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ORTHOGONALITY OF JACOBI POLYNOMIALS WITH GENERAL PARAMETERS*

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Abstract. In this paper we study the orthogonality conditions satisfied by Jacobi polynomials $P_n^{(\alpha,\beta)}$ when the parameters α and β are not necessarily > -1. We establish orthogonality on a generic closed contour on a Riemann surface. Depending on the parameters, this leads to either full orthogonality conditions on a single contour in the plane, or to multiple orthogonality conditions on a number of contours in the plane. In all cases we show that the orthogonality conditions characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Key words. Jacobi polynomials, orthogonality, Rodrigues formula, zeros.

AMS subject classifications. 33C45.

1. Introduction.

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are given explicitly by

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k},$$

or, equivalently, by the well-known Rodrigues formula

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!} (z-1)^{-\alpha} (z+1)^{-\beta} \left(\frac{d}{dz}\right)^n \left[(z-1)^{n+\alpha}(z+1)^{n+\beta}\right]. \tag{1.1}$$

These expressions show that $P_n^{(\alpha,\beta)}$ are analytic functions of the parameters α and β, and thus can be considered for general $\alpha, \beta \in \mathbb{C}$.

The classical Jacobi polynomials correspond to parameters $\alpha, \beta > -1$. For these parameters, the Jacobi polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $(1-z)^\alpha (1+z)^\beta$. As a result, all their zeros are simple and belong to the interval $(-1, 1)$. For general α, β, this is no longer valid. Indeed, the zeros can be non-real, and there can be multiple zeros. In fact, $P_n^{(\alpha,\beta)}$ may have a multiple zero at $z = 1$ if $\alpha \in \{-1, \dots, -n\}$, at $z = -1$ if $\beta \in \{-1, \dots, -n\}$ or, even, at $z = \infty$ (which means a degree reduction) if $n + \alpha + \beta \in \{-1, \dots, -n\}$.

More precisely, for $k \in \{1, \dots, n\}$, we have (see [29, formula (4.22.2)]),

$$P_n^{(-k,\beta)}(z) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+1-k)} \frac{(n-k)!}{n!} \left(\frac{z-1}{2}\right)^k P_{n-k}^{(k,\beta)}(z). \tag{1.2}$$

This implies in particular that $P_n^{(-k,\beta)}(z) \equiv 0$ if additionally $\max\{k, -\beta\} \le n \le k - \beta - 1$. Analogous relations hold for $P_n^{(\alpha,-l)}$ when $l \in \{1, \dots, n\}$. Thus, when both $k, l \in \mathbb{N}$ and $k + l \le n$, we have

$$P_n^{(-k,-l)}(z) = 2^{-k-l} (z-1)^k (z+1)^l P_{n-k-l}^{(k,l)}(z). \tag{1.3}$$

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Furthermore, when $n+\alpha+\beta=-k\in\{-1,\dots,-n\}$,

$$P_n^{(\alpha,\beta)}(z)=\frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha)}\,\frac{(k-1)!}{n!}\,P_{k-1}^{(\alpha,\beta)}(z)\,; \tag{1.4}$$

see [29, Eq. (4.22.3)]. See §4.22 of [29] for a more detailed discussion. Formulas (1.2)–(1.4) allow us to exclude these special integer parameters from our analysis.

In this paper we will show that for general $\alpha,\beta\in\mathbb{C}$, but excluding some special cases, the Jacobi polynomials $P_n^{(\alpha,\beta)}$ may still be characterized by orthogonality relations. The case $\alpha,\beta>-1$ is classical, and standard orthogonality on the interval $[-1,1]$ takes place, this being the key for the study of many properties of Jacobi polynomials. Thus, our goal is to establish orthogonality conditions for the remaining cases. We will show that $P_n^{(\alpha,\beta)}$ satisfies orthogonality conditions on certain curves in the complex plane. In some cases the orthogonality conditions on a single curve are enough to characterize the Jacobi polynomial, while in others a combination of orthogonality conditions on two or three curves is required. This last phenomenon is called multiple orthogonality; see e.g. [3, 27]. In some particular cases this orthogonality has been established before (see e.g. [4]–[9]), and used in the study of asymptotic behavior of these polynomials [25]. Similar orthogonality conditions, but for Laguerre polynomials, have been applied in [26] in order to study the zero distribution in the case of varying parameters, and in [23]–[24], in order to establish the strong asymptotics by means of the Riemann-Hilbert techniques. A different kind of orthogonality involving derivatives has been found for negative integer values of the parameters of $P_n^{(\alpha,\beta)}$; see [1]–[2].

We believe that these new orthogonality conditions can be useful in the study of the zeros of Jacobi polynomials. For general $\alpha,\beta\in\mathbb{C}$ the zeros are not confined to the interval $[-1,1]$ but they distribute themselves in the complex plane. K. Driver, P. Duren and collaborators [11]–[20] noted that the behavior of these zeros is very well organized; see also [25]. We believe that the orthogonality conditions we find, and in particular the Riemann-Hilbert problem derived from that (see Section 3 below) can be used to establish asymptotic properties of Jacobi polynomials. In particular this could explain the observed behavior of zeros.

2. Orthogonality on a Riemann surface.

Consider the path Γ encircling the points $+1$ and -1 first in a positive sense and then in a negative sense, as shown in Fig. 2.1. The point $\xi\in(-1,1)$ is the beginning and endpoint of Γ.

