori error estimators provided in [12] and [10] (or [13]). Our analysis, which makes use of the inverse inequalities in finite element subspaces and the localization technique based on bubble functions (see [18], [8], and [7]), even for the terms involving boundary integral operators, could also be extended to other dual-mixed variational formulations, such as the one studied in [3].

The rest of the paper is organized as follows. In Section 2 we present the boundary value problems from [2] and [10], and state the associated dual-mixed variational formulations. The mixed finite element schemes are described in Section 3, and the corresponding results on the unique solvability, stability, and a-priori error estimates are also established there. In Section 4 we recall from [12], [10], and [13] the reliable residual-based a-posteriori error estimators. Finally, the proofs of efficiency are given in Section 5. Throughout this paper, $c$ and $C$, with or without subscripts, denote positive constants, independent of the parameters and functions involved, which may take different values at different ocurrences.

## 2. The boundary value problems.

In this section we present the boundary value problems of interest, and provide the corresponding dual-mixed variational formulations.

### 2.1. An interior problem.

We describe here the boundary value problem and the corresponding dual-mixed variational formulation studied in [2]. In fact, let $\Omega$ be a simply connected domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$, and such that all its interior angles lie in $(0, 2\pi)$. Also, let $\Gamma_D$ and $\Gamma_N$ be disjoint open subsets of $\partial\Omega$, with $|\Gamma_D|\,,\,|\Gamma_N| \neq 0$, such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. Then, given $f \in L^2(\Omega)$, $g \in H_{00}^{-1/2}(\Gamma_N)$, and a matrix valued function $\boldsymbol{\kappa} \in C(\bar{\Omega})$ inducing a strongly elliptic differential operator, we consider the model boundary value problem: Find $u \in H^1(\Omega)$ such that

$$-\operatorname{div}(\boldsymbol{\kappa}\nabla u) = f \quad \text{in} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \Gamma_D\,, \quad \text{and} \quad (\boldsymbol{\kappa}\nabla u)\cdot\boldsymbol{\nu} = g \quad \text{on} \quad \Gamma_N\,, \tag{2.1}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\Gamma_N$. We recall that the Sobolev space $H_{00}^{-1/2}(\Gamma_N)$ is the dual of $H_{00}^{1/2}(\Gamma_N)$, where $H_{00}^{1/2}(\Gamma_N) := \{\, v|_{\Gamma_N} : \quad v \in H^1(\Omega)\,, \quad v = 0 \quad \text{on} \quad \Gamma_D \}$. The corresponding duality pairing with respect to the $L^2(\Gamma_N)-$inner product is denoted by $\langle\cdot,\cdot\rangle$.

For the derivation of the weak formulation, we define first the flux variable $\boldsymbol{\sigma} := \boldsymbol{\kappa}\nabla u$ in $\Omega$ as additional unknown. Then we integrate by parts in $\Omega$ and observe that the Dirichlet and Neumann conditions become now natural and essential boundary conditions, respectively. Thus, the latter is imposed weakly, which yields the introduction of the Lagrange multiplier $\lambda := -u|_{\Gamma_N} \in H_{00}^{1/2}(\Gamma_N)$.

In this way, as shown in [2], the dual-mixed variational formulation of (2.1) becomes:

Find $(\boldsymbol{\sigma}, u, \lambda) \in H(\mathrm{div}\,;\Omega) \times L^2(\Omega) \times H_{00}^{1/2}(\Gamma_N)$ such that

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathcal{B}(\boldsymbol{\tau}, (u, \lambda)) &= 0 \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}\,;\Omega)\,, \\
\mathcal{B}(\boldsymbol{\sigma}, (v, \mu)) &= -\int_\Omega f v\, dx + \langle g, \mu \rangle \quad \forall\, (v, \mu) \in L^2(\Omega) \times H_{00}^{1/2}(\Gamma_N)\,,
\end{aligned}
\tag{2.2}
$$

where $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$
\mathcal{A}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau}\, dx\,,
$$

and

$$
\mathcal{B}(\boldsymbol{\sigma}, (v, \mu)) := \int_\Omega v \,\mathrm{div}\, \boldsymbol{\sigma}\, dx + \langle \boldsymbol{\sigma} \cdot \nu, \mu \rangle\,.
$$

THEOREM 2.1. *There exists a unique* $(\boldsymbol{\sigma}, u, \lambda) \in H(\mathrm{div}\,;\Omega) \times L^2(\Omega) \times H_{00}^{1/2}(\Gamma_N)$ *solution of* (2.2)*, and the following continuous dependence result holds*

$$
\|(\boldsymbol{\sigma}, u, \lambda)\| \le C \left\{ \|f\|_{L^2(\Omega)} + \|g\|_{H_{00}^{-1/2}(\Gamma_N)} \right\}.
$$

*Proof*. See Theorem 2.1 in [2] for details. □

**2.2. An exterior transmission problem.** We now describe the boundary value problem and the corresponding dual-mixed variational formulation studied in [10] (see also [15] and [13]). Indeed, let $\Omega$ be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary $\Gamma_N$. Then, given $f \in L^2(\Omega)$, and a matrix valued function $\boldsymbol{\kappa} \in C(\bar{\Omega})$ as in Section 2.1, we consider the exterior problem: *Find* $u \in H^1_{loc}(\mathbb{R}^2)$ *such that*

$$
\begin{gathered}
-\,\mathrm{div}\,(\boldsymbol{\kappa}\,\nabla u) = f \quad \text{in} \quad \Omega\,, \qquad -\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\Omega}\,, \\
u(x) = O(1) \quad \text{as} \quad \|x\| \to +\infty\,,
\end{gathered}
\tag{2.3}
$$

whose partial differential equations in $\Omega$ and $\mathbb{R}^2 \setminus \bar{\Omega}$ are coupled by the following transmission conditions:

$$
\begin{aligned}
\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) &= \lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^2 \setminus \bar{\Omega}}} u(x)\,, \\
\lim_{\substack{x \to x_0 \\ x \in \Omega}} \boldsymbol{\kappa}(x) \nabla u(x) \cdot \boldsymbol{\nu}(x_0) &= \lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^2 \setminus \bar{\Omega}}} \nabla u(x) \cdot \boldsymbol{\nu}(x_0)\,,
\end{aligned}
\tag{2.4}
$$

for almost all $x_0 \in \Gamma_N$, where $\boldsymbol{\nu}(x_0)$ denotes the unit outward normal at $x_0$.

In order to establish the weak formulation, similarly as in the previous section, we first introduce the flux variable $\boldsymbol{\sigma} := \boldsymbol{\kappa}\,\nabla u$ in $\Omega$ and the trace $\lambda := u|_{\Gamma_N} \in H^{1/2}(\Gamma_N)$ as further unknowns. Then, we perform integration by parts in $\Omega$, and incorporate the boundary integral equations arising from Green's representation formula for $u$ in $\mathbb{R}^2 \setminus \bar{\Omega}$, which, because of (2.4) and the definitions of $\boldsymbol{\sigma}$ and $\lambda$, become

$$
\lambda = \left( \frac{1}{2}\mathbf{I} + \mathbf{K} \right) \lambda - \mathbf{V}\,(\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) + d \quad \text{on} \quad \Gamma_N\,,
\tag{2.5}
$$

and

$$
\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = -\,\mathbf{W}\lambda + \left( \frac{1}{2}\mathbf{I} - \mathbf{K}' \right) (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \quad \text{on} \quad \Gamma_N\,.
\tag{2.6}
$$

Hereafter, $d$ is a constant, $\mathbf{I}$ is the identity operator, and $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ are the boundary integral operators of the simple, double, adjoint of the double and hypersingular layer potentials, respectively, whose explicit definitions and main mapping properties can be seen, e.g., in [13]. Also, we let $\langle \cdot, \cdot \rangle$ be the duality pairing between $H^{-1/2}(\Gamma_N)$ and $H^{1/2}(\Gamma_N)$ with respect to the $L^2(\Gamma_N)-$inner product, and define the spaces $H_0^{1/2}(\Gamma_N) := \{\mu \in H^{1/2}(\Gamma_N) : \quad \langle 1, \mu \rangle = 0\}$, $H_0^{-1/2}(\Gamma_N) := \{\zeta \in H^{-1/2}(\Gamma_N) : \langle \zeta, 1 \rangle = 0\}$, and $H_0(\mathrm{div}\,;\Omega) := \{\boldsymbol{\tau} \in H(\mathrm{div}\,;\Omega) : \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle = 0\}$.

