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ASYMPTOTIC LOWER BOUNDS FOR EIGENVALUES BY NONCONFORMING FINITE ELEMENT METHODS*

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Abstract. We analyze the approximation obtained for the eigenvalues of the Laplace operator by the nonconforming piecewise linear finite element of Crouzeix-Raviart. For singular eigenfunctions, as those arising in nonconvex polygons, we prove that the eigenvalues obtained with this method give lower bounds of the exact eigenvalues when the mesh size is small enough.

Key words. finite elements, eigenvalue problems, nonconforming methods.

AMS subject classifications. 65N25, 65N30.

1. Introduction. For second order elliptic problems it is known that the eigenvalues computed using the standard conforming finite element method are always above the exact ones. Indeed this can be proved using the minimum-maximum characterization of the eigenvalues (see, for example, [4]). Therefore, it is an interesting problem to find methods which give lower bounds of the eigenvalues. However, as far as we know, only few results in this direction have been obtained and mainly for finite difference methods. Forsythe proved that the eigenvalue approximation obtained by standard five points finite differences is below the eigenvalue of the continuous problem, when the mesh-size is small enough, for some particular domains and smooth enough eigenfunctions (see [7], [8]). Since that finite difference method coincides with the standard piecewise linear finite elements with mass lumping on uniform meshes, one could expect that similar results hold for more general meshes. Although this has not been proved, several numerical experiments suggest that it is true (see [2]). On the other hand, Weinberger proved that lower bounds can be obtained applying finite differences on a domain slightly larger than the original one (see [12], [13]). However, the approximations obtained in this way are of lower order than those given by Forsythe.

In view of these results, a natural question to ask is whether it is possible to find a method which gives lower bounds, at least asymptotically, for eigenvalues corresponding to nonsmooth eigenfunctions. It seems reasonable to look among nonconforming methods. Indeed, if the finite element space is not contained in the Hilbert space where the continuous variational problem is formulated, one can not know in advance whether the computed eigenvalues are below or above the exact ones.

In this note we analyze the approximations obtained using the nonconforming piecewise linear finite element of Crouzeix-Raviart for the Laplace equation. We prove that, when the exact eigenfunction is singular, the eigenvalues computed with this method using quasi-uniform meshes are smaller than the exact ones for small enough mesh-size.

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We end the paper with some numerical examples which suggest that the sequence of eigenvalue approximations obtained by uniform refinement of an initial mesh is monotone increasing. In particular the numerical experiments show that, although our results are of an asymptotic character, the Crouzeix-Raviart method gives lower bounds for eigenvalues corresponding to singular eigenfunctions even with coarse meshes, which would be a reasonable starting point for an adaptive procedure.

2. The Eigenvalue Problem. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We consider the following eigenvalue problem:

$$
\begin{aligned}
-\Delta u &= \lambda u \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
\tag{2.1}
$$

We denote by $(\,,\,)$ the usual inner product in $L^2(\Omega)$. We will also use the standard notation for L^p based Sobolev spaces, namely, $W^{m,p}(\Omega)$ is the space of functions in $L^p(\Omega)$ such that all its derivatives up to the order m are in $L^p(\Omega)$ and for $p = 2$ we write $H^m(\Omega) = W^{m,2}(\Omega)$.

The variational problem associated with (2.1) is given by: Find λ and $u \in H_0^1(\Omega)$, $u \neq 0$, satisfying

$$
\begin{aligned}
a(u, v) &= \lambda(u, v) \quad \forall v \in H_0^1(\Omega), \\
\|u\|_{L^2(\Omega)} &= 1,
\end{aligned}
\tag{2.2}
$$

where $a(u, v) = \int_\Omega \nabla u \nabla v$, which is continuous on $H^1(\Omega)$ and coercive on $H_0^1(\Omega)$.

It is well-known that the solution of this problem is given by a sequence of pairs (λ_j, u_j), with positive eigenvalues λ_j diverging to $+\infty$. We assume the eigenvalues to be increasingly ordered: $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$. The associated eigenfunctions u_j belong to the Besov space $B_2^{1+r,\infty}(\Omega)$, and in particular to the Sobolev space $H^{1+r-\varepsilon}(\Omega)$ for $\varepsilon > 0$ (see, for example, [4] for the definition of these spaces), where $r = 1$ if Ω is convex and $r = \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω) otherwise (see [3]).

