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ETNA
Kent State University
etna@mcs.kent.edu

DISCRETE SOBOLEV AND POINCARÉ INEQUALITIES FOR PIECEWISE POLYNOMIAL FUNCTIONS*

SUSANNE C. BRENNER†

Abstract. Discrete Sobolev and Poincaré inequalities are derived for piecewise polynomial functions on two dimensional domains. These inequalities can be applied to classical nonconforming finite element methods and discontinuous Galerkin methods.

Key words. discrete Sobolev inequality, discrete Poincaré inequality, piecewise polynomial functions, nonconforming, discontinuous Galerkin.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\mathcal{T}_h$ be a family of quasi-uniform simplicial or quadrilateral triangulations of Ω indexed by the mesh size h. To streamline the presentation, we first introduce the following notation concerning $\mathcal{T}_h$:

- the generic subdomain in $\mathcal{T}_h$ is denoted by D, which is either a triangle or a convex quadrilateral,
- $\mathcal{E}_h$ is the set of the interior edges of $\mathcal{T}_h$,
- $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$,
- $\mathcal{V}_e$ is the set of the two endpoints of the edge e,
- $\mathcal{V}_T$ is the set of the three vertices of the triangle T,
- $\mathcal{E}_T$ is the set of the three edges of the triangle T,
- $\mathcal{E}_{p,h}$ is the set of the edges in $\mathcal{E}_h$ sharing the common endpoint $p \in \mathcal{V}_h$,
- $\mathcal{T}_{p,h}$ is the set of the triangles or quadrilaterals in $\mathcal{T}_h$ that share the point $p \in \bar{\Omega}$ in their closures,
- $\mathcal{T}_{e,h}$ is the set of the two triangles or quadrilaterals in $\mathcal{T}_h$ sharing the common edge $e \in \mathcal{E}_h$,
- $|D|$ is the area of the subdomain D,
- $|e|$ is the length of the edge e,
- $\pi_{0,e}$ is the orthogonal projection operator that maps $L_2(e)$ onto the space of constant functions on e,
- the jump of a function v across an edge $e \in \mathcal{E}_h$ is denoted by $[v]_e$.

Note that even though a jump can be measured in two ways that differ by a minus sign, this ambiguity does not affect the statements in this paper because the jumps always appear in squared terms.

Let k be a nonnegative integer. In the case of a simplicial triangulation $\mathcal{T}_h$, we define

$$V_h = \{v \in L_2(\Omega) : v_{_D} = v\big|_D \in P_k(D) \quad \forall\, D \in \mathcal{T}_h\}, \tag{1}$$

where $P_k(D)$ is the space of polynomials of total degree $\leq k$ restricted to the triangle D. In the case of a quadrilateral triangulation $\mathcal{T}_h$, there is a bilinear homeomorphism $F_D : S \longrightarrow D$

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†Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A. E-mail: `brenner@math.sc.edu`. This research was supported in part by the National Science Foundation under Grant No. DMS-03-11790

from the unit square $S = (0,1) \times (0,1)$ onto any given (convex) quadrilateral $D \in \mathcal{T}_h$. We denote by $Q_k(D)$ the space of functions v on D such that $v \circ F_D$ is a polynomial on S whose degree in each variable is $\leq k$. In other words, $v \in Q_k(D)$ is a polynomial of individual degree $\leq k$ in the curvilinear coordinates on D induced by F_D^{-1}. We then define

$$V_h = \{v \in L_2(\Omega) : v_{_D} = v\big|_D \in Q_k(D) \quad \forall\, D \in \mathcal{T}_h\}. \tag{2}$$

In order to state the discrete Poincaré inequality for piecewise polynomial functions we define the mean value $\{\!\{v\}\!\}_h(p)$ of a function $v \in V_h$ at a point $p \in \bar{\Omega}$ by

$$\{\!\{v\}\!\}_h(p) = \frac{1}{|\mathcal{T}_{p,h}|} \sum_{D \in \mathcal{T}_{p,h}} v_{_D}(p), \tag{3}$$

where $|\mathcal{T}_{p,h}|$ is the number of triangles or quadrilaterals in $\mathcal{T}_{p,h}$.