As shown in [10] (see also [15] and [13]), the dual-mixed variational formulation of (2.3)-(2.4) reduces to: Find $(\boldsymbol{\sigma}, \lambda, u) \in H_0(\mathrm{div}\,;\Omega) \times H_0^{1/2}(\Gamma_N) \times L^2(\Omega)$ such that

$$
\begin{aligned}
\mathcal{A}\,((\boldsymbol{\sigma},\lambda),(\boldsymbol{\tau},\mu)) \,+\, \mathcal{B}\,((\boldsymbol{\tau},\mu),u) \quad &= \; 0 \qquad \forall\,(\boldsymbol{\tau},\mu) \in H_0(\mathrm{div}\,;\Omega) \times H_0^{1/2}(\Gamma_N)\,,\\
\mathcal{B}\,((\boldsymbol{\sigma},\lambda),v) \quad &= \; -\int_\Omega f\,v\,dx \qquad \forall\, v \in L^2(\Omega)\,,
\end{aligned}
\tag{2.7}
$$

where $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{\sigma},\lambda),(\boldsymbol{\tau},\mu)) \,:=\, & \int_\Omega \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx \,+\, \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \mathbf{V}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\rangle \,+\, \langle \mathbf{W}\lambda, \mu\rangle\\
& -\langle\, \boldsymbol{\tau}\cdot\boldsymbol{\nu}, (\tfrac{1}{2}\mathbf{I}+\mathbf{K})\lambda\,\rangle \,+\, \langle\, (\tfrac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}\cdot\boldsymbol{\nu}), \mu\,\rangle\,,
\end{aligned}
$$

and

$$
\mathcal{B}((\boldsymbol{\sigma},\lambda),v) \,:=\, \int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx\,.
$$

THEOREM 2.2. *There exists a unique* $(\boldsymbol{\sigma}, \lambda, u) \in H_0(\mathrm{div}\,;\Omega) \times H_0^{1/2}(\Gamma_N) \times L^2(\Omega)$ *solution of* (2.7)*, and the following continuous dependence result holds*

$$
\|(\boldsymbol{\sigma},\lambda,u)\| \;\le\; C\;\|f\|_{L^2(\Omega)}\,.
$$

*Proof*. See [10] and [15]. □

**3. The mixed finite element methods.** In this section we recall from [2] and [10] (see also [15] and [13]) the Galerkin schemes associated with each one of the dual-mixed formulations (2.2) and (2.7). For this purpose, we introduce first some necessary notations.

Throughout the rest of the paper, and for each polygonal domain $\Omega$, we let $\{\mathcal{T}_h\}_{h>0}$ be a regular family (in the sense of [11]) of triangulations of $\bar{\Omega}$ by triangles $T$ of diameter $h_T$, where $h$ stands for $\max\{\, h_T : \; T \in \mathcal{T}_h \,\}$. We assume that $\mathcal{T}_h$ satisfies the minimum angle condition, which means that there exists $c > 0$ such that $c^{-1}\, h_T^2 \;\le\; |T| \;\le\; c\, h_T^2 \qquad \forall\, T \in \mathcal{T}_h$ , where $|T|$ is the area of $T$. Also, we let $E_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, denote by $h_e$ the diameter of each $e \in E_h$, and given $T \in \mathcal{T}_h$, $E(T)$ stands for the set of its edges. In addition, we write $E_h(\Omega) := \{e \in E_h : \quad e \subseteq \Omega\}$, and for any subset $G$ of $\partial\Omega$ we set $E_h(G) := \{e \in E_h : \quad e \subseteq G\}$. In addition, for each $T \in \mathcal{T}_h$ we let $\mathrm{RT}_0(T)$ be the local Raviart-Thomas space of order zero, that is

$$
\mathrm{RT}_0(T) \,:=\, \mathrm{span}\,\left\{ \begin{pmatrix} 1\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 1\end{pmatrix}, \begin{pmatrix} x_1\\ x_2\end{pmatrix} \right\}\,,
$$

and given a non-negative integer $k$ and a subset $S$ of $\mathbb{R}^2$, $\mathbf{P}_k(S)$ stands for the space of polynomials defined on $S$ of degree $\le k$.

ETNA
Kent State University
etna@mcs.kent.edu



**3.1. The mixed finite element method with Lagrange multipliers.** The Galerkin scheme associated with (2.2), which constitutes a mixed finite element method with Lagrange multipliers, is established here. We assume that all the points in $\bar{\Gamma}_D \cap \bar{\Gamma}_N$ become vertices of $\mathcal{T}_h$ for all $h > 0$. Then, the finite element subspaces employed in [2] for the unknowns $\boldsymbol{\sigma}$ and $u$ are given, respectively, by

$$H_h^{\boldsymbol{\sigma}} := \{ \boldsymbol{\tau} \in H(\mathrm{div}\,;\Omega) : \quad \boldsymbol{\tau}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h \} ,$$

and

$$H_h^u := \left\{ v \in L^2(\Omega) : \quad v|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \right\} .$$

In order to define the finite element subspace for $\lambda \in H_{00}^{1/2}(\Gamma_N)$, we introduce an independent partition $\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_m\}$ of $\Gamma_N$, denote $\tilde{h} := \max\{ |\tilde{e}_j| : \ j \in \{1, ..., m\} \}$, and suppose that there exists $C > 0$ such that $\tilde{h} \leq C\, h$. Then, we set

$$H_{\tilde{h}}^{\lambda} := \left\{ \mu \in H_{00}^{1/2}(\Gamma_N) : \quad \mu|_{\tilde{e}_j} \in \mathbf{P}_1(\tilde{e}_j) \quad \forall\, j \in \{1, ..., m\} \right\} .$$

In this way, the Galerkin scheme associated with the continuous formulation (2.2) reads as follows: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}}) \in H_h^{\boldsymbol{\sigma}} \times H_h^u \times H_{\tilde{h}}^{\lambda}$ such that

$$\begin{aligned} A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) \,+\, B(\boldsymbol{\tau}, (u_h, \lambda_{\tilde{h}})) \quad &= 0 \qquad \forall\, \boldsymbol{\tau} \in H_h^{\boldsymbol{\sigma}} , \\ B(\boldsymbol{\sigma}_h, (v, \mu)) \quad &= -\int_\Omega f\, v\, dx \,+\, \langle g, \mu \rangle \quad \forall\, (v, \mu) \,\in\, H_h^u \times H_{\tilde{h}}^{\lambda} . \end{aligned} \tag{3.1}$$

THEOREM 3.1. *Assume that the independent partition on $\Gamma_N$ and the one induced by $\mathcal{T}_h$ are uniformly regular. Then there exists $C_0 \in (0, 1]$ such that for all $h \leq C_0\, \tilde{h}$ the discrete scheme (3.1) has a unique solution $(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}}) \in H_h^{\boldsymbol{\sigma}} \times H_h^u \times H_{\tilde{h}}^{\lambda}$. Moreover, there exist $c\,,\ C > 0$, independent of $h$ and $\tilde{h}$, such that*

$$||(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}})|| \,\leq\, c\, \{ \, ||f||_{L^2(\Omega)} \,+\, ||g||_{H_{00}^{-1/2}(\Gamma_N)} \, \} ,$$

