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ON HERMITE INTERPOLATION IN R_d*

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Abstract. In this article, we deal with the problem of "Minimal Hermite Interpolation." That is, given a number k of distinct points in R_d and the values of several derivatives at this point, we want to find a subspace of minimal dimension, where this interpolation problem has a solution, independent of the choice of points. In Section 2, we present some results on such subspaces in the particular cases of two points and some or all partial derivatives of the first order. In Section 3, we obtain some general upper bounds on the dimension of interpolation subspaces.

Key words. Hermite interpolation, Lagrange interpolation.

AMS subject classifications. 41A05, 41A63, 65D05.

1. Introduction. When describing a problem of Hermite interpolation in several variables, one is confronted with a necessary evil of unbearably cumbersome notations. I am hoping to avoid this assault on the psyche of the reader by confining the discussion to a few simple examples, where an explicit description of the question circumvents the need for a general scheme. So, let us agree to use the words "Lagrange Interpolation" for the problem of finding a function with prescribed values at finitely many points. "Hermite Interpolation" then refers to the situation when not only the values of the function, but the values of some of the partial derivatives of the function are prescribed.

So, given k points: $u_1, u_2, \dots u_k \in R_d$; k scalars $a_1, a_2, \dots a_k \in R$ and a finite dimensional subspace $F \subset C(R_d)$, we wish to find a function $f \in F$ such that $f(u_j) = a_j$ for all $j = 1, \dots, k$. If that is possible for all choices of scalars $a_1, a_2, \dots a_k \in R$, we will say that the space F interpolates the functionals $\Phi_k : \{\delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k}\}$. (Here δ_u refers to the functional on $C(R_d)$ defined as $\delta_u(f) = f(u)$.) Let $f_1, f_2, \dots f_N$ be bases for F. Then F interpolates the functionals $\delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k}$ if and only if the rank of the $k \times N$ matrix $[f_j(u_m)]$; $m = 1, \dots k$; $j = 1, \dots N$, is equal to k. The three types of problems in the Lagrange interpolation are as follows:

1. Given a space F, find all configurations of points $u_1, u_2, \dots u_k \in R_d$ such that F interpolates $\delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k}$.
2. Given $\Phi_k := \{\delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k}\}$, find the space F that interpolates Φ_k.
3. Let $I(\Phi_k) = \min\{\dim F : F \text{ interpolates } \delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k} \text{ for all choices of distinct } u_1, u_2, \dots u_k \in R_d\}$. Find $I(\Phi_k)$ and a space F that interpolates $\delta_{u_1}, \delta_{u_2}, \dots, \delta_{u_k}$ for all choices of distinct $u_1, u_2, \dots u_k \in R_d$ such that $\dim F = I(\Phi_k)$. We will refer to this problem as the "Minimal Interpolation Problem."

For Hermite interpolation, we once again start with k points: $u_1, u_2, \dots u_k \in R_d$ and a set $\bar{n}$ of k integers $n_1, n_2, \dots n_k$. For each integer n_j, we prescribe a family of differential operators $\left\{D_j^a : a = 1, \dots, n_j\right\}$. Let $\Phi(j) = \left\{\delta_{u_j} \circ D_j^a : a = 1, \dots, n_j\right\}$ be the family of corresponding functionals at the point u_j and let $\Phi_k(\bar{n}) = \bigcup_{j=1}^{k} \Phi(j)$. We are interested in a space $F \subset C^\infty(R_d)$ that interpolates the functionals $\Phi_k(\bar{n})$. The corresponding three types of problems in Hermite interpolation are:

1. Given a space F, find all configurations of points $u_1, u_2, \dots u_k \in R_d$ such that F interpolates $\Phi_k(\bar{n})$.
2. Given $\Phi_k(\bar{n})$, find the space F that interpolates $\Phi_k(\bar{n})$.

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3. Let $I\left(\Phi_k\left(\bar{n}\right)\right) = \min\{\dim F : F$ interpolates $\Phi_k\left(\bar{n}\right)$ for all choices of distinct $u_1, u_2, \ldots u_k \in R_d\}$. Find $I\left(\Phi_k\left(\bar{n}\right)\right)$ and a space F that interpolates $\Phi_k\left(\bar{n}\right)$ for all choices of distinct $u_1, u_2, \ldots u_k \in R_d$ such that $\dim F = I\left(\Phi_k\left(\bar{n}\right)\right)$. We will refer to this problem as the "Minimal Hermite Interpolation Problem."