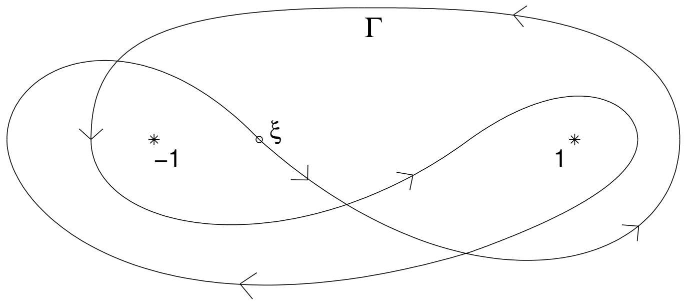

FIG. 2.1. *Path* Γ.

For $\alpha,\beta\in\mathbb{C}$, define

$$w(z;\alpha,\beta):=(1-z)^\alpha(z+1)^\beta=\exp[\alpha\log(1-z)+\beta\log(z+1)].$$

It is a multi-valued function with branch points at ∞ and ± 1. However, if we start with a value of $w(z;\alpha,\beta)$ at a particular point of Γ, and extend the definition of $w(z;\alpha,\beta)$ continuously along Γ, then we obtain a single-valued function $w(z;\alpha,\beta)$ on Γ if we view Γ as a contour on the Riemann surface for the function $w(z;\alpha,\beta)$. For definiteness, we assume that the "starting point" is $\xi \in (-1,1)$, and that the branch of w is such that $w(\xi;\alpha,\beta) > 0$.

In the sequel we prefer to view Γ as a subset of the complex plane. Then Γ has points of self-intersection; see Fig. 2.1. At points of self-intersection the value of $w(z;\alpha,\beta)$ is not well-defined.

The following is our main result. It shows that the Jacobi polynomials satisfy non-hermitian orthogonality conditions on Γ.

THEOREM 2.1. *Let $\alpha,\beta \in \mathbb{C}$, and let Γ, $w(z;\alpha,\beta)$ have the meaning as described above. Then for $k \in \{0,1,\dots,n\}$,*

$$\int_\Gamma t^k\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt = \frac{-\pi^2 2^{2n+\alpha+\beta+3} e^{\pi i(\alpha+\beta)}}{\Gamma(2n+\alpha+\beta+2)\Gamma(-n-\alpha)\Gamma(-n-\beta)}\, \delta_{kn}\,. \tag{2.1}$$

Proof. In the proof we use $f^{(k)}$ to denote the k-th derivative of f.

By the Rodrigues formula (1.1),

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}\, \frac{w^{(n)}(t;n+\alpha,n+\beta)}{w(t;\alpha,\beta)}. \tag{2.2}$$

Integrating in (2.1) n times by parts and using (2.2), we get

$$\begin{aligned} \int_\Gamma t^k P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt &= \frac{(-1)^n}{2^n n!} \sum_{j=0}^{n-1} (-1)^j \left[t^k\right]^{(j)} w^{(n-j-1)}(t;n+\alpha,n+\beta)\Big|_\Gamma \\ &\quad + \frac{1}{2^n n!} \int_\Gamma \left[t^k\right]^{(n)} w(t;n+\alpha,n+\beta)\, dt\,. \end{aligned} \tag{2.3}$$

Since $w(z;\alpha,\beta)$ is single-valued on Γ,

$$\left[t^k\right]^{(j)} w^{(n-j-1)}(t;n+\alpha,n+\beta)\Big|_\Gamma = 0\,, \qquad \text{for } 0 \le j \le n-1\,.$$

Thus, if $k \le n-1$, all the terms in the right-hand side of (2.3) vanish, which proves that the integral in (2.1) is 0 for $k = 0,1,\dots,n-1$. Furthermore, for $k = n$, we get

$$I_n(\alpha,\beta) := \int_\Gamma t^n\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt = 2^{-n} \int_\Gamma w(t;n+\alpha,n+\beta)\, dt\,.$$

Observe that $I_n(\alpha,\beta)$ is an analytic function of α and β, so that we may compute it for a certain range of parameters and then extend it analytically elsewhere. Following [30, §12.43], we assume $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and deform the path Γ, tautening it between -1 and $+1$. Thus, Γ will become the union of two small circles around ± 1 and two straight lines along $[-1,1]$, each piece traversed twice, once in a positive direction and once in a negative direction.

Since the integrand is bounded in the neighborhoods of ± 1, $I_n(\alpha,\beta)$ splits into the following four integrals:

$$\begin{aligned} I_n(\alpha,\beta) &= \int_{-1}^1 f(t)\, dt - e^{2\pi i(\alpha+n)} \int_{-1}^1 f(t)\, dt \\ &\quad + e^{2\pi i(\alpha+\beta+2n)} \int_{-1}^1 f(t)\, dt - e^{2\pi i(\beta+n)} \int_{-1}^1 f(t)\, dt\,, \end{aligned}$$

where

$$f(t) = 2^{-n} w(t; n+\alpha, n+\beta) > 0 \ \text{ for } \ t \in (-1,1)\,.$$

Thus,

$$\begin{aligned} I_n(\alpha,\beta) &= \left(1 - e^{2\pi i(\alpha+n)}\right) \left(1 - e^{2\pi i(\beta+n)}\right) \int_{-1}^{1} f(t)\,dt \\ &= -4e^{\pi i(\alpha+\beta)} \sin(\pi\alpha)\, \sin(\pi\beta) \int_{-1}^{1} f(t)\,dt\,. \end{aligned}$$