*and*

$$||(\boldsymbol{\sigma}, u, \lambda) \,-\, (\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}})|| \,\leq\, C \inf_{(\boldsymbol{\tau}, v, \mu) \in H_h^{\boldsymbol{\sigma}} \times H_h^u \times H_{\tilde{h}}^{\lambda}} ||(\boldsymbol{\sigma}, u, \lambda) \,-\, (\boldsymbol{\tau}, v, \mu)|| .$$

*Proof*. See Lemmata 3.1, 3.2, 3.3, and Theorem 3.4 in [2]. □

Due to the sufficient but not necessary condition $h \leq C_0 \tilde{h}$, and since, as proved in [2], the constant $C_0$ is only known to live in $(0, 1]$, we assume from now on that each edge $e \in E_h(\Gamma_N)$ is contained in an edge $\tilde{e}_j$, for some $j \in \{1, ..., m\}$. Certainly, this implicitly requires that the end points of $\tilde{e}_j$ be vertices of $\mathcal{T}_h$, which is also assumed in what follows. Then, for each $e \in E_h(\Gamma_N)$ we set $\tilde{h}_e := |\tilde{e}_j|$, where $\tilde{e}_j$ is the segment containing edge $e$.

**3.2. The coupling of mixed finite element and boundary element methods.** The Galerkin scheme associated with (2.7), which becomes a coupling of mixed finite element and boundary element methods, is provided now. Indeed, the finite element subspaces used in [10], [15], and [13] for the unknowns $\boldsymbol{\sigma}$, $\lambda$, and $u$ are given, respectively, by

$$H_h^{\boldsymbol{\sigma}} := \{ \boldsymbol{\tau} \in H_0(\mathrm{div}\,;\Omega) : \quad \boldsymbol{\tau}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h \} ,$$

ETNA
Kent State University
etna@mcs.kent.edu



$$H_h^\lambda := \{ \mu \in H_0^{1/2}(\Gamma_N) : \quad \mu|_e \in \mathbf{P}_1(e) \quad \forall\, e \in E_h(\Gamma_N) \},$$

and

$$H_h^u := \{ v \in L^2(\Omega) : \quad v|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \},$$

whence the discrete scheme reads: Find $(\boldsymbol{\sigma}_h, \lambda_h, u_h) \in H_h^{\boldsymbol{\sigma}} \times H_h^\lambda \times H_h^u$ such that

$$\begin{aligned} \mathcal{A}((\boldsymbol{\sigma}_h,\lambda_h),(\boldsymbol{\tau},\mu)) + \mathcal{B}((\boldsymbol{\tau},\mu),u_h) &= 0 \qquad \forall\,(\boldsymbol{\tau},\mu) \in H_h^{\boldsymbol{\sigma}} \times H_h^\lambda, \\ \mathcal{B}((\boldsymbol{\sigma}_h,\lambda_h),v) &= -\int_\Omega f\, v\, dx \qquad \forall\, v \in H_h^u. \end{aligned} \tag{3.2}$$

THEOREM 3.2. *The discrete scheme* (3.2) *has a unique solution* $(\boldsymbol{\sigma}_h, \lambda_h, u_h) \in H_h^{\boldsymbol{\sigma}} \times H_h^\lambda \times H_h^u$*. Moreover, there exist* $c$, $C > 0$*, independent of* $h$*, such that*

$$||(\boldsymbol{\sigma}_h, \lambda_h, u_h)|| \le c\, ||f||_{L^2(\Omega)},$$

*and*

$$||(\boldsymbol{\sigma},\lambda,u) - (\boldsymbol{\sigma}_h,\lambda_h,u_h)|| \le C \inf_{(\boldsymbol{\tau},\mu,v)\in H_h^{\boldsymbol{\sigma}} \times H_h^\lambda \times H_h^u} ||(\boldsymbol{\sigma},\lambda,u) - (\boldsymbol{\tau},\mu,v)||.$$

*Proof.* See [10] and [15]. □

**4. The residual-based a-posteriori error estimates.** In this section we recall from [12] and [10] (see also [13]) the reliable a-posteriori error estimates for the discrete schemes (3.1) and (3.2), respectively. To this end, we need to specify some notations. Given a vector-valued function $\boldsymbol{\tau} := (\tau_1, \tau_2)^t$ defined in $\Omega$, an edge $e \in E(T) \cap E_h(\Omega)$, and the unit tangential vector $\mathbf{t}_T$ along $e$, we let $J[\boldsymbol{\tau} \cdot \mathbf{t}_T]$ be the corresponding jump across $e$, that is $J[\boldsymbol{\tau} \cdot \mathbf{t}_T] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e \cdot \mathbf{t}_T$, where $T'$ is the other triangle of $\mathcal{T}_h$ having $e$ as edge. Here, the tangential vector $\mathbf{t}_T$ is given by $(-\nu_2, \nu_1)^t$ where $\boldsymbol{\nu}_T := (\nu_1, \nu_2)^t$ is the unit outward normal to $\partial T$. Also, for each $e \in E_h$ we denote by $\pi_e$ the orthogonal projection from $L^2(e)$ onto $\mathbf{P}_0(e)$, that is $\pi_e(\xi) := \dfrac{1}{h_e}\displaystyle\int_e \xi\, ds \quad \forall\, \xi \in L^2(e)$, for which there exists $c > 0$, independent of $e$, such that the following approximation property holds:

$$||\xi - \pi_e(\xi)||_{L^2(e)} \le c\, h_e\, ||\xi||_{H^1(e)} \qquad \forall\, \xi \in H^1(e).$$

In addition, given vector and scalar functions $\boldsymbol{\tau}$ and $v$, respectively, we let $\mathrm{curl}\,(\boldsymbol{\tau})$ be the scalar $\frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}$, and we denote by $\mathbf{curl}(v)$ the vector $\left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right)^t$.

THEOREM 4.1. *Let* $(\boldsymbol{\sigma}, u, \lambda) \in H(\mathrm{div}\,;\Omega) \times L^2(\Omega) \times H_{00}^{1/2}(\Gamma_N)$ *and* $(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde h}) \in H_h^{\boldsymbol{\sigma}} \times H_h^u \times H_{\tilde h}^\lambda$ *be the unique solutions of the continuous and discrete formulations* (2.2) *and* (3.1)*, respectively, and assume that the Neumann data* $g \in L^2(\Gamma_N)$*. Then there exists* $C_1 > 0$*, independent of* $h$ *and* $\tilde h$*, such that*

$$\left|\left|(\boldsymbol{\sigma},u,\lambda) - (\boldsymbol{\sigma}_h,u_h,\lambda_{\tilde h})\right|\right| \le C_1\, \boldsymbol{\eta} := C_1 \left\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \right\}^{1/2}, \tag{4.1}$$

*where for any triangle* $T \in \mathcal{T}_h$ *we define*

$$\eta_T^2 := ||f + \mathrm{div}\,\boldsymbol{\sigma}_h||_{L^2(T)}^2 + h_T^2\, \left|\left|\mathrm{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\right|\right|_{L^2(T)}^2 + h_T^2\, \left|\left|\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\right|\right|_{[L^2(T)]^2}^2$$