The goal of this paper is to establish the following inequalities, where we use the standard notation for Sobolev spaces [7, 4] and the positive constant C is independent of h:

$$\|v\|^2_{L_\infty(\Omega)} \leq C(1 + |\ln h|)\Big\{\sum_{D \in \mathcal{T}_h} \|v\|^2_{H^1(D)} + \sum_{e \in \mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|^2_{L_2(e)}\Big\} \tag{4}$$

and

$$\|v\|^2_{L_\infty(\Omega)} \leq C(1 + |\ln h|)\Big\{\sum_{D \in \mathcal{T}_h} \|v\|^2_{H^1(D)} + \sum_{e \in \mathcal{E}_h}\sum_{p \in \mathcal{V}_e} \big([v]_e(p)\big)^2\Big\} \tag{5}$$

for all $v \in V_h$;

$$\|v\|^2_{L_2(\Omega)} \leq C(1 + |\ln h|)\Big\{\sum_{D \in \mathcal{T}_h} |v|^2_{H^1(D)} + \sum_{e \in \mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|^2_{L_2(e)}\Big\} \tag{6}$$

and

$$\|v\|^2_{L_2(\Omega)} \leq C(1 + |\ln h|)\Big\{\sum_{D \in \mathcal{T}_h} |v|^2_{H^1(D)} + \sum_{e \in \mathcal{E}_h}\sum_{p \in \mathcal{V}_e} \big([v]_e(p)\big)^2\Big\} \tag{7}$$

for all $v \in V_h$ such that $\{\!\{v\}\!\}_h(p) = 0$ for a given point $p \in \bar{\Omega}$. The inequalities (4)–(7) generalize the well-known discrete Sobolev and Poincaré inequalities [2, 14, 16] for finite element functions in $H^1(\Omega)$.

To avoid the proliferation of constants, we will use the notation $A \lesssim B$ to represent the inequality $A \leq (\text{constant}) \times B$, where the constant is independent of h. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

We begin by establishing the discrete Sobolev inequality (4) for piecewise constant functions with respect to a simplicial triangulation $\mathcal{T}_h$, i.e., for the case where $k = 0$ in (1) and $\mathcal{T}_h$ consists of triangles. Let W_h be the space of piecewise constant functions with respect to $\mathcal{T}_h$ and let $\tilde{W}_h \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}_h$. We define a linear map $E_h : W_h \longrightarrow \tilde{W}_h$ by

$$(E_h w)(p) = \{\!\{w\}\!\}_h(p) \qquad \forall\, w \in W_h,\ p \in \mathcal{V}_h. \tag{8}$$

LEMMA 1. *Let $p \in \mathcal{V}_h$ be a vertex of the triangle $T \in \mathcal{T}_h$. The following estimate holds*:

$$\big(w_{T}(p) - (E_h w)(p)\big)^2 \lesssim \sum_{e \in \mathcal{E}_{p,h}} |e|^{-1} \|[w]_e\|^2_{L_2(e)} \qquad \forall\, w \in W_h.$$

Proof. From (3) and (8) we have

$$w_{T}(p) - (E_h w)(p) = \frac{1}{|\mathcal{T}_{p,h}|} \sum_{T' \in \mathcal{T}_{p,h}} \big(w_{T}(p) - w_{T'}(p)\big)$$

which together with the Cauchy-Schwarz inequality implies that

$$\begin{aligned}\big(w_{T}(p) - (E_h w)(p)\big)^2 &\leq \frac{1}{|\mathcal{T}_{p,h}|} \sum_{T' \in \mathcal{T}_{p,h}} \big(w_{T}(p) - w_{T'}(p)\big)^2 \\ &\lesssim \sum_{e \in \mathcal{E}_{p,h}} [w]_e^2 = \sum_{e \in \mathcal{E}_{p,h}} |e|^{-1} \|[w]_e\|^2_{L_2(e)}.\end{aligned}$$