The approximations of the eigenvalue λ and its associated eigenfunction u are obatined as follows:

Let $\{\mathcal{T}_h\}$ be a triangulation of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge and let h be the mesh-size; namely, $h = \max_{T \in \mathcal{T}_h} h_T$, with h_T being the diameter of the triangle T. We suppose that the family of triangulations $\mathcal{T}_h$ satisfies the usual shape regularity condition, i.e, there exists a constant $\sigma > 0$ such that $\frac{h_T}{\rho_T} \leq \sigma$, where ρ_T is the diameter of the largest ball contained in T.

Let V_h be the nonconforming piecewise linear finite element space of Crouzeix-Raviart given by:

$$
\begin{aligned}
V_h = \{v :\ v|_T \in \mathcal{P}_1 &\text{ is continuous in the midpoints of the edges of } T \quad \forall T \in \mathcal{T}_h \\
&\text{and } v = 0 \text{ at the midpoints on } \partial\Omega\},
\end{aligned}
$$

where $\mathcal{P}_1$ denotes the space of polynomials of degree less than or equal to 1.

Since $V_h \not\subseteq H_0^1(\Omega)$ we define the following bilinear form on $V_h + H_0^1(\Omega)$

$$
a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \nabla v_h.
$$

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So, the nonconforming approximation problem is given by: Find λ_h and $u_h \in V_h$, $u_h \neq 0$, such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h) \qquad \forall v_h \in V_h, \tag{2.3}$$
$$\|u_h\|_{L^2(\Omega)} = 1.$$

It is well-known that the form $a_h(\cdot,\cdot)$ is positive-definite on V_h (see, for example, [5]). Therefore, the approximation problem reduces to a generalized eigenvalue problem involving positive definite symmetric matrices. It attains a finite number of eigenpairs $(\lambda_{h,j}, u_{h,j})_{1\leq j\leq N_h}$, $N_h = \dim V_h$, with positive eigenvalues which we assume increasingly ordered: $\lambda_{h,1} \leq \cdots \leq \lambda_{h,N_h}$.

In order to obtain an expression for the difference between λ_j and its nonconforming approximation $\lambda_{h,j}$, we will use the "edge average" interpolant $I_h : H_0^1(\Omega) \to V_h$ defined as follows:

For any $u \in H_0^1(\Omega)$, $I_h(u) \in V_h$ is given by

$$\int_\ell I_h(u) = \int_\ell u \quad \forall \ell, \tag{2.4}$$

where ℓ denotes any edge of any triangle $T \in \mathcal{T}_h$.

In the next lemma we give some error estimates for this interpolation which will be used in our subsequent analysis.

LEMMA 2.1. *There exists a constant C independent of h and u such that,*

$$\|u - I_h(u)\|_{L^2(\Omega)} \leq Ch^m \|u\|_{H^m(\Omega)} \qquad \textit{for } m = 1, 2, \tag{2.5}$$

$$\|u - I_h(u)\|_{L^2(\Omega)} \leq Ch^{1+r} \|u\|_{B_2^{1+r,\infty}(\Omega)} \qquad \textit{for } 0 < r < 1, \tag{2.6}$$

$$\|u - I_h(u)\|_{L^1(\Omega)} \leq Ch^2 \|u\|_{W^{2,1}(\Omega)}, \tag{2.7}$$

and,

$$\|I_h(u)\|_{L^\infty(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}. \tag{2.8}$$

Proof. From the definition of I_h we have that for any constant vector $\mathbf{k} \in \mathbb{R}^2$

$$\int_T \nabla(u - I_h(u)) \cdot \mathbf{k} = \int_{\partial T} (u - I_h(u))\mathbf{k} \cdot n = 0. \tag{2.9}$$

In particular, for $j = 1, 2$, $\frac{\partial I_h(u)}{\partial x_j}$ is the average of $\frac{\partial u}{\partial x_j}$ on each $T \in \mathcal{T}_h$, and therefore, it follows from the well-known Poincaré inequality that

$$\|\nabla u - \nabla I_h(u)\|_{L^2(T)} \leq Ch^{m-1} \|u\|_{H^m(T)}, \qquad m = 1, 2. \tag{2.10}$$