ETNA
Kent State University
etna@mcs.kent.edu



$$+ \sum_{e \in E(T) \cap E_h(\Omega)} h_e \left|\left| J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \right|\right|^2_{L^2(e)} + \sum_{e \in E(T) \cap E_h(\Gamma_D)} h_e \left|\left| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T \right|\right|^2_{L^2(e)}$$

$$+ \log[1 + C_{\tilde{h}}(\Gamma_N)] \sum_{e \in E(T) \cap E_h(\Gamma_N)} \tilde{h}_e \, ||g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{L^2(e)}$$

$$(4.2) + \sum_{e \in E(T) \cap E_h(\Gamma_N)} h_e \left\{ \left|\left| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T + \frac{d\lambda_{\tilde{h}}}{d\mathbf{t}_T} \right|\right|^2_{L^2(e)} + \left|\left| \lambda_{\tilde{h}} - \pi_e(\lambda_{\tilde{h}}) \right|\right|^2_{L^2(e)} \right\},$$

*with*

$$(4.3) \qquad C_{\tilde{h}}(\Gamma_N) := \max \left\{ \frac{|\tilde{e}_i|}{|\tilde{e}_j|} : \quad |i - j| = 1 \,, \quad i, j \in \{1, ..., m\} \right\}.$$

*Proof.* See Theorem 3.1 in [12]. □

We remark here that the first four terms in (4.2) are the standard and well known ones for the mixed finite element method without Lagrange multipliers (see, e.g. [7]). Since we are using Raviart-Thomas subspaces of order zero for $\boldsymbol{\sigma}_h$, we also observe that when $\boldsymbol{\kappa}$ is a piecewise constant diagonal matrix, the second term in the definition of $\eta_T$ vanishes.

THEOREM 4.2. *Let* $(\boldsymbol{\sigma}, \lambda, u) \in H_0(\mathrm{div}\,; \Omega) \times H_0^{1/2}(\Gamma_N) \times L^2(\Omega)$ *and* $(\boldsymbol{\sigma}_h, \lambda_h, u_h) \in H_h^{\boldsymbol{\sigma}} \times H_h^{\lambda} \times H_h^{u}$ *be the unique solutions of the continuous and discrete formulations* (2.7) *and* (3.2)*, respectively. Then there exists* $C_2 > 0$*, independent of* $h$*, such that*

$$(4.4) \qquad ||(\boldsymbol{\sigma}, \lambda, u) \, - \, (\boldsymbol{\sigma}_h, \lambda_h, u_h)|| \, \leq \, C_2 \, \boldsymbol{\theta} \, := \, C_2 \left\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \right\}^{1/2},$$

*where for any triangle* $T \in \mathcal{T}_h$ *we define*

$$\theta_T^2 := ||f + \mathrm{div}\, \boldsymbol{\sigma}_h||^2_{L^2(T)} + h_T^2 \left|\left| \mathrm{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \right|\right|^2_{L^2(T)} + h_T^2 \left|\left| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h \right|\right|^2_{[L^2(T)]^2}$$

$$+ \sum_{e \in E(T) \cap E_h(\Omega)} h_e \left|\left| J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \right|\right|^2_{L^2(e)}$$

$$+ \sum_{e \in E(T) \cap E_h(\Gamma_N)} h_e \left\{ \left\| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T - \frac{d\xi_h}{d\mathbf{t}_T} \right\|^2_{L^2(e)} + ||\boldsymbol{\zeta}_h||^2_{L^2(e)} + ||\xi_h - \pi_e(\xi_h)||^2_{L^2(e)} \right\},$$

(4.5)

*with*

$$\xi_h := \left( \frac{1}{2}\mathbf{I} + \mathbf{K} \right) \lambda_h - \mathbf{V}(\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}) \qquad \textit{and} \qquad \boldsymbol{\zeta}_h := \left( \frac{1}{2}\mathbf{I} + \mathbf{K}' \right) (\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}) + \mathbf{W}\lambda_h \,.$$

*Proof.* See Theorem 3 in [10] or Theorem 4.1 in [13]. □

It is important to remark, as stated in Section 1, that not much details are provided neither in [10] nor in [13] for the efficiency of $\boldsymbol{\theta}$, and the readers are just referred in [10] to the related analysis in [6], [7], and [9]. In particular, it is mentioned that the arguments for quasi-uniform meshes on the boundary given in [6] can also be adopted in this case.

ETNA
Kent State University
etna@mcs.kent.edu



**5. Efficiency of the a-posteriori error estimates.** In this section we give a unified proof for the efficiency of the a-posteriori error estimates $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. In other words, we show the existence of $C_3,\, C_4 > 0$, independent of $h$, such that

$$C_3\,\boldsymbol{\eta} \;\le\; \|(\boldsymbol{\sigma}, u, \lambda) \,-\, (\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}})\| \;+\; \mathrm{h.o.t.}\,, \tag{5.1}$$

and

$$C_4\,\boldsymbol{\theta} \;\le\; \|(\boldsymbol{\sigma}, \lambda, u) \,-\, (\boldsymbol{\sigma}_h, \lambda_h, u_h)\| \;+\; \mathrm{h.o.t.}\,, \tag{5.2}$$

where h.o.t. denotes one or several terms of higher order. Similarly as in [7] and [8], our analysis is based on the localization technique introduced in [18] (see also [1]) and the inverse inequalities in finite element subspaces (see [11]). This procedure is even applied to the terms in $\boldsymbol{\theta}$ involving boundary integral operators.

**5.1. Preliminaries.** We first recall from [17] that given $k \in \mathbb{N} \cup \{0\}$, $T \in \mathcal{T}_h$, and $e \in E(T)$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathbf{P}_k(T)$ and $L(p)\big|_e = p \quad \forall\, p \in \mathbf{P}_k(e)$. In addition, we define $w_e \,:=\, \cup\,\{\, T' \in \mathcal{T}_h : \quad e \in E(T')\,\}$ and let $\psi_T$ and $\psi_e$ be the usual triangle-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [18]), which satisfy $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T \in \mathbf{P}_3(T)$, $\psi_T = 0$ on $\partial T$, $0 \le \psi_T \le 1$ in $T$, $\mathrm{supp}(\psi_e) \subseteq w_e$, $\psi_e|_T \in \mathbf{P}_2(T) \;\; \forall\, T \subseteq w_e$, $\psi_e = 0$ on $\partial T \setminus e$, and $0 \le \psi_e \le 1$ in $w_e$. Additional properties of $\psi_T$, $\psi_e$, and $L$, are collected in the following lemma.

LEMMA 5.1. *There exist positive constants $c_1$, $c_2$, and $c_3$, depending only on $k$ and the shape of the triangles, such that for all $q \in \mathbf{P}_k(T)$ and $p \in \mathbf{P}_k(e)$, there hold*

$$\|\psi_T\, q\|^2_{L^2(T)} \;\le\; \|q\|^2_{L^2(T)} \;\le\; c_1\,\|\psi_T^{1/2}\, q\|^2_{L^2(T)}\,, \tag{5.3}$$

$$\|\psi_e\, L(p)\|^2_{L^2(T)} \;\le\; \|p\|^2_{L^2(e)} \;\le\; c_2\,\|\psi_e^{1/2}\, p\|^2_{L^2(e)}\,, \tag{5.4}$$

$$c_1^{-1}\, h_e\, \|p\|^2_{L^2(e)} \;\le\; \|\psi_e^{1/2}\, L(p)\|^2_{L^2(T)} \;\le\; c_3\, h_e\, \|p\|^2_{L^2(e)}\,. \tag{5.5}$$

*Proof*. See Lemma 1.3 in [17]. □

The following inverse estimate will also be used.

LEMMA 5.2. *Let $l$, $m \in \mathbb{N} \cup \{0\}$ such that $l \le m$. Then there exists a positive constant c, depending only on $k$, $l$, $m$, and the shape of the triangles, such that*

$$|q|_{H^m(T)} \;\le\; c\, h_T^{l-m}\, |q|_{H^l(T)} \qquad \forall\, q \,\in\, \mathbf{P}_k(T)\,. \tag{5.6}$$

*Proof*. See Theorem 3.2.6 in [11]. □

**5.2. Upper bounds for the terms defining $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$.** In this section we bound each one of the terms defining the reliable a-posteriori error estimates $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. To this respect, we observe that the first four terms defining $\boldsymbol{\eta}$ coincide with those of $\boldsymbol{\theta}$, and hence the proofs of the corresponding upper bounds serve for both estimates.