Changing the variable, $t = 2x - 1$, we get immediately the integral defining the beta function, and as a consequence

$$I_n(\alpha,\beta) = -2^{n+\alpha+\beta+3}\, e^{\pi i(\alpha+\beta)}\, \sin(\pi\alpha)\, \sin(\pi\beta)\, \frac{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{\Gamma(2n+\alpha+\beta+2)}\,.$$

Using $\sin(\pi x)\Gamma(x)\Gamma(1-x) = \pi$, we obtain (2.1) for $k = n$ and for α and β with $\operatorname{Re}\alpha$ and $\operatorname{Re}\beta$ positive. By analytic continuation the identity (2.1) holds for every $\alpha, \beta \in \mathbb{C}$. ☐

Observe that the right hand side in (2.1) vanishes for $k = n$ if and only if either $-2n - \alpha - \beta - 2$, or $n + \alpha$ or $n + \beta$ is a non-negative integer. In some of these cases the zero comes from integrating a single-valued analytic function along a curve in the region of analyticity; other values of α and β correspond to the special cases mentioned before when there is a zero at ± 1.

3. A Riemann-Hilbert problem for Jacobi polynomials. In this section we construct a Riemann-Hilbert problem whose solution is given in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with parameters satisfying

$$-n-\alpha-\beta \notin \mathbb{N}, \quad \text{and} \quad n+\alpha \notin \mathbb{N}, \quad \text{and} \quad n+\beta \notin \mathbb{N}. \tag{3.1}$$

We consider Γ as a curve in $\mathbb{C}$ with three points of self-intersection. We let Γ^o be the curve without the points of self-intersection. The orientation of Γ, as in Fig. 2.1, induces a $+$ and $-$ side in a neighborhood of Γ, where the $+$ side is on the left while traversing Γ according to its orientation and the $-$ side is on the right. We say that a function Y on $\mathbb{C} \setminus \Gamma$ has a boundary value $Y_+(t)$ for $t \in \Gamma^o$ if the limit of $Y(z)$ as $z \to t$ with z on the $+$ side of Γ exist. Similarly for $Y_-(t)$.

The Riemann-Hilbert problem asks for a 2×2 matrix valued function $Y : \mathbb{C}\setminus\Gamma \to \mathbb{C}^{2\times 2}$ such that the following four conditions are satisfied.

(a) Y is analytic on $\mathbb{C} \setminus \Gamma$.

(b) Y has continuous boundary values on Γ^o, denoted by Y_+ and Y_-, such that

$$Y_+(t) = Y_-(t) \begin{pmatrix} 1 & w(t;\alpha,\beta) \\ 0 & 1 \end{pmatrix} \qquad \text{for } t \in \Gamma^o.$$

(c) As $z \to \infty$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};$$

(d) $Y(z)$ remains bounded as $z \to t \in \Gamma \setminus \Gamma^o$.

This Riemann-Hilbert problem is similar to the Riemann-Hilbert problem for orthogonal polynomials due to Fokas, Its, and Kitaev [21]; see also [10]. Also the solution is similar. It is built out of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$. For parameters satisfying (3.1), the polynomial $P_n^{(\alpha,\beta)}$ has degree n; recall that there is a degree reduction if and only if $-n-\alpha-\beta \in \{1,\dots,n\}$. Therefore there is a constant c_n such that

$$c_n P_n^{(\alpha,\beta)} \quad \text{is a monic polynomial.} \tag{3.2}$$

From Theorem 2.1 and the condition (3.1) on the parameters, it follows that $P_{n-1}^{(\alpha,\beta)}$ satisfies

$$I_{n-1}(\alpha,\beta) = \int_\Gamma t^{n-1} P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta) dt \neq 0.$$

Thus we can define the constant $d_{n-1} = -2\pi i / I_{n-1}(\alpha,\beta)$ such that

$$d_{n-1} \int_\Gamma t^{n-1} P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta) dt = -2\pi i. \tag{3.3}$$

Then we can state the following result.

PROPOSITION 3.1. *The unique solution of the Riemann-Hilbert problem is given by*

$$Y(z) = \begin{pmatrix} c_n P_n^{(\alpha,\beta)}(z) & \frac{c_n}{2\pi i} \int_\Gamma \frac{P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)}{t-z} dt \\ d_{n-1} P_{n-1}^{(\alpha,\beta)}(z) & \frac{d_{n-1}}{2\pi i} \int_\Gamma \frac{P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta)}{t-z} dt \end{pmatrix}. \tag{3.4}$$