∎

In view of the elementary facts that

$$\begin{aligned}\|v\|_{L_\infty(T)} &= \max_{p \in \mathcal{V}_T} |v(p)| && \forall\, v \in P_1(T), \\ \|v\|^2_{L_2(T)} &\approx |T| \sum_{p \in \mathcal{V}_T} v(p)^2 && \forall\, v \in P_1(T), \\ |v|^2_{H^1(T)} &\lesssim \sum_{p \in \mathcal{V}_T} v(p)^2 && \forall\, v \in P_1(T), \\ |T| &\approx |e|^2 && \forall\, e \in \mathcal{E}_T,\end{aligned}$$

the estimates below follow immediately from Lemma 1:

$$\|w - E_h w\|^2_{L_\infty(\Omega)} \lesssim \max_{e \in \mathcal{E}_h} |e|^{-1} \|[w]_e\|^2_{L_2(e)} \qquad \forall\, w \in W_h, \tag{9}$$

$$\|w - E_h w\|^2_{L_2(\Omega)} \lesssim \sum_{e \in \mathcal{E}_h} |e|\, \|[w]_e\|^2_{L_2(e)} \qquad \forall\, w \in W_h, \tag{10}$$

$$\sum_{T \in \mathcal{T}_h} |w - E_h w|^2_{H^1(T)} \lesssim \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[w]_e\|^2_{L_2(e)} \qquad \forall\, w \in W_h. \tag{11}$$

The following lemma establishes (4) for the special case of piecewise constant functions with respect to a family of quasi-uniform simplicial triangulations.

LEMMA 2. *The following inequality holds:*

$$\|w\|^2_{L_\infty(\Omega)} \lesssim (1 + |\ln h|) \Big\{ \|w\|^2_{L_2(\Omega)} + \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[w]_e\|^2_{L_2(e)} \Big\} \qquad \forall\, w \in W_h. \tag{12}$$

Proof. From the discrete Sobolev inequality [2, 14, 16] for P_1 finite element functions in $H^1(\Omega)$, we have

$$\|E_h w\|^2_{L_\infty(\Omega)} \lesssim (1 + |\ln h|) \|E_h w\|^2_{H^1(\Omega)}. \tag{13}$$

Combining (9)–(11) and (13) we find that

$$\|w\|^2_{L_\infty(\Omega)} \lesssim \|w - E_h w\|^2_{L_\infty(\Omega)} + \|E_h w\|^2_{L_\infty(\Omega)}$$

$$
\begin{aligned}
&\lesssim \max_{e\in\mathcal{E}_h} |e|^{-1}\|[w]_e\|^2_{L_2(e)} + (1+|\ln h|)\|E_h w\|^2_{H^1(\Omega)} \\
&\lesssim \sum_{e\in\mathcal{E}_h} |e|^{-1}\|[w]_e\|^2_{L_2(e)} + (1+|\ln h|)\sum_{T\in\mathcal{T}_h}\left(\|w\|^2_{H^1(T)} + \|w-E_h w\|^2_{H^1(T)}\right) \\
&\lesssim (1+|\ln h|)\Big\{\|w\|^2_{L_2(\Omega)} + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|[w]_e\|^2_{L_2(e)}\Big\}.
\end{aligned}
$$

□

The next lemma shows that the result in Lemma 2 is also valid for quadrilateral triangulations, i.e., (4) is valid for the case where $k = 0$ in (2).