Throughout this section, we assume for simplicity that $(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)|_T$, $(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T|_e$, $g|_e$, $\xi_h|_e$, and $\boldsymbol{\zeta}_h|_e$ are polynomials for each $T \in \mathcal{T}_h$, $e \in E_h$, and $e \in E_h(\Gamma_N)$ (last 3 functions), respectively. Otherwise, additional higher order terms, given by the errors arising

ETNA
Kent State University
etna@mcs.kent.edu



from suitable polynomial approximations, which is guaranteed by the corresponding local regularity of the discrete solutions, will appear in the bounds below.

Since $\mathrm{div}\,\boldsymbol{\sigma} = -f$ in $\Omega$, we note that $\|f + \mathrm{div}\,\boldsymbol{\sigma}_h\|_{L^2(T)} = \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(T)}$, and hence

$$\sum_{T\in\mathcal{T}_h} \|f + \mathrm{div}\,\boldsymbol{\sigma}_h\|^2_{L^2(T)} = \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)} . \tag{5.7}$$

In order to bound other components of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$, we consider a Helmholtz decomposition of the error $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$. In fact, let $z \in H^1(\Omega)$ be the weak solution of the boundary value problem: $-\,\mathrm{div}\,(\boldsymbol{\kappa}\nabla z) = -\,\mathrm{div}\,\boldsymbol{\sigma}_h$ in $\Omega$, $z = u$ on $\partial\Omega$. Since $\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\kappa}\nabla z) = 0$ in $\Omega$, and $\Omega$ is simply connected, it follows that there exists $\varphi \in H^1(\Omega)$, with $\int_\Omega \varphi\,dx = 0$, such that $\boldsymbol{\sigma}_h - \boldsymbol{\kappa}\nabla z = \mathbf{curl}(\varphi)$. Thus, using that $\boldsymbol{\sigma} = \boldsymbol{\kappa}\,\nabla u$ and that $\mathbf{curl}(\varphi)$ is $[L^2(\Omega)]^2$-orthogonal to $\nabla(H^1_0(\Omega))$, we find that

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\kappa}\,\nabla(u - z) - \mathbf{curl}(\varphi) \tag{5.8}$$

and

$$\|\boldsymbol{\kappa}^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{[L^2(\Omega)]^2} = \|\boldsymbol{\kappa}^{-1/2}\nabla(u - z)\|^2_{[L^2(\Omega)]^2} + \|\boldsymbol{\kappa}^{-1/2}\,\mathbf{curl}(\varphi)\|^2_{[L^2(\Omega)]^2} . \tag{5.9}$$

The following three lemmata provide the corresponding upper bounds for the remaining three terms that are common to $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$.

LEMMA 5.3. *There exists $c > 0$, independent of $h$, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2\,\|\,\mathrm{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\|^2_{L^2(T)} \le c\,\|\boldsymbol{\kappa}^{-1/2}\,\mathbf{curl}(\varphi)\|^2_{L^2(T)} . \tag{5.10}$$

*Proof*. See Lemma 6.1 in [7]. □

LEMMA 5.4. *There exists $c > 0$, independent of $h$, such that for each $e \in E_h(\Omega)$ there holds*

$$h_e\,\|\,J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T]\,\|^2_{L^2(e)} \le c\,\|\boldsymbol{\kappa}^{-1/2}\,\mathbf{curl}(\varphi)\|^2_{L^2(w_e)} . \tag{5.11}$$

*Proof*. See Lemma 6.2 in [7]. □

As a consequence of the estimates (5.9) - (5.11), and using the fact that the number of triangles in $w_e$ is bounded, independently of $h$, which follows from the minimum angle condition satisfied by $\mathcal{T}_h$, we deduce that

$$\sum_{T\in\mathcal{T}_h} h_T^2\,\|\,\mathrm{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\|^2_{L^2(T)} + \sum_{e\in E_h(\Omega)} h_e\,\|\,J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T]\,\|^2_{L^2(e)} \le C\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} . \tag{5.12}$$

LEMMA 5.5. *There exists $c > 0$, independent of $h$, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2\,\|\,\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \le c\,\left\{ \|u - u_h\|^2_{L^2(T)} + h_T^2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \right\} . \tag{5.13}$$

**ETNA**
Kent State University
etna@mcs.kent.edu



*Proof*. We adapt the proof of Lemma 6.3 in [7]. In fact, applying the estimate (5.3), using that $\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma} = \nabla u$ and that $\nabla u_h = 0$ on $T$, and integrating by parts, we find that

$$\| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \leq c_1 \|\psi_T^{1/2} (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} = c_1 \int_T \psi_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\, dx$$

$$= c_1 \int_T \psi_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot [\nabla(u - u_h) - \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)]\, dx$$

$$= -c_1 \left\{ \int_T \operatorname{div}(\psi_T(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h))\,(u - u_h)\, dx + \int_T \psi_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\, dx \right\} .$$

Then, Cauchy-Schwarz's inequality, the inverse inequality (5.6), and the estimate (5.3) yield

$$\| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \leq C \left\{ h_T^{-1} \|u - u_h\|_{L^2(T)} + \|\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{[L^2(T)]^2} \right\} \| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|_{[L^2(T)]^2} ,$$

whence

$$\| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \leq C \left\{ h_T^{-1} \|u - u_h\|_{L^2(T)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \right\} ,$$

which implies (5.13) and completes the proof. □

It follows easily from the previous lemma that

$$\sum_{T\in\mathcal{T}_h} h_T^2 \| \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \leq c \left\{ \|u - u_h\|^2_{L^2(\Omega)} + h^2 \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} \right\} . \tag{5.14}$$

The following lemma provides the upper bound for the fifth term defining $\boldsymbol{\eta}$.

LEMMA 5.6. *There exists $c > 0$, independent of $h$, such that for each $e \in E_h(\Gamma_D)$ there holds*

$$h_e \| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T \|^2_{L^2(e)} \leq c \|\boldsymbol{\kappa}^{-1/2} \,\mathbf{curl}(\varphi)\|^2_{L^2(w_e)} . \tag{5.15}$$

*Proof*. Since $u = 0$ on $\Gamma_D$ we observe that $(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}) \cdot \mathbf{t}_T = \nabla u \cdot \mathbf{t}_T = 0$, and hence $(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T = \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{t}_T$ on $\Gamma_D$. Then, denoting $v_e = (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T$ on $e$, applying the estimate (5.4), considering that $L(v_e) = v_e$ on $e$ and that $\psi_e = 0$ on $\partial T \setminus e$, and using Gauss's formula, we find that

$$\| v_e \|^2_{L^2(e)} \leq c_2 \|\psi_e^{1/2} v_e \|^2_{L^2(e)} = c_2 \int_{\partial w_e} \psi_e L(v_e)\, (\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{t}_T)\, ds$$

$$= c_2 \left\{ \int_{w_e} \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{curl}(\psi_e L(v_e))\, dx + \int_{w_e} \psi_e L(v_e) \operatorname{curl}(\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}))\, dx \right\}$$

$$= c_2 \left\{ \int_{w_e} \boldsymbol{\kappa}^{-1} \mathbf{curl}(\varphi) \cdot \mathbf{curl}(\psi_e L(v_e))\, dx + \int_{w_e} \psi_e L(v_e) \operatorname{curl}(\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}))\, dx \right\} ,$$

where the last equality makes use from (5.8) that $\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) = \boldsymbol{\kappa}^{-1}\,\mathbf{curl}(\varphi) - \nabla(u - z)$, and that $\mathbf{curl}(\psi_e L(v_e))$ is $[L^2(\Omega)]^2$-orthogonal to $\nabla(H_0^1(\Omega))$. We also observe, since $\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma} = \nabla u$, that $\operatorname{curl}(\boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})) = \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)$.