Obviously $I\left(\Phi_k; B\right) \geq k$ for Lagrange interpolation and $I\left(\Phi_k\left(\bar{n}\right)\right) \geq \#\Phi_k\left(\bar{n}\right)$ for Hermite interpolation. The rest depends on $d = \dim R_d$ and, as we will see in the next section, on the choice of operators D_j^a. For instance, for the Minimal Lagrange interpolation problem in one variable, $I\left(\Phi_k\right) = k$ and the space of polynomials Π_{k-1}^1 is an example of a Φ_k-interpolating space of minimal dimension.

Interpolation in several variables has received serious attention in the literature only recently (cf. survey papers [2] and [4]). Specifically, for "Minimal Lagrange Interpolation" in R_2, the following striking result was proved by F. Cohen and D. Handle [1] and rediscovered by Vasiliev [7]:

THEOREM 1.1. *Let $I(k,2)$ be the least dimension of a space $F \subset C\left(R_2\right)$ that interpolates at arbitrary k points in R_2. Then $2k - \nu(k) \leq I\left(\Phi_k\right) \leq 2k-1$, where $\nu(k)$ is the number of 1's in the binary representation of an integer k.*

Despite the flurry of activity around Hermite interpolation, the "Minimal Interpolation" in several variables had been investigated only for Lagrange Interpolation. This paper is a modest attempt to attract attention to the "Minimal Hermite Interpolation." One way to do so is to compare Lagrange and Hermite interpolation, which we will do in the next section, by means of simple, yet peculiar, examples. We hope that these examples will emphasize the topological nature of the problems. In Section 3 we will prove some general estimates for the quantity $I\left(\Phi_k\left(\bar{n}\right)\right)$.

We will use the notation Π_N^d to denote the space of polynomials of degree N of d variables.

2. Case Study of $(2,1)$ Interpolation.

These are the cases of Minimal Hermite interpolation at two distinct points in R_2 and some of the partial derivatives of the first order at those points. The two distinct points are denoted as u and v, and the point evaluation functionals are δ_u, δ_v. Hence in this section we are interested in finding a subspace $F \subset C^\infty\left(R_2\right)$ of minimal dimension that interpolates the functionals δ_u, δ_v, and some or all of the functionals $\delta_u \circ \frac{\partial}{\partial x}$, $\delta_u \circ \frac{\partial}{\partial y}$, $\delta_v \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial y}$. We will often use (x,y) to denote the point $u = (x,y) \in R_2$. Hence, it will be understood that for instance the function $f(u) = x^2 - y$ is the function $f(x,y) = x^2 - y$. Similarly, $\left(\frac{\partial}{\partial x} f\right)(u) = f_x(u)$ is the derivative of the function f with respect to the first variable, evaluated at the point u.

Case 1: Interpolation of δ_u, δ_v. This is a classical case of Lagrange interpolation in R_2.

THEOREM 2.1 ([5]). *For any two-dimensional subspace $F = \operatorname{span}\left[f_1, f_2\right] \subset C\left(R_2\right)$ there exists a pair of distinct points $u, v \in R_2$ such that the space F does not interpolate at these points. Hence $I\left(\Phi_k\right) = 3$ and the space $F = \operatorname{span}[1, x, y]$ is a minimal $I\left(\Phi_k\right)$-interpolation space.*

A nice proof is due to Mairhuber [5]. Since we will use the idea of this proof over and over, we will refer to it as the Mairhuber Argument.

Proof. Position two points u, v on diametrically opposite ends of a circle and consider the determinant

$$D[u,v] = \begin{vmatrix} f_1(u) & f_2(u) \\ f_1(v) & f_2(v) \end{vmatrix}.$$

As we rotate the diameter, the points u and v switch positions and hence $D[u,v]$ changes sign. By the intermediate value theorem, there exists a pair u, v such that $D[u,v] = 0$ and

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hence F is not interpolating at these points.