Proof. The proof that (3.4) satisfies the Riemann-Hilbert problem is similar to the proof for usual orthogonal polynomials; see e.g. [10, 22]. The condition (a) is obviously satisfied by (3.4). The jump condition (b) follows from the Sokhotskii-Plemelj formula

$$f_+(t) = f_-(t) + v(t), \qquad t \in \Gamma^o,$$

which is satisfied for $f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{v(t)}{t-z} dt$. The asymptotic condition (c) follows because of the normalizations (3.2)–(3.3) and the orthogonality conditions given in Theorem 2.1. For example, for the $(2,2)$ entry of Y the condition (c) is $Y_{22}(z) = z^{-n} + O(z^{-n-1})$ as $z \to \infty$, and this is satisfied by the $(2,2)$ entry of (3.4) since

$$\begin{aligned} \frac{d_{n-1}}{2\pi i} \int_\Gamma \frac{P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta)}{t-z} dt &= -\sum_{k=0}^{\infty} \left(\frac{d_{n-1}}{2\pi i} \int_\Gamma t^k P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta) dt \right) z^{-k-1} \\ &= \left(-\frac{d_{n-1}}{2\pi i} \int_\Gamma t^{n-1} P_{n-1}^{(\alpha,\beta)}(t) w(t;\alpha,\beta) dt \right) z^{-n} + O(z^{-n-1}) \qquad \text{as } z \to \infty \\ &= z^{-n} + O(z^{-n-1}) \quad \text{as } z \to \infty. \end{aligned}$$

For the first equality we used the Laurent expansion of $\frac{1}{t-z}$ around $z = \infty$, for the second equality we used the orthogonality conditions satisfied by $P_{n-1}^{(\alpha,\beta)}$ on Γ, and for the third equality we used (3.3). Finally, the boundedness condition (d) is certainly satisfied for the first column of (3.4). It is also satisfied by the second column, since by analyticity we may deform the contour Γ from which it follows that the entries in the second column have analytic continuations across Γ. Then they are certainly bounded.

The proof of uniqueness is as in [10, 22] and we omit it here. □

Using the Riemann-Hilbert problem, we can easily prove that the orthogonality conditions in Theorem 2.1 characterize the Jacobi polynomial in case the parameters satisfy (3.1).

THEOREM 3.2. *Assume that* $\alpha, \beta \in \mathbb{C}$ *satisfy* (3.1)*, and that* Γ*,* $w(z;\alpha,\beta)$ *are as above. Then the monic Jacobi polynomial* $c_n P_n^{(\alpha,\beta)}$ *is the only monic polynomial* p_n *of degree* n *that satisfies*

$$\int_\Gamma t^k p_n(t) w(t;\alpha,\beta)\,dt = 0, \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{3.5}$$

Proof. The orthogonality conditions (3.5) are what is necessary to fill the first row of Y. That is, if p_n is a monic polynomial of degree n, satisfying (3.5), then

$$\begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\int_\Gamma \frac{p_n(t)w(t;\alpha,\beta)}{t-z}dt \\ d_{n-1}P_{n-1}^{(\alpha,\beta)}(z) & \frac{d_{n-1}}{2\pi i}\int_\Gamma \frac{P_{n-1}^{(\alpha,\beta)}(t)w(t;\alpha,\beta)}{t-z}dt \end{pmatrix}$$

satisfies all conditions in the Riemann-Hilbert problem for Y. Since the solution is unique and given by (3.4) it follows that $p_n(z) = Y_{11}(z) = c_n P_n^{(\alpha,\beta)}(z)$. □

4. Non-hermitian quasiorthogonality. In the rest of this paper we assume for simplicity that α and β are real, although extension to non-real parameters is possible. We also take $\alpha, \beta, \alpha+\beta \notin \mathbb{Z}$, so that (3.1) is automatically guaranteed.

Since in (2.1) we are integrating an analytic function, we may deform the universal path Γ freely within the region of analyticity. In particular, if the integrand is integrable in the neighborhood of a branch point (± 1 or ∞), we may allow Γ to pass through this point, taking care of using the correct branch of the integrand.

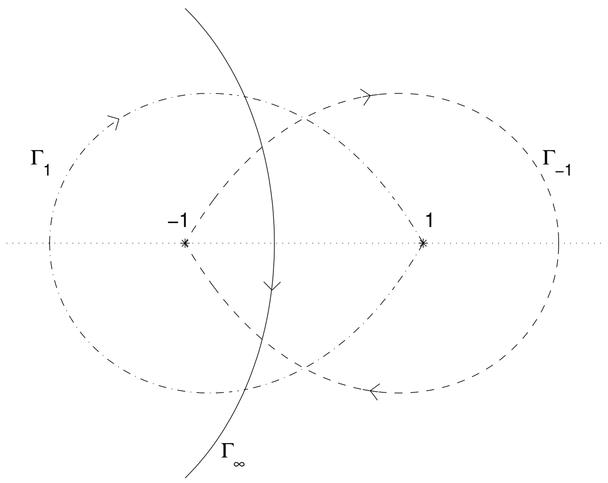

FIG. 4.1. *Paths of integration.*

We define the following paths of integration (see Fig. 4.1): Γ_1 will be an arbitrary curve oriented clockwise, connecting $1 - i0$ with $1 + i0$ and lying entirely in $\mathbb{C} \setminus [-1, +\infty)$, except for its endpoints. The circle $\{z \in \mathbb{C} : |z+1| = 2\}$, oriented clockwise, is a good instance of a curve Γ_1. Analogously, Γ_{-1} will be an arbitrary curve oriented clockwise, connecting

$-1+i0$ with $-1-i0$ and lying entirely in $\mathbb{C}\setminus(-\infty,1]$, except for its endpoints. Finally, Γ_∞ is a curve in $\mathbb{C}\setminus\big((-\infty,-1]\cup[1,+\infty)\big)$, extending from $+i\infty$ to $-i\infty$ (for example, the imaginary axis, oriented downward, might do as Γ_∞).