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Since $u - I_h(u)$ has vanishing mean value on the sides ℓ of T, it follows from a Poincaré type inequality for this class of functions (see, for example, [1]) that

$$\|u - I_h(u)\|_{L^2(T)} \leq Ch\|\nabla u - \nabla I_h(u)\|_{L^2(T)}.$$

Combining this estimate with (2.10), and summing up the squares of the norms over all the triangles, we obtain (2.5). Now, (2.6) follows by interpolation of Banach spaces in view of the definition of the Besov spaces (see [4]).

The estimate (2.7) can be proved exactly with the same arguments used above applied now to L^1.

Finally, in order to prove (2.8), we recall that the basis function of the Crouzeix-Raviart elements associated with the midpoint of a side ℓ can be written as $\lambda_1 + \lambda_2 - \lambda_3$, where λ_1 and λ_2 are the barycentric coordinates corresponding to the vertices of ℓ and λ_3 is that corresponding to the opposite vertex. Therefore, the absolute value of any basis function is bounded by 2. Then, (2.8) follows immediately from the fact that the absolute value of the degrees of freedom defining $I_h(u)$ (see (2.4)) are bounded by $\|u\|_{L^\infty(\Omega)}$. $\square$

In what follows we will use the notation $\|.\|_h$ for the norm associated with a_h, namely,

$$\|v\|_h = \sqrt{a_h(v,v)}.$$

The next lemma gives a relation between the errors in the eigenvalue and eigenfunction approximations. We will use the following relation which follows from property (2.9):

$$a_h(I_h(u), v) = a_h(u, v) \quad \forall v \in V_h. \tag{2.11}$$

LEMMA 2.2. *Let (λ_j, u_j) and $(\lambda_{h,j}, u_{h,j})$ be the solutions of problems (2.2) and (2.3), respectively. Then we have*

$$\begin{aligned} \lambda_j - \lambda_{h,j} = \|u_j - u_{h,j}\|_h^2 - \lambda_{h,j}\|I_h(u_j) - u_{h,j}\|_{L^2(\Omega)}^2 \\ + \lambda_{h,j}\left(\|I_h(u_j)\|_{L^2(\Omega)}^2 - \|u_j\|_{L^2(\Omega)}^2\right). \end{aligned} \tag{2.12}$$

Proof. Using (2.11) and the fact that $\|u_{h,j}\|_{L^2(\Omega)} = \|u_j\|_{L^2(\Omega)} = 1$, we have

$$\begin{aligned} \lambda_j + \lambda_{h,j} &= a_h(u_j - u_{h,j}, u_j - u_{h,j}) + 2a_h(u_j, u_{h,j}) \\ &= a_h(u_j - u_{h,j}, u_j - u_{h,j}) + 2a_h(I_h(u_j), u_{h,j}) \\ &= \|u_j - u_{h,j}\|_h^2 + 2\lambda_{h,j}(I_h(u_j), u_{h,j}) \\ &= \|u_j - u_{h,j}\|_h^2 - \lambda_{h,j}\|I_h(u_j) - u_{h,j}\|_{L^2(\Omega)}^2 + \lambda_{h,j}\|u_{h,j}\|_{L^2(\Omega)}^2 + \lambda_{h,j}\|I_h(u_j)\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore,

$$\lambda_j + \lambda_{h,j} = \|u_j - u_{h,j}\|_h^2 - \lambda_{h,j}\|I_h(u_j) - u_{h,j}\|_{L^2(\Omega)}^2 + 2\lambda_{h,j} + \lambda_{h,j}(\|I_h(u_j)\|_{L^2(\Omega)}^2 - \|u_j\|_{L^2(\Omega)}^2),$$

and (2.12) follows. ☐

As mentioned above, when Ω is not convex, the eigenfunctions of problem (2.2) are usually singular.

We will prove that in the singular case the approximation given by the nonconforming method (2.3) is below the corresponding exact eigenvalue given by (2.2), i.e., $\lambda_{h,j} \leq \lambda_j$, $1 \leq j \leq N_h$, for h small enough.