LEMMA 3. *The inequality (12) holds for piecewise constant functions w with respect to a quasi-uniform family of quadrilateral triangulations.*

Proof. Let $\tilde{\mathcal{T}}_h$ be the family of simplicial triangulations obtained from $\mathcal{T}_h$ by adding the two diagonals of each quadrilateral in $\mathcal{T}_h$. Then $\tilde{\mathcal{T}}_h$ is a quasi-uniform family of simplicial triangulations. Let w be an arbitrary piecewise constant function with respect to $\mathcal{T}_h$. Since w is also a piecewise constant function with respect to $\tilde{\mathcal{T}}_h$, Lemma 2 implies that

$$
\|w\|^2_{L_\infty(\Omega)} \lesssim (1+|\ln h|)\Big\{\|w\|^2_{L_2(\Omega)} + \sum_{e\in\tilde{\mathcal{E}}_h} |e|^{-1}\|[w]_e\|^2_{L_2(e)}\Big\}, \tag{14}
$$

where $\tilde{\mathcal{E}}_h$ is the set of interior edges of $\tilde{\mathcal{E}}_h$.

The inequality (12) follows from (14) and the observation that

$$
[w]_e = 0 \quad \forall\, e\in\tilde{\mathcal{E}}_h\setminus\mathcal{E}_h.
$$

□

We can now establish the general case of (4).

THEOREM 4. *The inequality (4) holds for a quasi-uniform family of simplicial or quadrilateral triangulations.*

Proof. Let $v \in V_h$ be arbitrary and $\Pi_0 v$ be the L_2 orthogonal projection of v into the space of piecewise constant functions, i.e.,

$$
(\Pi_0 v)\big|_D = \frac{1}{|D|}\int_D v\,dx \qquad \forall\, D\in\mathcal{T}_h.
$$

The following estimate [7, 4] is well-known:

$$
\|v-\Pi_0 v\|_{L_2(D)} \lesssim (\operatorname{diam} D)|v|_{H^1(D)} \qquad \forall\, v\in V_h. \tag{15}
$$

Moreover it follows from (15) and a standard inverse estimate [7, 4] that

$$
\|v-\Pi_0 v\|_{L_\infty(\Omega)} \lesssim \max_{D\in\mathcal{T}_h} |v|_{H^1(D)} \qquad \forall\, v\in V_h. \tag{16}
$$

For $e\in\mathcal{E}_h$, since $[\Pi_0 v]_e$ is a constant, the trace theorem (with scaling) and (15) imply that

$$
\begin{aligned}
|e|^{-1}\|[\Pi_0 v]_e - \pi_{0,e}[v]_e\|^2_{L_2(e)} &= |e|^{-1}\|\pi_{0,e}([\Pi_0 v]_e - [v]_e)\|^2_{L_2(e)} \\
&\le |e|^{-1}\|[\Pi_0 v]_e - [v]_e\|^2_{L_2(e)} \\
&\lesssim \sum_{D\in\mathcal{T}_{e,h}}\left(\frac{1}{|D|}\|\Pi_0 v - v\|^2_{L_2(D)} + |\Pi_0 v - v|^2_{H^1(D)}\right) \lesssim \sum_{D\in\mathcal{T}_{e,h}} |v|^2_{H^1(D)}.
\end{aligned} \tag{17}
$$

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Kent State University
etna@mcs.kent.edu

Combining Lemma 2, Lemma 3, and (15)–(17) we have

$$
\begin{aligned}
\|v\|_{L_\infty(\Omega)}^2 &\lesssim \|v-\Pi_0 v\|_{L_\infty(\Omega)}^2 + \|\Pi_0 v\|_{L_\infty(\Omega)}^2 \\
&\lesssim \max_{D\in\mathcal{T}_h} |v|_{H^1(D)}^2 + (1+|\ln h|)\Big\{\|\Pi_0 v\|_{L_2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|[\Pi_0 v]_e\|_{L_2(e)}^2\Big\} \\
&\lesssim \sum_{D\in\mathcal{T}_h} |v|_{H^1(D)}^2 + (1+|\ln h|)\Big\{\|v\|_{L_2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|_{L_2(e)}^2\Big\} \\
&\qquad + (1+|\ln h|)\sum_{e\in\mathcal{E}_h} |e|^{-1}\|[\Pi_0 v]_e - \pi_{0,e}[v]_e\|_{L_2(e)}^2 \\
&\lesssim (1+|\ln h|)\Big\{\sum_{D\in\mathcal{T}_h} \|v\|_{H^1(D)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|_{L_2(e)}^2\Big\}.
\end{aligned}
$$