In what follows, we denote by $\mathbb{P}_n$ the set of all algebraic polynomials with complex coefficients of degree $\le n$ (for $n<0$ we assume $\mathbb{P}_n=\emptyset$), and by $\mathbb{P}_n^*=\mathbb{P}_n\setminus\mathbb{P}_{n-1}$, the subset of polynomials of degree exactly n. Also, for $c\in\mathbb{R}$, we use $[c]$ to denote the largest integer $\le c$, as usual.

The following theorem shows that for certain combinations of parameters a number of orthogonality relations are valid on Γ_{-1}, Γ_1, or Γ_∞. These are called quasi-orthogonality relations, since in general there are less than n conditions on the polynomial $P_n^{(\alpha,\beta)}$, so that these relations do not characterize the polynomial. When integrating over Γ_s, $s\in\{-1,1,\infty\}$, we mean by $w(t;\alpha,\beta)$ a branch of the weight function $(1-t)^\alpha(t+1)^\beta$ which is continuous on $\Gamma_s\setminus\{s\}$.

THEOREM 4.1. *Assume that* $\alpha,\beta,\alpha+\beta\in\mathbb{R}\setminus\mathbb{Z}$.

i) *If* $n+\beta>-1$, *and* $k=\max\{0,[-\beta]\}$, *then*

$$\int_{\Gamma_{-1}} q(t)\,P_n^{(\alpha,\beta)}(t)w(t;\alpha,k+\beta)\,dt\begin{cases}=0, & \forall\, q\in\mathbb{P}_{n-k-1}\,,\\ \neq 0, & \forall\, q\in\mathbb{P}^*_{n-k}\,.\end{cases}\tag{4.1}$$

ii) *If* $n+\alpha>-1$, *and* $k=\max\{0,[-\alpha]\}$, *then*

$$\int_{\Gamma_{1}} q(t)\,P_n^{(\alpha,\beta)}(t)w(t;k+\alpha,\beta)\,dt\begin{cases}=0, & \forall\, q\in\mathbb{P}_{n-k-1}\,,\\ \neq 0, & \forall\, q\in\mathbb{P}^*_{n-k}\,.\end{cases}\tag{4.2}$$

iii) *If* $n+\alpha+\beta<-1$ *and* $m=\min\{n-1,[-(n+\alpha+\beta+1)]\}$, *then*

$$\int_{\Gamma_\infty} q(t)\,P_n^{(\alpha,\beta)}(t)w(t;\alpha,\beta)\,dt=0\,,\quad \forall\, q\in\mathbb{P}_m\,.\tag{4.3}$$

If additionally, $n+\alpha+\beta<-n-1$, *then*

$$\int_{\Gamma_\infty} q(t)\,P_n^{(\alpha,\beta)}(t)w(t;\alpha,\beta)\,dt\begin{cases}=0\,, & \forall\, q\in\mathbb{P}_{n-1}\,,\\ \neq 0\,, & \forall\, q\in\mathbb{P}^*_{n}\,.\end{cases}$$

Proof. Assume that for a polynomial $q\in\mathbb{P}_n$ the function $f(t)=q(t)P_n^{(\alpha,\beta)}(t)\,w(t;\alpha,\beta)$ is integrable at $t=-1$. Then we can deform the path Γ in (2.1) into Γ_{-1} traversed twice (in opposite directions), so that

$$\int_\Gamma f(t)\,dt=\left(e^{2\pi i\beta}-1\right)\int_{\Gamma_{-1}} f(t)\,dt\,.$$

If $n+\beta>-1$, then $k=\max\{0,[-\beta]\}\in\{0,1,\dots,n\}$, $k+\beta>-1$, and the integrability of f at -1 is guaranteed taking

$$q(t)=(t+1)^k r(t)\,,\quad r\in\mathbb{P}_{n-k}\,.$$

Then,

$$\begin{aligned}\int_\Gamma q(t)P_n^{(\alpha,\beta)}(t)w(t;\alpha,\beta)\,dt&=\left(e^{2\pi i\beta}-1\right)\int_{\Gamma_{-1}} q(t)P_n^{(\alpha,\beta)}(t)w(t;\alpha,\beta)\,dt\\&=\left(e^{2\pi i\beta}-1\right)\int_{\Gamma_{-1}} r(t)P_n^{(\alpha,\beta)}(t)w(t;\alpha,k+\beta)\,dt\,,\end{aligned}$$

and (4.1) follows from Theorem 2.1.

The proof of (4.2) follows by reversing the roles of α and β.

The proof of part iii) uses similar arguments. If for a polynomial q, $n+\alpha+\beta+\deg q < -1$, then $f = qP_n^{(\alpha,\beta)} w(\cdot;\alpha,\beta)$ is integrable at ∞, and we can deform the path Γ in (2.1) into Γ_∞ traversed four times (two in each direction), each time with a different branch of the integrand. Then again

$$\int_\Gamma q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt = c \int_{\Gamma_\infty} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt\,, \quad c \neq 0.$$

In particular, $n+\alpha+\beta+m < -1$, and we may apply (2.1) with $k = 0,\dots,m \le n-1$ to establish (4.3). Furthermore, if $n+\alpha+\beta < -n-1$, then (2.1) can be used up to $k = n$. This concludes the proof. □

REMARKS: If $-1 < n+\beta < 0$, then $\max\{0,[-\beta]\} = n$, and (4.1) is reduced to a single condition,

$$\int_{\Gamma_{-1}} P_n^{(\alpha,\beta)}(t) w(t;\alpha,n+\beta)\, dt \neq 0\,.$$

An analogous degenerate situation is observed when $-1 < n+\alpha < 0$. On the other hand, if $-n-1 < n+\alpha+\beta < -1$, the integral in (4.3) diverges for $q \in \mathbb{P}_{m+1}^*$.