We will make use of error estimates for the approximation of spectral problems by the nonconforming elements of Crouzeix-Raviart. These estimates follow from the general theory and have been obtained in [6]. In particular, it is known that there exists a constant C, which depends on u_j and λ_j but is independent of h, such that,

$$\|u_{h,j} - u_j\|_h \leq Ch^r,$$

$$\|u_{h,j} - u_j\|_{L^2(\Omega)} \leq Ch^{2r}, \tag{2.13}$$

with $r = \frac{\pi}{\omega}$ where ω is the maximum angle of Ω.

In [3] the possible singularities in the solution of the Dirichlet problem on polygonal domains are characterized exactly in terms of the angles of the domain. Assume that $\omega > \pi$ and, for simplicity, that the other angles are strictly smaller than ω (see [3] for the more general case). It follows from the results of [3] that the solution of problem (2.2) can be written, in polar coordinates (ρ, θ) centered at the point corresponding to the angle ω, as $u_j = k\rho^{\frac{\pi}{\omega}}\phi(\theta) + v$, where k is a constant, ϕ is a smooth function, and v is a function smoother than the first term. Moreover, it is also proved in [3] that $\rho^{\frac{\pi}{\omega}}\phi(\theta) \in B_2^{1+r,\infty}(\Omega) \setminus B_2^{1+s,\infty}(\Omega)$ for any $s > r$.

From this regularity result it follows that u_j can be approximated in the $\| \,.\, \|_h$ norm by functions in V_h with order h^r and in particular the error estimates (2.13) can be obtained.

On the other hand, in [14] and [3] inverse type results were proved which say that, whenever a function is approximated in the H^1 norm with order h^s by finite element functions on a suitable family of meshes, then the function is in $B_2^{1+s,\infty}(\Omega)$. The arguments of [14] can be extended to the nonconforming case considered here to show that if a function u is approximated with order h^s by functions in V_h for an appropriate family of meshes, then the function is in $B_2^{1+s,\infty}(\Omega)$. Therefore, whenever the constant k is different from 0 (i.e., the solution u_j is singular), which is usually (although not always) the case in practice, it is natural to assume that $\|u_{h,j} - u_j\|_h \geq ch^r$, and this is the assumption that we make in the following theorem, which gives the main result of this paper.

THEOREM 2.3. *Let λ_j and $\lambda_{h,j}$ be the eigenvalues of problems (2.2) and (2.3), respectively. If $u_j \in B_2^{1+r,\infty}(\Omega)$ and there exists a constant c such that $\|u_{h,j} - u_j\|_h \geq ch^r$, with $r < 1$, then, for h small enough, we have that*

$$\lambda_{h,j} \leq \lambda_j. \tag{2.14}$$

Proof. From Lemma (2.2) we know that

$$\begin{aligned}\lambda_j - \lambda_{h,j} = \|u_{h,j} - u_j\|_h^2 - \lambda_{h,j}\|I_h(u_j) - u_{h,j}\|^2_{L^2(\Omega)}\\ + \lambda_{h,j}\left(\|I_h(u_j)\|^2_{L^2(\Omega)} - \|u_j\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{2.15}$$

Since $u_j \in B_2^{1+r,\infty}(\Omega)$, we know from (2.6) that

$$\|I_h(u_j) - u_j\|_{L^2(\Omega)} \le Ch^{1+r},$$

and therefore, by (2.13), we conclude that

$$\|I_h(u_j) - u_{h,j}\|_{L^2(\Omega)} \le Ch^{2r}.$$

Consider now the third term of (2.15). Using (2.8) we have

$$\left|\|I_h(u_j)\|^2_{L^2(\Omega)} - \|u_j\|^2_{L^2(\Omega)}\right| \le \int_\Omega |I_h(u_j) - u_j||I_h(u_j) + u_j| \le C\|u_j\|_{L^\infty(\Omega)}\|I_h(u_j) - u_j\|_{L^1(\Omega)}.$$

Now, from known a priori estimates for elliptic problems on polygonal domains (see, for example, [10]) it follows that $\|u_j\|_{2,p} \le C\lambda_j\|u_j\|_{0,p}$ for some $p > 1$. In particular, we have that, for any polygonal domain, $u_j \in W^{2,1}(\Omega)$. Then, using now (2.7) , we obtain that

$$\left|\|I_h(u_j)\|^2_{L^2(\Omega)} - \|u_j\|^2_{L^2(\Omega)}\right| \le Ch^2,$$

with C depending on u_j but independent of h.