On the other hand, it is easy to verify that the 3-dimensional space spanned by $[1, x, y]$ interpolates at two arbitrary points. Indeed the

$$\operatorname{rank}\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\end{bmatrix} = 2$$

if (and only if) $(x_1, y_1) \neq (x_2, y_2)$, i.e., $u \neq v$. □

CONCLUSION 1. *The minimal dimension of the space that interpolates δ_u, δ_v is three, one more than the number of interpolating conditions.*

Case 2: Interpolation of δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial x}$. This is the first case of Hermite Interpolation. Suppose that we now want to find a three-dimensional space $F = \operatorname{span}[f, g, h] \subset C^\infty(R_2)$ such that for any two points $u, v \in R_2$ the space F interpolates the functionals δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial x}$. Observe that in this case the Mairhuber Argument does not work. For in the determinant

$$\begin{vmatrix} f(u) & g(u) & h(u)\\ f(v) & g(v) & h(v)\\ \frac{\partial f}{\partial x}(u) & \frac{\partial g}{\partial x}(u) & \frac{\partial g}{\partial x}(u)\end{vmatrix},$$

as u and v switch positions, the last row changes as well and so the conclusion is indeed not warranted. Nevertheless, any three-dimensional space $F \subset C^\infty(R_2)$ does not interpolate δ_u, δ_v, and $\delta_u \circ \frac{\partial}{\partial x}$ for arbitrary distinct u and v.

THEOREM 2.2. *The minimal dimension of the space $F \subset C^\infty(R_2)$ that interpolates functionals δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial x}$ for arbitrary $u, v \in R_2$ is 4. The space $F = \operatorname{span}\left[1, x, y, x^2 - y^2\right]$ is an example of such space.*

Proof. Contrary to the conclusion, let $F = \operatorname{span}[f, g, h]$ interpolate δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial x}$ for arbitrary $u \neq v \in R_2$, i.e., the determinant

$$D(u,v) := \begin{vmatrix} f(u) & g(u) & h(u)\\ f(v) & g(v) & h(v)\\ f_x(u) & g_x(u) & h_x(u)\end{vmatrix} = 0 \text{ if and only if } u = v. \tag{2.1}$$

Let $\phi : R_2 \to R_3$ be defined by $\phi(u) = (f(u), g(u), h(u))$. Then ϕ is one-to-one, for otherwise, the first two rows in determinant (2.1) would be equal for some $u \neq v$. Hence $\phi(R_2)$ is a smooth 2-manifold in R_3. Let $T(u)$ be a plane defined as $\operatorname{span}[\phi(u), \phi_x(u)]$. Then for every $u \in R_2$, $T(u)$ is a two-dimensional plane passing through the origin, otherwise the first and the third rows in the determinant $D(u, v)$ would be linearly dependent, which would contradict (2.1). Furthermore,

$$\phi(R_2) \cap T(u) = \{\phi(u)\}. \tag{2.2}$$

Indeed, if that was not so, then we could find $\phi(v) \in \phi(R_2) \cap T(u)$ for some $v \neq u$. Hence the vectors $\phi(u)$, $\phi(v)$, $\phi_x(u)$ all belong to a two-dimensional space $T(u)$ which, once again, would imply that $\phi(u)$, $\phi(v)$, $\phi_x(u)$ are linearly dependent and $D(u, v) = 0$, in contradiction to (2.1).

Since $\phi(R_2)$ is a smooth two-dimensional manifold without a boundary, (2.2) implies that $T(u)$ is a tangent space to $\phi(R_2)$ at every point $\phi(u) \in \phi(R_2)$. The important conclusion is:

For every point $w \in \phi(R_2)$, the tangent space T_w passes through the origin.

That actually implies that $\phi(R_2)$ is either a plane or, locally, a cone. This means that there is a straight line in $\phi(R_2)$ that belongs to $T_w = T(u)$, which contradicts (2.2). Since I couldn't find a convenient reference to this last fact in the literature, let me verify it directly. Let $P(w)$ be an arbitrary plane passing through the origin and a point $w = \phi(u) \in \phi(R_2)$

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such that $w \neq 0$. Then $r(w) = P \cap \phi(R_2)$ is a one-dimensional smooth curve and the tangent line $t(w)$ to the curve $r(w)$ belongs to $P(w) \cap T_w$ and, in particular, passes through the origin. Let $r(s) = (x(s), y(s), z(s))$ be a parametrization for the curve $r(w)$. Since the tangent line to $r(s)$ passes through the origin for all s, we have

$$\frac{x'(s)}{x(s)} = \frac{y'(s)}{y(s)} = \frac{z'(s)}{z(s)}.$$

It is now easy to solve this differential equation to conclude that $r(s)$ is a straight line. Hence $r(s) \in \phi(R_2) \cap T(u)$. That contradicts (2.1) and proves the first part of the theorem.