5. Orthogonality on a single contour. Sometimes it is possible to obtain a full set of orthogonality conditions on Γ_{-1}, Γ_1 or Γ_∞.

THEOREM 5.1 (Non-hermitian orthogonality). *If for $\alpha,\beta,\alpha+\beta \in \mathbb{R}\setminus\mathbb{Z}$, at least one of the following conditions is fulfilled:*

$$\alpha > -1, \qquad \beta > -1, \qquad 2n+\alpha+\beta < 0, \tag{5.1}$$

then Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfy a full set of non-hermitian (complex) orthogonality conditions:

$$\int_\gamma q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-1}\,, \\ \neq 0\,, & q \in \mathbb{P}_n^*, \end{cases} \tag{5.2}$$

where

$$\gamma = \begin{cases} \Gamma_1\,, & \text{if } \alpha > -1, \\ \Gamma_{-1}\,, & \text{if } \beta > -1, \\ \Gamma_\infty\,, & \text{if } 2n+\alpha+\beta < 0\,. \end{cases}$$

(If $-1 < 2n+\alpha+\beta < 0$ the integral in (5.2) diverges for $\gamma = \Gamma_\infty$ and $q \in \mathbb{P}_n^*$).

The conditions (5.2) characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. This is an immediate consequence of Theorem 4.1, since under our assumptions, $k = 0$ in (4.1)–(4.2). It is straightforward to show that orthogonality conditions in (5.2) are equivalent to (3.5), and thus by Theorem 3.2 characterize the polynomial. □

The following result, describing the real orthogonality of Jacobi polynomials, is classical, but we put it within the general framework.

COROLLARY 5.2 (Real orthogonality). *If for $\alpha, \beta, \alpha+\beta \in \mathbb{R} \setminus \mathbb{Z}$, two of the three conditions in (5.1) are fulfilled, then Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfy a full set of orthogonality conditions on the real line:*

$$\int_\Delta q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-1}, \\ \neq 0\,, & q \in \mathbb{P}_n^*, \end{cases} \tag{5.3}$$

where

$$\Delta = \begin{cases} [-1,1]\,, & \text{if } \alpha, \beta > -1, \\ (-\infty,-1]\,, & \text{if } 2n+\alpha+\beta < 0 \text{ and } \beta > -1, \\ [1,+\infty)\,, & \text{if } 2n+\alpha+\beta < 0 \text{ and } \alpha > -1\,. \end{cases} \tag{5.4}$$

(If $-1 < 2n+\alpha+\beta < 0$ the integral in (5.3) diverges for $q \in \mathbb{P}_n^*$).

The conditions (5.3) characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. This is a consequence of Theorem 5.1. For instance, if $2n+\alpha+\beta < 0$, by (5.2) we have non-hermitian orthogonality on Γ_∞; if additionally $\alpha > -1$, we can deform Γ_∞ into $[1,+\infty)$ traversed twice, and the statement follows. □

6. Multiple orthogonality. Theorem 5.1 provides orthogonality conditions, characterizing Jacobi polynomials when their parameters belong to the region in the (α,β)-plane given by at least one of the conditions (5.1).

For other combinations of parameters we still have some orthogonality relations according to Theorem 4.1, but each of the three orthogonality relations (4.1), (4.2), (4.3) does not give enough conditions to determine $P_n^{(\alpha,\beta)}$ by itself. However, the three relations taken together give n or more relations for $P_n^{(\alpha,\beta)}$. They constitute what we call a set of multiple orthogonality conditions. In many cases there will be more than n conditions, so that the relations of Theorem 4.1 overdetermine $P_n^{(\alpha,\beta)}$.

We are going to discuss this in more detail now.

6.1. Multiple orthogonality as an alternative to orthogonality on a single contour. We will consider the following subcases of the situation described in Theorem 5.1.

THEOREM 6.1. *Let $\alpha, \beta, \alpha+\beta \in \mathbb{R} \setminus \mathbb{Z}$ such that exactly one of the conditions (5.1) is satisfied, and such that either $-n < \alpha < -1$ or $-n < \beta < -1$. Then*

$$\int_\gamma q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt = 0\,, \quad q \in \mathbb{P}_{k-1}\,, \tag{6.1}$$

and

$$\int_\Delta q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha_1,\beta_1)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-k-1}, \\ \neq 0\,, & q \in \mathbb{P}_{n-k}^*, \end{cases} \tag{6.2}$$

where the corresponding parameters are gathered in the following table:

Cases	γ	Δ	α_1	β_1	k
$\alpha > -1$, $-n < \beta < -1$	Γ_1	$[-1,1]$	α	$\beta + [-\beta]$	$[-\beta]$
$\beta > -1$, $-n < \alpha < -1$	Γ_{-1}	$[-1,1]$	$\alpha + [-\alpha]$	β	$[-\alpha]$
$2n+\alpha+\beta < 0$, $-n < \beta < -1$	Γ_∞	$(-\infty,-1]$	α	$\beta + [-\beta]$	$[-\beta]$
$2n+\alpha+\beta < 0$, $-n < \alpha < -1$	Γ_∞	$[1,+\infty)$	$\alpha + [-\alpha]$	β	$[-\alpha]$

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(If $-1 < 2n+\alpha+\beta < 0$ the integral in (6.2) diverges for $q \in \mathbb{P}^*_{n-k}$).