From our hypothesis, the first term on the right hand side of (2.15) is greater than a constant times h^{2r}. So, the second and third terms are of higher order (h^{4r} and h^2, respectively). Therefore, if h is small enough, the sign of $\lambda_j - \lambda_{h,j}$ is given by the first term in (2.15), so, we conclude the proof. □

3. Numerical Examples. In this section we present the numerical approximations of the first eigenvalue of problem (2.2) for different domains Ω. In all the examples the corresponding eigenfunction is known to be singular and the hypotheses of Theorem 2.3 are satisfied.

In all the cases we refine the initial mesh in a uniform way (each triangle is divided in four similar triangles). We recall that our goal is to obtain lower bounds of the eigenvalues and this is why we use uniform refinement. In practical applications one should combine this method with an adaptive procedure. A lower bound (combined with upper bounds obtained with conforming methods) could be used to have an estimate of the error in order to decide at which refinement level the adaptive procedure should be started.

The results suggest that the sequence of eigenvalue approximations obtained in this way is monotone increasing.

First we consider the case of an L-domain. For this domain, it is known that the first eigenfunction is singular. In Figure 3.1 we show the initial mesh.

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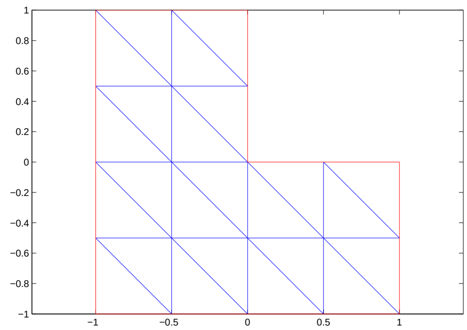

FIG. 3.1. *Initial mesh for the L-domain*

number of nodes	$\lambda_{h,1}$
44	9.02916234407
160	9.20540571806
608	9.46626945159
2368	9.57515200626

Table 1

In the next table we present the numerical approximation of the first eigenvalue.

In our next two examples we take Ω as nonconvex polygons which are approximations of different levels to the fractal Koch domain. Also in these cases it is known that the first eigenfunctions are singular (see [9] , [11]). In Figure 3.2 and Figure 3.3 we show the first meshes for the two examples.

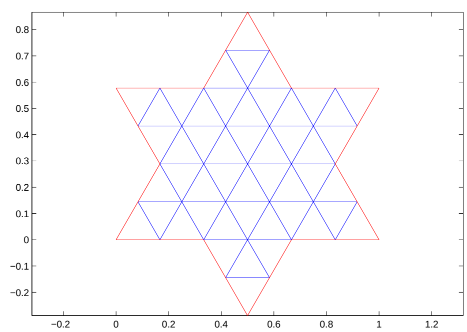

FIG. 3.2. *Initial mesh for level 1 approximation of the Koch domain*

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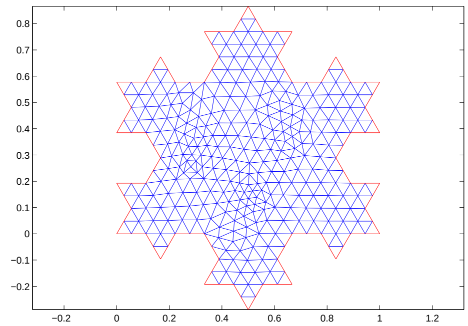

FIG. 3.3. *Initial mesh for level 2 approximation of the Koch domain*

In tables 2 and 3 we present the numerical approximation of the first eigenvalues for the domains of Figures 3.2 and 3.3, respectively.

number of nodes	$\lambda_{h,1}$
84	37.00124133068
312	38.84043356529
1200	39.74253482521

Table 2

number of nodes	$\lambda_{h,1}$
888	38.875778741698
3233	39.80755771713

Table 3

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