The fact that the space $F = \operatorname{span}\left[1, x, y, x^2 - y^2\right]$ interpolates functionals δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial x}$ for arbitrary $u, v \in R_2$ will be verified in the next case. □

CONCLUSION 2. *The minimal dimension of a subspace that interpolates δ_u, δ_v and $\delta_u \circ \frac{\partial}{\partial_x} = 4$, is one more than the number of interpolation conditions.*

Case 3: Interpolation of δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial y}$. Surprisingly in this case the minimal dimension of the interpolation space is the same as the number of interpolation conditions.

PROPOSITION 2.3. *The space $F := \operatorname{span}\left[1, x, y, x^2 - y^2\right] \subset C(R_2)$ interpolates the functionals δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial y}$ for arbitrary choice of distinct points $u, v \in R_2$.*

Proof. Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$. Then the corresponding determinant

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 - y_1^2 \\ 1 & x_2 & y_2 & x_2^2 - y_2^2 \\ 0 & 1 & 0 & 2x_1 \\ 0 & 0 & 1 & -2y_2 \end{vmatrix}$$

$$= -2x_2x_1 + y_2^2 + x_2^2 + x_1^2 - 2y_1y_2 + y_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

and equals to 0 only if $x_1 = x_2$ and $y_1 = y_2$; that is, when $u = v$. □

CONCLUSION 3. *The minimal dimension of a space that interpolates δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial y}$ is 4.*

Case 4: Interpolation of δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_u \circ \frac{\partial}{\partial y}$. Once again, we have

PROPOSITION 2.4. *The four-dimensional space $F = \operatorname{span}\left[1, x, y, x^2 + y^2\right]$ interpolates the functionals δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_u \circ \frac{\partial}{\partial y}$ for arbitrary choice of distinct points $u, v \in R_2$.*

Proof. For $u = (x_1, y_1)$ and $v = (x_2, y_2)$, the corresponding determinant

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 + y_1^2 \\ 1 & x_2 & y_2 & x_2^2 + y_2^2 \\ 0 & 1 & 0 & 2x_1 \\ 0 & 0 & 1 & 2y_1 \end{vmatrix}$$

$$= -2x_2x_1 - 2y_2y_1 + x_2^2 + y_2^2 + x_1^2 + y_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

and equals to 0 if and only if $x_1 = x_2$ and $y_1 = y_2$; that is, when $u = v$. □

CONCLUSION 4. *The minimal dimension of a space that interpolates δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_u \circ \frac{\partial}{\partial y}$ is 4.*

Case 5: Interpolation of δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial x}$. In contrast to the previous two cases, I was unable to find a four-dimensional space that interpolates these functionals for arbitrary $u \neq v \in R_2$.

CONJECTURE 1. *No four-dimensional space interpolates functionals δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$ and $\delta_v \circ \frac{\partial}{\partial x}$ for arbitrary choice of distinct points $u, v \in R_2$.*

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Case 6: Interpolation of δ_u, δ_v, $\delta_u \circ \frac{\partial}{\partial x}$, $\delta_v \circ \frac{\partial}{\partial_x}$, $\delta_u \circ \frac{\partial}{\partial y}$ and $\delta_v \circ \frac{\partial}{\partial y}$.