□

As in the case of finite element functions belonging to $H^1(\Omega)$, the discrete Poincaré inequality (6) follows from the discrete Sobolev inequality (4).

THEOREM 5. *The inequality (6) holds for a quasi-uniform family of simplicial or quadrilateral triangulations.*

Proof. Let $v \in V_h$ be arbitrary and

$$\bar{v} = \frac{1}{|\Omega|}\int_\Omega v\,dx$$

be the mean of v over Ω. From the Poincaré-Friedrichs inequality for piecewise H^1 functions [3] we have

$$\|v-\bar{v}\|_{L_2(\Omega)}^2 \lesssim \sum_{D\in\mathcal{T}_h} |v|_{H^1(D)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|_{L_2(e)}^2 \tag{18}$$

which together with (4) yields

$$
\begin{aligned}
\|v-\bar{v}\|_{L_\infty(\Omega)}^2 &\lesssim (1+|\ln h|)\Big\{\sum_{D\in\mathcal{T}_h} \|v-\bar{v}\|_{H^1(D)}^2 \\
&\qquad\qquad + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v-\bar{v}]_e\|_{L_2(e)}^2\Big\} \\
&\lesssim (1+|\ln h|)\Big\{\sum_{D\in\mathcal{T}_h} |v|_{H^1(D)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|_{L_2(e)}^2\Big\}.
\end{aligned} \tag{19}
$$

On the other hand, since $\{\!\{v\}\!\}_h(p) = 0$, we have, by (3),

$$\|\bar{v}\|_{L_2(\Omega)} \lesssim |\bar{v}| = \big|\{\!\{v-\bar{v}\}\!\}_h(p)\big| = \frac{1}{|\mathcal{T}_{p,h}|}\Big|\sum_{T\in\mathcal{T}_{p,h}} \big(v_T(p)-\bar{v}\big)\Big| \le \|v-\bar{v}\|_{L_\infty(\Omega)}. \tag{20}$$

The estimate (6) now follows from (18)–(20) and the triangle inequality. □

REMARK 6. *The inequality (6) clearly remains valid if we replace the condition* $\{\!\{v\}\!\}_h(p) = 0$ *by the more general condition that*

$$\sum_{D\in\mathcal{T}_{p,h}} \omega_D v_D(p) = 0,$$

where the nonnegative weights ω_D *satisfy* $\sum_{D\in\mathcal{T}_{p,h}} \omega_D = 1$.

We now turn to the alternative forms (5) and (7) of the discrete Sobolev and Poincaré inequalities.

THEOREM 7. *The inequalities (5) and (7) hold for a quasi-uniform family of simplicial or quadrilateral triangulations.*

Proof. In view of (4) and (6), it suffices to show that

$$\sum_{e\in\mathcal{E}_h} |e|^{-1}\|\pi_{0,e}[v]_e\|^2_{L_2(e)} \lesssim \sum_{D\in\mathcal{T}_h} |v|^2_{H^1(D)} + \sum_{e\in\mathcal{E}_h}\sum_{p\in\mathcal{V}_e} \big([v]_e(p)\big)^2 \qquad \forall\, v\in V_h. \tag{21}$$

Let $v \in V_h$ be arbitrary and let $[v]^I_e \in P_1(e)$ agree with $[v]_e$ at the endpoints of e. By the trace theorem (with scaling), we have

$$|e|^{-1}\big\|\pi_{0,e}\big([v]_e - [v]^I_e\big)\big\|^2_{L_2(e)} \lesssim \sum_{D\in\mathcal{T}_{e,h}} \Big(\frac{1}{|D|}\|v_D - v^I_D\|^2_{L_2(D)} + |v_D - v^I_D|^2_{H^1(D)}\Big), \tag{22}$$

where v^I_D is either the polynomial in $P_1(D)$ that agrees with v at the vertices of the triangle D or the curvilinear polynomial in $Q_1(D)$ that agrees with v at the vertices of the quadrilateral D.