In each case, these multiple orthogonality conditions characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. Assume for instance that $\beta > -1$ and $-n < \alpha < -1$. Then we have on the one hand by Theorem 5.1 that

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-1}\,, \\ \neq 0\,, & q \in \mathbb{P}^*_n\,, \end{cases} \tag{6.3}$$

and on the other hand by Theorem 4.1 ii), that

$$\int_{\Gamma_1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+k,\beta)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-k-1}\,, \\ \neq 0\,, & q \in \mathbb{P}^*_{n-k}\,, \end{cases} \tag{6.4}$$

where $k = [-\alpha]$. By Euclid's algorithm, any $q \in \mathbb{P}_m$, $m \geq k$, may be represented in the form $q(t) = (t-1)^k r(t) + s(t)$ with $r \in \mathbb{P}_{m-k}$ and $s \in \mathbb{P}_{k-1}$. Thus, (6.3) is equivalent to (6.1) and

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+k,\beta)\, dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-k-1}\,, \\ \neq 0\,, & q \in \mathbb{P}^*_{n-k}. \end{cases} \tag{6.5}$$

Since in (6.4)–(6.5) we can deform the path of integration and the functions are integrable at the singularities ± 1, these two identities are equivalent to (6.2) with $\Delta = [-1,1]$, $\alpha_1 = \alpha + [-\alpha]$, and $\beta_1 = \beta$. Thus, (6.1)–(6.2) are equivalent to (6.3), and they characterize $P_n^{(\alpha,\beta)}$ up to a constant factor by Theorem 5.1.

The other cases are handled in a similar fashion. □

THEOREM 6.2. *Let $\alpha,\beta,\alpha+\beta \in \mathbb{R}\setminus\mathbb{Z}$ be such that $-n < \alpha+\beta+n < -1$ and either $\alpha > -1$ or $\beta > -1$. Then with $m = [-(n+\alpha+\beta+1)]$,*

$$\int_{\Delta} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt = 0\,, \quad q \in \mathbb{P}_m\,,$$

and

$$\int_{\gamma} t^k\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt \begin{cases} = 0\,, & \text{if } k \in \mathbb{N},\ m+1 \leq k \leq n-1\,, \\ \neq 0\,, & \text{if } k = n\,, \end{cases}$$

where

$$\begin{cases} \Delta = [1,+\infty),\ \gamma = \Gamma_1\,, & \text{if } \alpha > -1, \\ \Delta = (-\infty,-1],\ \gamma = \Gamma_{-1}\,, & \text{if } \beta > -1. \end{cases}$$

In each case, these multiple orthogonality conditions characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. This is a corollary of Theorem 5.1 and (4.3). □

6.2. Multiple orthogonality when there is no orthogonality on a single contour. As we have seen, in the cases analyzed so far the multiple orthogonality was in a certain sense "optional": whenever at least one of the conditions in (5.1) is satisfied, we can restrict ourselves to orthogonality on a single contour Γ_{-1}, Γ_1 or Γ_∞, as in Theorem 5.1. In the remaining cases there are quasi-orthogonality conditions on Γ_{-1}, Γ_1 and Γ_∞ as specified in Theorem 4.1, but we need at least two of these sets of quasi-orthogonality relations to characterize the Jacobi polynomial.

So in this part we assume that $\alpha < -1$, $\beta < -1$ and $2n + \alpha + \beta > 0$. In our next result we consider cases where the parameters are such that a combination of two of the cases in Theorem 4.1 give at least n orthogonality conditions. The theorem says how to obtain from that n conditions that characterize $P_n^{(\alpha,\beta)}$.

THEOREM 6.3. *Let $\alpha, \beta, \alpha+\beta \in \mathbb{R}\setminus\mathbb{Z}$ such that $\alpha < -1$, $\beta < -1$ and $2n+\alpha+\beta > 0$.*

i) *If $\alpha + \beta + n > -1$, then*

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta+[-\beta])\,dt = 0\,, \quad q \in \mathbb{P}_{[-\alpha]-1}\,, \tag{6.6}$$

$$\int_{\Gamma_1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+[-\alpha],\beta)\,dt = 0\,, \quad q \in \mathbb{P}_{[-\beta]-1}\,, \tag{6.7}$$

and

$$\int_{-1}^{1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+[-\alpha],\beta+[-\beta])\,dt \begin{cases} = 0\,, & q \in \mathbb{P}_{n-[-\alpha]-[-\beta]-1}\,, \\ \neq 0\,, & q \in \mathbb{P}_{n-[-\alpha]-[-\beta]}\,. \end{cases} \tag{6.8}$$

ii) *If $\alpha < -n$, then*

$$\int_{-\infty}^{-1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta+[-\beta])\,dt = 0\,, \quad q \in \mathbb{P}_{[-\alpha]-n-1}\,, \tag{6.9}$$

$$\int_{\Gamma_\infty} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\,dt = 0\,, \quad q \in \mathbb{P}_{[-\beta]-1}\,, \tag{6.10}$$