ETNA
Kent State University
etna@mcs.kent.edu



Next, applying Cauchy-Schwarz's inequality, Lemma 5.3, the inverse estimate (5.6), and then (5.5), we obtain that

$$\| v_e \|^2_{L^2(e)} \;\le\; C \Big\{ \|\boldsymbol{\kappa}^{-1/2}\, \mathbf{curl}(\varphi)\|_{[L^2(w_e)]^2}\; |\psi_e\, L(v_e)|_{H^1(w_e)}$$

$$+\; \|\psi_e\, L(v_e)\|_{L^2(w_e)}\; \|\,\mathrm{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\|_{L^2(w_e)} \Big\}$$

$$\le\; C\; h_e^{-1}\; \|\psi_e\, L(v_e)\|_{L^2(w_e)}\; \|\boldsymbol{\kappa}^{-1/2}\, \mathbf{curl}(\varphi)\|_{[L^2(w_e)]^2}$$

$$\le\; C\; h_e^{-1/2}\; \|v_e\|_{L^2(e)}\; \|\boldsymbol{\kappa}^{-1/2}\, \mathbf{curl}(\varphi)\|_{[L^2(w_e)]^2}\,, \tag{5.16}$$

whence

$$h_e^{1/2}\; \|\, v_e\, \|_{L^2(e)} \;\le\; C\; \|\boldsymbol{\kappa}^{-1/2}\, \mathbf{curl}(\varphi)\|_{[L^2(w_e)]^2}\,,$$

which yields (5.15) and finishes the proof. □

Applying (5.15), the fact that the number of triangles in $w_e$ is bounded, independently of $h$, and using (5.9), we get

$$\sum_{e\in E_h(\Gamma_D)} h_e\, \|\, (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T\, \|^2_{L^2(e)} \;\le\; C\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2}\,. \tag{5.17}$$

The following two lemmata give the upper bounds for two similar terms appearing in the definitions of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$, respectively.

LEMMA 5.7. *There exists $C > 0$, independent of $h$ and $\tilde{h}$, such that*

$$\sum_{e\in E_h(\Gamma_N)} h_e\, \left\| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T + \frac{d\lambda_{\tilde{h}}}{d\mathbf{t}_T} \right\|^2_{L^2(e)} \le C\, \Big\{ \|\lambda - \lambda_{\tilde{h}}\|^2_{H^{1/2}_{00}(\Gamma_N)} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} \Big\}\,. \tag{5.18}$$

*Proof*. Let us define $v_e := (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T + \dfrac{d\lambda_{\tilde{h}}}{d\mathbf{t}_T}$ on $e \in E_h(\Gamma_N)$. Then, using that $\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma} = \nabla u$ in $\Omega$ and that $u = -\,\lambda$ on $\Gamma_N$, we deduce that $v_e = \dfrac{d}{d\mathbf{t}_T}(\lambda_{\tilde{h}} - \lambda) + \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot \mathbf{t}_T$ on $e$. Hence, applying the estimate (5.4) and the fact that $L(v_e) = v_e$ on $e$, we obtain that

$$h_e\, \|v_e\|^2_{L^2(e)} \;\le\; c_2\, \left\{ h_e \int_e \psi_e v_e\, \frac{d}{d\mathbf{t}_T}(\lambda_{\tilde{h}} - \lambda)\, ds \,+\, h_e \int_e \psi_e\, L(v_e)\, \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot \mathbf{t}_T\, ds \right\}. \tag{5.19}$$

For the second integral on the right hand side of (5.19) we proceed exactly as in the proof of Lemma 5.6 and find (see (5.16)) that

$$h_e \int_e \psi_e\, L(v_e)\, \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot \mathbf{t}_T\, ds \;\le\; C\, h_e^{1/2}\; \|v_e\|_{L^2(e)}\; \|\boldsymbol{\kappa}^{-1/2}\, \mathbf{curl}(\varphi)\|_{[L^2(w_e)]^2}\,,$$

which, using Cauchy-Schwarz's inequality, the fact that the number of triangles in $w_e$ is bounded, independently of $h$, and equation (5.9), implies that

$$\sum_{e\in E_h(\Gamma_N)} h_e \int_e \psi_e L(v_e) \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot \mathbf{t}_T\, ds \le C \left\{ \sum_{e\in E_h(\Gamma_N)} h_e \|v_e\|^2_{L^2(e)} \right\}^{1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2}. \tag{5.20}$$

On the other hand, let us define $\psi := h_e\,\psi_e\,v_e$ on each $e \in E_h(\Gamma_N)$, and introduce the trivial extensions $\hat{\psi} := \begin{cases} \psi & \text{on} \quad \Gamma_N \\ 0 & \text{on} \quad \Gamma_D \end{cases}$ and $\hat{\lambda} := \begin{cases} \lambda_{\tilde{h}} - \lambda & \text{on} \quad \Gamma_N \\ 0 & \text{on} \quad \Gamma_D \end{cases}$. Then, it is easy to see that $\hat{\psi}$ and $\hat{\lambda}$ belong to $H^{1/2}(\Gamma)$, and that $\|\hat{\psi}\|_{H^{1/2}(\Gamma)}$ and $\|\hat{\lambda}\|_{H^{1/2}(\Gamma)}$ are equivalent to $\|\psi\|_{H^{1/2}_{00}(\Gamma_N)}$ and $\|\lambda_{\tilde{h}} - \lambda\|_{H^{1/2}_{00}(\Gamma_N)}$, respectively.

According to the above notations, using an inverse inequality for the piecewise polynomial $\hat{\psi}$, applying the boundedness of the tangential derivative (as an operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$), and noting that $0 \le \psi_e \le 1$ and that $h_e \le h$, we get

$$\sum_{e\in E_h(\Gamma_N)} h_e \int_e \psi_e v_e \,\frac{d}{d\mathbf{t}_T}(\lambda_{\tilde{h}} - \lambda)\,ds \;=\; \int_\Gamma \hat{\psi}\,\frac{d\hat{\lambda}}{d\mathbf{t}_T}\,ds \;\le\; \|\hat{\psi}\|_{H^{1/2}(\Gamma)}\;\left\|\frac{d\hat{\lambda}}{d\mathbf{t}_T}\right\|_{H^{-1/2}(\Gamma)}$$

$$\le\; C\,h^{-1/2}\,\|\hat{\psi}\|_{L^2(\Gamma)}\;\|\hat{\lambda}\|_{H^{1/2}(\Gamma)} \;\le\; C\,h^{-1/2}\,\|\hat{\psi}\|_{L^2(\Gamma)}\;\|\hat{\lambda}_{\tilde{h}} - \lambda\|_{H^{1/2}_{00}(\Gamma_N)}$$

$$\le\; C\,\left\{\sum_{e\in E_h(\Gamma_N)} h_e\,\|v_e\|^2_{L^2(e)}\right\}^{1/2}\;\|\hat{\lambda}_{\tilde{h}} - \lambda\|_{H^{1/2}_{00}(\Gamma_N)}\,. \tag{5.21}$$

Finally, it is not difficult to see that (5.18) follows from (5.19), (5.20), and (5.21). ◻

LEMMA 5.8. *Let* $\xi_h := (\frac{1}{2}\mathbf{I} + \mathbf{K})\lambda_h - \mathbf{V}(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$. *Then there exists* $C > 0$, *independent of* $h$, *such that*

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\left\|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T - \frac{d\xi_h}{d\mathbf{t}_T}\right\|^2_{L^2(e)} \le\; C\,\left\{\|\lambda - \lambda_h\|^2_{H^{1/2}(\Gamma_N)} \;+\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{H(\mathrm{div}\,;\Omega)}\right\}. \tag{5.22}$$