It follows from standard interpolation error estimates and inverse estimates [7, 4] that

$$\frac{1}{|D|}\|v_D - v^I_D\|^2_{L_2(D)} + |v_D - v^I_D|^2_{H^1(D)} \lesssim (\operatorname{diam} D)^2|v|^2_{H^2(D)} \lesssim |v|^2_{H^1(D)}. \tag{23}$$

Furthermore a direct calculation yields

$$|e|^{-1}\|\pi_{0,e}[v]^I_e\|^2_{L_2(e)} \le |e|^{-1}\|[v]^I_e\|^2_{L_2(e)} \lesssim \sum_{p\in\mathcal{V}_e}\big([v]_e(p)\big)^2. \tag{24}$$

The estimate (21) follows from (22)–(24) and the triangle inequality. ☐

The discrete Sobolev inequality (4) and the discrete Poincaré inequality (6) for piecewise polynomial functions can be applied to many classical nonconforming finite element functions [9, 11, 12, 13, 10, 6, 5] that satisfy the weak continuity condition

$$0 = \int_e [v]_e\, ds = |e|\,(\pi_{0,e}[v]_e) \qquad \forall\, e\in\mathcal{E}_h. \tag{25}$$

For such functions the inequalities (4) and (6) simplify to

$$\|v\|^2_{L_\infty(\Omega)} \le C(1+|\ln h|)\sum_{D\in\mathcal{T}_h}\|v\|^2_{H^1(D)} \tag{26}$$

and

$$\|v\|^2_{L_2(\Omega)} \le C(1+|\ln h|)\sum_{D\in\mathcal{T}_h}|v|^2_{H^1(D)}, \tag{27}$$

respectively.

On the other hand, it follows from the alternative inequalities (5) and (7) that (26) and (27) are also valid for nonconforming finite element functions [17, 19] that do not satisfy the weak continuity condition (25) but are continuous at the vertices of $\mathcal{T}_h$.

The inequalities (4) and (6) can also be applied to discontinuous Galerkin methods. Indeed, by dropping the orthogonal projection operator $\pi_{0,e}$ in (4) and (6) we immediately arrive at the inequalities

$$\|v\|^2_{L_\infty(\Omega)} \le C(1+|\ln h|)\Big\{\sum_{D\in\mathcal{T}_h}\|v\|^2_{H^1(D)} + \sum_{e\in\mathcal{E}_h}|e|^{-1}\|[v]_e\|^2_{L_2(e)}\Big\} \tag{28}$$

and

$$\|v\|^2_{L_2(\Omega)} \leq C(1+|\ln h|)\Big\{\sum_{D\in\mathcal{T}_h}|v|^2_{H^1(D)}+\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[v]_e\|^2_{L_2(e)}\Big\}. \tag{29}$$

The sums in (28) and (29) involving the jumps of v now appear naturally in many discontinuous Galerkin methods [8, 1].

REMARK 8. *The discrete Sobolev and Poincaré inequalities for finite element functions in $H^1(\Omega)$ are useful for example in the analysis of the L_∞ stability of finite element methods for parabolic problems* [16] *and the analysis of various nonoverlapping domain decomposition methods* [2, 15, 18, 4]. *The inequalities* (26)–(29) *enable similar analyses to be carried out for classical nonconforming finite element methods and discontinuous Galerkin methods.*

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