PROPOSITION 2.5. *There exists a seven-dimensional space that interpolates at two arbitrary points, and all of the first-order partial derivates at these points. No six-dimensional space F has this property.*

Proof. Consider the space F spanned by the first seven harmonic polynomials: $1, x, y, x^2 - y^2, -2yx, x^3 - 3y^2 x, -3x^2 y + y^3$. We need to show that for every $(x_1, y_1) \neq (x_2, y_2)$

$$\operatorname{rank} \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 - y_1^2 & -2y_1x_1 & x_1^3 - 3y_1^2x_1 & -3x_1^2y_1 + y_1^3 \\ 1 & x_2 & y_2 & x_2^2 - y_2^2 & -2x_2y_2 & x_2^3 - 3y_2^2x_2 & -3x_2^2y_2 + y_2^3 \\ 0 & 1 & 0 & 2x_1 & -2y_1 & 3x_1^2 - 3y_1^2 & -6y_1x_1 \\ 0 & 1 & 0 & 2x_2 & -2y_2 & 3x_2^2 - 3y_2^2 & -6x_2y_2 \\ 0 & 0 & 1 & -2y_1 & -2x_1 & -6y_1x_1 & -3x_1^2 + 3y_1^2 \\ 0 & 0 & 1 & -2y_2 & -2x_2 & -6x_2y_2 & -3x_2^2 + 3y_2^2 \end{bmatrix} = 6.$$

This can be easily done by direct computations:

$$\det \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 - y_1^2 & -2y_1x_1 & x_1^3 - 3y_1^2x_1 \\ 1 & x_2 & y_2 & x_2^2 - y_2^2 & -2x_2y_2 & x_2^3 - 3y_2^2x_2 \\ 0 & 1 & 0 & 2x_1 & -2y_1 & 3x_1^2 - 3y_1^2 \\ 0 & 1 & 0 & 2x_2 & -2y_2 & 3x_2^2 - 3y_2^2 \\ 0 & 0 & 1 & -2y_1 & -2x_1 & -6y_1x_1 \\ 0 & 0 & 1 & -2y_2 & -2x_2 & -6x_2y_2 \end{bmatrix}$$
$$= -2(x_2 - x_1)\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)\left((x_2 - x_1)^2 - 3(y_2 - y_1)^2\right)$$

and

$$\det \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 - y_1^2 & -2y_1x_1 & -3x_1^2y_1 + y_1^3 \\ 1 & x_2 & y_2 & x_2^2 - y_2^2 & -2x_2y_2 & -3x_2^2y_2 + y_2^3 \\ 0 & 1 & 0 & 2x_1 & -2y_1 & -6y_1x_1 \\ 0 & 1 & 0 & 2x_2 & -2y_2 & -6x_2y_2 \\ 0 & 0 & 1 & -2y_1 & -2x_1 & -3x_1^2 + 3y_1^2 \\ 0 & 0 & 1 & -2y_2 & -2x_2 & -3x_2^2 + 3y_2^2 \end{bmatrix}$$
$$= -2(-y_2 + y_1)\left(3(x_1 - x_2)^2 - (-y_2 + y_1)^2\right)\left((x_1 - x_2)^2 + (-y_2 + y_1)^2\right).$$

It is now easy to check that the two are equal to zero simultaneously if and only if $u = v$.

On the other hand, we can use the same Mairhuber argument to show that a six-dimensional subspace cannot interpolate the functionals in question. Once again, let $f_1, f_2, \ldots, f_6$ be a basis for such a space. Consider the determinant

$$\begin{vmatrix} f_1(u) & f_2(u) & \cdots & f_6(u) \\ f_1(v) & f_2(v) & \cdots & f_6(v) \\ \frac{\partial f_1}{\partial x}(u) & \frac{\partial f_2}{\partial x}(u) & \cdots & \frac{\partial f_6}{\partial x}(u) \\ \frac{\partial f_1}{\partial x}(v) & \frac{\partial f_2}{\partial x}(v) & \cdots & \frac{\partial f_6}{\partial x}(u) \\ \frac{\partial f_1}{\partial y}(u) & \frac{\partial f_2}{\partial y}(u) & \cdots & \frac{\partial f_6}{\partial y}(u) \\ \frac{\partial f_1}{\partial y}(v) & \frac{\partial f_2}{\partial y}(v) & \cdots & \frac{\partial f_6}{\partial y}(v) \end{vmatrix}.$$

As u and v are rotated into each other, three consecutive pairs of rows alternate and hence, the sign of the determinant changes. Once again, by the indeterminate value theorem, we conclude the existence of u and v for which the above determinant is zero. □

This result generalizes to k points interpolation, which we will prove in the next section.