*Proof*. We define now $v_e := (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T - \dfrac{d\xi_h}{d\mathbf{t}_T}$ on $e \in E_h(\Gamma_N)$. Since $\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma} = \nabla u$ in $\Omega$ and $u = \lambda = (\frac{1}{2}\mathbf{I} + \mathbf{K})\lambda - \mathbf{V}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu}) + d$ on $\Gamma_N$ (cf. (2.5)), we deduce that $v_e = \dfrac{d}{d\mathbf{t}_T}(\xi - \xi_h) + \boldsymbol{\kappa}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{t}_T$ on $e$, where $\xi := (\frac{1}{2}\mathbf{I} + \mathbf{K})\lambda - \mathbf{V}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})$. Thus, using the same arguments of the proof of Lemma 5.7, one can show that

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\left\|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T - \frac{d\xi_h}{d\mathbf{t}_T}\right\|^2_{L^2(e)} \;\le\; C\,\left\{\|\xi - \xi_h\|^2_{H^{1/2}(\Gamma_N)} \;+\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2}\right\}.$$

Next, replacing $\xi_h$ by $(\frac{1}{2}\mathbf{I} + \mathbf{K})\lambda_h - \mathbf{V}(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$, and applying the continuity properties of the boundary integral operators $\mathbf{K}$ and $\mathbf{V}$, we find that

$$\|\xi - \xi_h\|^2_{H^{1/2}(\Gamma_N)} \;\le\; c\,\left\{\|\lambda - \lambda_h\|^2_{H^{1/2}(\Gamma_N)} \;+\; \|\boldsymbol{\sigma}\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma_N)}\right\}, \tag{5.23}$$

which completes the proof. ◻

The upper bounds for the remaining two terms defining $\boldsymbol{\eta}$ are given in the following two lemmata.

ETNA
Kent State University
etna@mcs.kent.edu



LEMMA 5.9. *There exists $c > 0$, independent of $h$ and $\tilde{h}$, such that for each $e \in E_h(\Gamma_N)$ there holds*

$$\tilde{h}_e \, \|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|^2_{L^2(e)} \;\le\; c \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(w_e)]^2} \,+\, h^2 \,\|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L^2(w_e)} \right\} . \tag{5.24}$$

*Proof*. We adapt the proof of Lemma 6.5 in [8]. Applying the estimate (5.4) and Gauss's formula, noting that $\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = g$ on $\Gamma_N$ and that $L(g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}) = (g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})$ on $e$, we deduce that

$$\|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|^2_{L^2(e)} \;\le\; c_2 \, \|\psi_e^{1/2} \, (g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})\|^2_{L^2(e)} \;=\; c_2 \int_e \psi_e \, (g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})^2 \, ds$$

$$= c_2 \left\{ \int_T \mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \, \psi_e \, L(g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}) \, dx \,+\, \int_T (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \nabla(\psi_e \, L(g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})) \, dx \right\} ,$$

where $T$ is any triangle contained in $w_e$. Next, Cauchy-Schwarz's inequality, estimate (5.5), and inverse inequality (5.6), give

$$\|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|^2_{L^2(e)} \;\le\; C \Big\{ \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(T)} \, \|\psi_e^{1/2} L(g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})\|_{L^2(T)}$$

$$+ \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \, |\psi_e L(g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})|_{H^1(T)} \Big\}$$

$$\le \, C \left\{ h_e^{1/2} \, \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(T)} \,+\, h_T^{-1} \, h_e^{1/2} \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \right\} \, \|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|_{L^2(e)} \, ,$$

which, using that $h_e \le h_T$, yields

$$\|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|_{L^2(e)} \;\le\; C \left\{ h_e^{1/2} \, \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(T)} \,+\, h_T^{-1/2} \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \right\} .$$

The above inequality and the fact that $h_e \,\le\, \tilde{h}_e \,\le\, \tilde{h} \,\le\, C\,h$, imply (5.24) and complete the proof of the lemma. ☐

As a consequence of the estimate (5.24), and using that the independent partition of $\Gamma_N$ is uniformly regular, we deduce the existence of $C > 0$, independent of $h$ and $\tilde{h}$, such that

$$\log[1 + C_{\tilde{h}}(\Gamma_N)] \sum_{e \in E(T) \cap E_h(\Gamma_N)} \tilde{h}_e \, \|g - \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\|^2_{L^2(e)}$$

$$\le \, C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} \,+\, h^2 \, \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)} \right\} . \tag{5.25}$$

LEMMA 5.10. *Assume that $\lambda$ is locally smooth, say $\lambda|_e \,\in\, H^1(e) \quad \forall\, e \,\in\, E_h(\Gamma_N)$. Then there exists $c > 0$, independent of $h$ and $\tilde{h}$, such that for each $e \in E_h(\Gamma_N)$ there holds*

$$\|\lambda_{\tilde{h}} - \pi_e(\lambda_{\tilde{h}})\|^2_{L^2(e)} \;\le\; 2 \, \|\lambda - \lambda_{\tilde{h}}\|^2_{L^2(e)} \,+\, c \, h_e^2 \, \|\lambda\|^2_{H^1(e)} \, . \tag{5.26}$$

*Proof*. Applying the approximation property satisfied by $\pi_e$, we easily obtain

$$\|\lambda_{\tilde{h}} - \pi_e(\lambda_{\tilde{h}})\|^2_{L^2(e)} \;\le\; \|\lambda_{\tilde{h}} - \pi_e(\lambda)\|^2_{L^2(e)} \;\le\; 2 \, \|\lambda - \lambda_{\tilde{h}}\|^2_{L^2(e)} \,+\, 2 \, \|\lambda - \pi_e(\lambda)\|^2_{L^2(e)}$$

$$\leq 2\,\|\lambda - \lambda_{\tilde{h}}\|^2_{L^2(e)} \,+\, c\, h_e^2\, \|\lambda\|^2_{H^1(e)}\,,$$

which finishes the proof. ☐

The estimate (5.26) yields the existence of $C > 0$, independent of $h$ and $\tilde{h}$, such that

$$\begin{aligned} &\sum_{e\in E_h(\Gamma_N)} h_e\,\|\lambda_{\tilde{h}} - \pi_e(\lambda_{\tilde{h}})\|^2_{L^2(e)} \\ &\leq C\left\{ h\,\|\lambda - \lambda_{\tilde{h}}\|^2_{H^{1/2}_{00}(\Gamma_N)} \,+\, h^3 \sum_{e\in E_h(\Gamma_N)} \|\lambda\|^2_{H^1(e)} \right\}. \end{aligned} \tag{5.27}$$

In this way, the efficiency of $\boldsymbol{\eta}$ (cf. (5.1)) is obtained from the estimates (5.7), (5.12), (5.14), (5.17), (5.18), (5.25), and (5.27). In particular, we conclude from (5.27) that the corresponding h.o.t. is of $O(h^{3/2})$.

The following two lemmata provide the upper bounds for the remaining terms defining $\boldsymbol{\theta}$.