REMARK 1. *To contrast Minimal Hermite interpolation with regular Hermite interpolation, compare the "success" of harmonic polynomials to failure of regular six-dimensional*

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space polynomials Π_2^2 *that do not interpolate functions* $\delta_u, \delta_v,\ \delta_u \circ \frac{\partial}{\partial x},\ \delta_u \circ \frac{\partial}{\partial y},\ \delta_v \circ \frac{\partial}{\partial x}$, $\delta_v \circ \frac{\partial}{\partial y}$ *for any points* u *and* v.

CONCLUSION 5. *The minimal dimension of a subspace that interpolates* $\delta_u, \delta_v,\ \delta_u \circ \frac{\partial}{\partial x}$, $\delta_u \circ \frac{\partial}{\partial y},\ \delta_v \circ \frac{\partial}{\partial x},\ \delta_v \circ \frac{\partial}{\partial y}$ *is* 7.

3. General Estimates. In this section, we will establish some upper bounds for the quantity $I\left(\Phi_k\left(\bar{n}\right)\right)$. These bounds give an accurate estimate of the asymptotics of $I\left(\Phi_k\left(\bar{n}\right)\right)$ with respect to parameters k, m and d but, as can be seen from examples in the previous section, are far from being exact.

For Lagrange interpolation, the following result was established in [6]

THEOREM 3.1. *There exists a subspace* $F \subset C\left(R_d\right)$ *with* $\dim F = (d+1)k$ *such that* F *interpolates at arbitrary* k *distinct points* $u_1, u_2, \ldots, u_k \in R_d$.

Next,we give a similar theorem for the case of Hermite interpolation, and an outline of its proof. The details are analogous to the proof of Theorem 3.1, and require the following definition of transversality (cf. [3]):

DEFINITION 3.2. *Let* A *and* B *be two differential manifolds and* G *be a smooth mapping* $G : A \to B$. *Let* C *be a submanifold of* B *and* $x \in A$. *We say that* G *is transversal to* C *at the point* x *if*

$$G(x) \notin C \text{ or } T_{G(s)}B = T_{G(s)}C + (DG)_x A.$$

We say that G *is transversal to* C, $(G \pitchfork C)$ *if it is transversal to* C *at every point* $x \in A$.

Here, $T_x U$ denotes the tangent plane to a manifold U at the point x, and $(DG)_x$ is the derivative of the function G at the point x.

THEOREM 3.3. *Let* $u_1, u_2, \ldots, u_k \in R_d$ *be an arbitrary collection of* k *distinct points. For each* j, *consider a collection of* n_j *distinct functionals* $\Phi(j) = \left\{\delta_{u_j} \circ D_1^{(j)}, \ldots, \delta_{u_j} \circ D_{nj}^{(j)}\right\}$ *where* $D_l^{(j)}$ *are arbitrary operators on* $C^\infty\left(R_d\right)$. *Let* $\Phi_k\left(\bar{n}\right) = \cup\Phi(j)$ *and let* $m = \#\Phi_k\left(\bar{n}\right)$, *the cardinality of* $\Phi_k\left(\bar{n}\right)$. *Then there exists a subspace* $F \subset C\left(R_d\right)$ *with* $\dim F = dk + m$ *that interpolates* Φ *for arbitrary choice of distinct points* $u_1, u_2, \ldots, u_k \in R_d$.

Proof. Let $N = dk + m$, and let M be an integer large enough so that the interpolation problem for functionals $\Phi_k\left(\bar{n}\right)$ is solvable in the space of polynomials Π_M^d. Let $E = \left(\Pi_M^d\right)^N$ be a differential manifold (vector space) which is a direct product of N copies of Π_M^d and $A = A(d,k)$ be a "configuration manifold," i.e., the set of all distinct k-tuples of points $u_1, u_2, \ldots, u_k \in R_d$. Finally, let $B = B(N,m)$ be the space of all $m \times N$ matrices and $C \subset B(N,m)$ be the submanifold of matrices of rank less than m.