LEMMA 5.11. *Let* $\boldsymbol{\zeta}_h := (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}) + \mathbf{W}\lambda_h$. *Then there exists* $C > 0$, *independent of* $h$, *such that*

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\|\boldsymbol{\zeta}_h\|^2_{L^2(e)} \;\leq\; C\left\{ \|\boldsymbol{\sigma}\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma_N)} \,+\, \|\lambda - \lambda_h\|^2_{H^{1/2}(\Gamma_N)} \right\}. \tag{5.28}$$

*Proof*. We first recall from (2.6) that $(\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}\cdot\boldsymbol{\nu}) + \mathbf{W}\lambda = 0$ on $\Gamma_N$. Hence, including this null expression into the definition of $\boldsymbol{\zeta}_h$, and then applying the estimate (5.4), we find that

$$h_e\,\|\boldsymbol{\zeta}_h\|^2_{L^2(e)} \;\leq\; c_2\, h_e\, \|\psi_e^{1/2}\,\boldsymbol{\zeta}_h\|^2_{L^2(e)}$$

$$= c_2\, h_e \int_e \psi_e\,\boldsymbol{\zeta}_h \left\{ (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}\cdot\boldsymbol{\nu}) \,+\, \mathbf{W}(\lambda_h - \lambda) \right\} ds\,.$$

Then, we let $\psi := h_e\,\psi_e\,\boldsymbol{\zeta}_h$ on each $e \in E_h(\Gamma_N)$, and observe that $\psi \in H^{1/2}(\Gamma_N)$. Hence, using an inverse inequality for the piecewise polynomial $\psi$, and applying the continuity properties of the boundary integral operators $\mathbf{K}'$ and $\mathbf{W}$, we deduce that

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\|\boldsymbol{\zeta}_h\|^2_{L^2(e)} \;\leq\; c_2 \left\langle \psi, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}\cdot\boldsymbol{\nu}) \,+\, \mathbf{W}(\lambda_h - \lambda) \right\rangle$$

$$\leq C\,\|\psi\|_{H^{1/2}(\Gamma_N)} \left\| \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}\cdot\boldsymbol{\nu}) \,+\, \mathbf{W}(\lambda_h - \lambda) \right\|_{H^{-1/2}(\Gamma_N)}$$

$$\leq\; C\,h^{-1/2}\,\|\psi\|_{L^2(\Gamma_N)} \left\{ \|\boldsymbol{\sigma}\cdot\boldsymbol{\nu} - \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}\|_{H^{-1/2}(\Gamma_N)} \,+\, \|\lambda - \lambda_h\|_{H^{1/2}(\Gamma_N)} \right\},$$

which, noting that $h_e \leq h$ and $0 \leq \psi_e \leq 1$, yields (5.28) and finishes the proof. ☐

LEMMA 5.12. *Assume that* $\lambda$ *is locally smooth, say* $\lambda|_e \in H^1(e) \quad \forall\, e \in E_h(\Gamma_N)$, *and let* $\xi_h := (\frac{1}{2}\mathbf{I} + \mathbf{K})\lambda_h - \mathbf{V}(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$. *Then there exists* $C > 0$, *independent of* $h$, *such*

**ETNA**
Kent State University
etna@mcs.kent.edu



*that*

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\|\xi_h-\pi_e(\xi_h)\|^2_{L^2(e)}$$

$$\leq\; C\left\{h\,\|\lambda-\lambda_h\|^2_{H^{1/2}(\Gamma_N)}\,+\,h\,\|\boldsymbol{\sigma}\cdot\boldsymbol{\nu}-\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma_N)}\,+\,h^3\sum_{e\in E_h(\Gamma_N)}\|\lambda\|^2_{H^1(e)}\right\}. \tag{5.29}$$

*Proof*. As in Lemma 5.8, we let $\xi\;:=\;(\frac{1}{2}\mathbf{I}+\mathbf{K})\lambda-\mathbf{V}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})$ on $\Gamma_N$, and note from (2.5) that $\lambda-\pi_e(\lambda)=\xi-\pi_e(\xi)$. Then, using the approximation property of $\pi_e$, we get

$$h_e\,\|\xi_h-\pi_e(\xi_h)\|^2_{L^2(e)}\;\leq\;h_e\,\|\xi_h-\pi_e(\xi)\|^2_{L^2(e)}$$

$$\leq\;2\,h_e\left\{\|\xi_h-\xi\|^2_{L^2(e)}+\|\lambda-\pi_e(\lambda)\|^2_{L^2(e)}\right\}\;\leq\;c\left\{h_e\,\|\xi_h-\xi\|^2_{L^2(e)}\,+\,h_e^3\,\|\lambda\|^2_{H^1(e)}\right\},$$

and hence

$$\sum_{e\in E_h(\Gamma_N)} h_e\,\|\xi_h-\pi_e(\xi_h)\|^2_{L^2(e)}\;\leq\;C\left\{h\,\|\xi_h-\xi\|^2_{H^{1/2}(\Gamma_N)}\,+\,h^3\sum_{e\in E_h(\Gamma_N)}\|\lambda\|^2_{H^1(e)}\right\},$$

which together with (5.23) complete the proof. □

Consequently, the efficiency of $\boldsymbol{\theta}$ (cf. (5.2)) is obtained from the estimates (5.7), (5.12), (5.14), (5.22), (5.28), and (5.29). In particular, we conclude from (5.29) that the corresponding h.o.t. is of $O(h^{3/2})$.

## REFERENCES

[1] A. ALONSO, *Error estimators for a mixed method*, Numer. Math., 74 (1996), pp. 385–395.

[2] I. BABUŠKA AND G. N. GATICA, *On the mixed finite element method with Lagrange multipliers*, Numer. Methods Partial Differential Equations, 19 (2003), no. 2, pp. 192–210.

[3] M. A. BARRIENTOS, G. N. GATICA, AND M. MAISCHAK, *A-posteriori error estimates for linear exterior problems via mixed-FEM and DtN mappings*, Math. Model. Numer. Anal., 36 (2002), no. 2, pp. 241–272.

[4] D. BRAESS AND R. VERFÜRTH, *A posteriori error estimators for the Raviart-Thomas element*, SIAM J. Numer. Anal., 33 (1996), pp. 2431–2444.

[5] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[6] C. CARSTENSEN, *Efficiency of a posteriori BEM error estimates for first kind integral equations on uniform meshes*, Math. Comp., 65 (1996), pp. 69-84.

[7] C. CARSTENSEN, *A posteriori error estimate for the mixed finite element method*, Math. Comp., 66 (1997), no. 218, pp. 465–476.

[8] C. CARSTENSEN AND G. DOLZMANN, *A posteriori error estimates for mixed FEM in elasticity*, Numer. Math., 81 (1998), pp. 187–209.

[9] C. CARSTENSEN AND S. A. FUNKEN, *Coupling of nonconforming finite elements and boundary elements II: a posteriori estimates and adaptive mesh-refinement*, Computing, 62 (1999), pp. 243–259.

[10] C. CARSTENSEN AND S. A. FUNKEN, *Coupling of mixed finite elements and boundary elements*, IMA J. Numer. Anal., 20 (2000), pp. 461–480.

[11] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[12] G. N. GATICA AND M. MAISCHAK, *A-posteriori error estimates for the mixed finite element method with Lagrange multipliers*, Preprint, 2002-18, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, 2002.

[13] G. N. GATICA AND S. MEDDAHI, *An a-posteriori error estimate for the coupling of BEM and mixed-FEM*, Numer. Funct. Anal. Optim., 20 (1999), no. 5&6, pp. 449–472.

[14] R. H. W. HOPPE AND B. I. WOHLMUTH, *A comparison of a posteriori error estimators for mixed finite element discretizations by Raviart-Thomas elements*, Math. Comp. 68 (1999), no. 228, pp. 1347–1378.

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



[15] S. MEDDAHI, J. VALDÉS, O. MENÉNDEZ, AND P. PÉREZ, *On the coupling of boundary integral and mixed finite element methods*, J. Comput. Appl. Math., 69 (1996), pp. 113–124.
[16] R. VERFÜRTH, *A posteriori error estimators for the Stokes problem*, Numer. Math., 55 (1989), pp. 309–325.
[17] R. VERFÜRTH, *A posteriori error estimates and adaptive mesh-refinement techniques*, J. Comput. Appl. Math., 50 (1994), pp. 67–83.
[18] R. VERFÜRTH, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.