Define a mapping $Q : E \times A(d,k) \to B(N,m)$ by

$$Q\left(p_1, p_2, \ldots, p_N; u_1, u_2, \ldots, u_k\right) = \left[\phi\left(p_j\right)\right]; \quad j = 1, \ldots, N; \quad \phi \in \Phi_k\left(\bar{n}\right). \tag{3.1}$$

Our immediate goal is to show that $Q \pitchfork C$, i.e., to verify that

$$T_{Q(p,u)}B = T_{Q(p,u)}C + (DQ)_{(p,u)}(E \times A). \tag{3.2}$$

Since B is a linear space, its tangent space is itself and is sufficient to show that

$$\operatorname{rank}(DQ)_{(p,u)} = \dim T_{Q(p,u)}B = \dim B = mN. \tag{3.3}$$

The Jacobian $(DQ)_{(p,u)}$ is an augmented matrix $L = [L_1, L_2]$, where $L_1 = \frac{\partial Q}{\partial p}$ and $L_2 = \frac{\partial Q}{\partial \phi}$. Differentiating with respect to polynomials $p \in \Pi_M^d$, we can easily find (cf. [6]) that

$$L_1 = \begin{bmatrix} H & 0 & \cdots & 0 \\ 0 & H & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H \end{bmatrix}$$

where H is the matrix of $\Phi_k(\bar{n})$ interpolation in the space Π_M^d. Since Π_M^d is $\Phi_k(\bar{n})$ interpolating, hence $\operatorname{rank} H = m$ and, thus, $\operatorname{rank} L \geq \operatorname{rank} L_1 = mN$ and $Q \pitchfork C$.

By the Transversality Theorem (cf. [3]), we conclude that for almost all fixed polynomials $p_1, p_2, \ldots, p_N \in \Pi_M^d$, the mapping $P := Q(p_1, p_2, p_N; u_1, u_2, \ldots, u_k) : A \to B$ is transversal to C. Let $F = \operatorname{span}[p_1, p_2, \ldots, p_N]$. We claim that the space F is $\Phi_k(\bar{n})$-interpolating. Indeed, since $P \pitchfork C$, hence either $P(A) \cap C = \emptyset$, that is, the matrix $[\phi(p_j)]$; $j = 1, \ldots, N$; $\phi \in \Phi(\bar{n})$ has rank m, or

$$T_{P(u)}B = T_{P(u)}C + (DP)_u A. \tag{3.4}$$

A simple dimension count shows that (3.4) is not possible: $\dim TB = \dim B = Nm$; $\operatorname{co}\dim C = N - m + 1$ and $\dim(DP)_x A \leq \dim A = dk$. Substituting $dk + m$ for N we have $dk + 1 \leq dk$. ☐

Comparing Theorems 3.1 and 3.3, one can observe that for Lagrange interpolation $m = k$ and, hence, Theorem 3.1 follows from Theorem 3.3.

Our next and last theorem establishes an analog of the right-hand side estimate in Theorem 1.1:

THEOREM 3.4. *There exists a $4k-1$ dimensional space $F \subset C(R_2)$ that interpolates at any k distinct points in R_2 and all of its partial derivatives of the first order at those points.*

Proof. We wish to prove that the space F of the first $4k - 1$ harmonic polynomials interpolate the values of the function and all of its partial derivatives at an arbitrary k points. Consider the k distinct points $u_1, u_2, \ldots, u_k \in R_2$ as complex numbers $u_j = (x_j, y_j) = x_j + iy_j$. Let $\{\alpha_j, \beta_j, \gamma_j \in R : j = 1, \ldots, k\}$ be given. Let $p(z) = a_0 + a_1 z + \cdots + a_{2k-1} z^{2k-1} = h(x, y) + ig(x, y)$ be a complex polynomial such that

$$p(u_j) = \alpha_j, \quad p'(u_j) = \beta_j - i\gamma_j.$$

Then $h(u_j) = \alpha_j$; and by the Cauchy-Riemann equation we have

$$f'(u_j) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(h(x, y) + ig(x, y)) = \frac{\partial h}{\partial x}(u_j) - i\frac{\partial h}{\partial y} = \beta_j - i\gamma_j.$$

Hence, h is the harmonic polynomial with the desired property. ☐

REMARK 2. *Unfortunately, the usefulness of harmonic polynomials ends at this point. If the second derivatives are involved, the equation $\Delta h = 0$ renders them useless for interpolation.*

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