and

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) t^{-n+[-\alpha]} w(t;\alpha,\beta+[-\beta])\,dt \begin{cases} = 0\,, & q \in \mathbb{P}_{2n-[-\alpha]-[-\beta]-1}\,, \\ \neq 0\,, & q \in \mathbb{P}_{2n-[-\alpha]-[-\beta]}\,. \end{cases} \tag{6.11}$$

iii) *If $\beta < -n$, then*

$$\int_{1}^{+\infty} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+[-\alpha],\beta)\,dt = 0\,, \quad q \in \mathbb{P}_{[-\beta]-n-1}\,, \tag{6.12}$$

$$\int_{\Gamma_\infty} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\,dt = 0\,, \quad q \in \mathbb{P}_{[-\alpha]-1}\,, \tag{6.13}$$

and

$$\int_{\Gamma_1} q(t)\, P_n^{(\alpha,\beta)}(t) t^{-n+[-\beta]} w(t;\alpha+[-\alpha],\beta)\,dt \begin{cases} = 0\,, & q \in \mathbb{P}_{2n-[-\alpha]-[-\beta]-1}\,, \\ \neq 0\,, & q \in \mathbb{P}_{2n-[-\alpha]-[-\beta]}\,. \end{cases} \tag{6.14}$$

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In all three cases, these orthogonality conditions characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. Indeed, by parts i) and ii) of Theorem 4.1 we have

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta+[-\beta])\, dt \begin{cases} = 0, & q \in \mathbb{P}_{n-[-\beta]-1}\,, \\ \neq 0, & q \in \mathbb{P}^*_{n-[-\beta]}\,, \end{cases} \tag{6.15}$$

and

$$\int_{\Gamma_1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+[-\alpha],\beta)\, dt \begin{cases} = 0, & q \in \mathbb{P}_{n-[-\alpha]-1}\,, \\ \neq 0, & q \in \mathbb{P}^*_{n-[-\alpha]}\,. \end{cases} \tag{6.16}$$

Since $[-\alpha]+[-\beta] \leq n$, we obtain (6.6) and (6.7). For $q \in \mathbb{P}_{n-[-\alpha]-[-\beta]}$, we take the polynomial $(t-1)^{[-\alpha]}q(t)$ in (6.6) and deform the contour Γ_{-1} to $[-1,1]$, to obtain (6.8) as well.

To prove that the conditions characterize the Jacobi polynomial, let p_n be a polynomial of degree $\leq n$ that satisfies (6.6), (6.7), (6.8). These conditions imply (6.15) and (6.16), and passing to the contour Γ, we obtain

$$\int_\Gamma q(t)(t+1)^{[-\beta]} p_n(t) w(t,\alpha,\beta) dt = 0, \quad q \in \mathbb{P}_{n-[-\beta]-1},$$

and

$$\int_\Gamma q(t)(t-1)^{[-\alpha]} p_n(t) w(t,\alpha,\beta) dt = 0, \quad q \in \mathbb{P}_{n-[-\alpha]-1}.$$

Thus

$$\int_\Gamma q(t) p_n(t) w(t,\alpha,\beta) dt = 0 \tag{6.17}$$

for every $q \in \mathbb{P}_{n-1}$ of the form $q(t) = (t+1)^{[-\beta]}r(t) + (t-1)^{[-\alpha]}s(t)$ with $r \in \mathbb{P}_{n-[-\beta]-1}$ and $s \in \mathbb{P}_{n-[-\alpha]-1}$. The linear mapping

$$\mathbb{P}_{n-[-\beta]-1} \times \mathbb{P}_{n-[-\alpha]-1} \to \mathbb{P}_{n-1} : (r,s) \mapsto q(t) = (t+1)^{[-\beta]}r(t) + (t-1)^{[-\alpha]}s(t) \tag{6.18}$$

has a kernel consisting of pairs (r,s) such that $(t+1)^{[-\beta]}r(t)+(t-1)^{[-\alpha]}s(t) \equiv 0$. Then it is easy to see that $r(t) = (t-1)^{[-\alpha]}p(t)$ and $s(t) = -(t+1)^{[-\beta]}p(t)$ with $p \in \mathbb{P}_{n-[-\alpha]-[-\beta]-1}$. So the kernel of the mapping (6.18) has dimension $n-[-\alpha]-[-\beta]$. Then by the dimension theorem from linear algebra the range of the mapping (6.18) has dimension $(n-[-\beta])+(n-[-\alpha])-(n-[-\alpha]-[-\beta]) = n$, and so the mapping is surjective. It follows that (6.17) holds for every $q \in \mathbb{P}_{n-1}$, and then it follows from Theorem 3.2 that p_n is a multiple of the Jacobi polynomial of degree n, so that the orthogonality conditions (6.6), (6.7), (6.8) characterize $P_n^{(\alpha,\beta)}$ up to a constant factor. This proves part i) of the theorem.

The other parts are proved in a similar way. □

What remains is the case where the parameters are such that we need all three cases of Theorem 4.1 in order to obtain at least n conditions on $P_n^{(\alpha,\beta)}$.

THEOREM 6.4. *Let $\alpha,\beta,\alpha+\beta \in \mathbb{R}\setminus\mathbb{Z}$ be such that $\alpha > -n$, $\beta > -n$ and $\alpha+\beta+n < -1$. Then the following hold:*

$$\int_{\Gamma_{-1}} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta+[-\beta])\, dt \;=0\,, \quad q \in \mathbb{P}_{n-[-\beta]-1}\,, \tag{6.19}$$

$$\int_{\Gamma_1} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha+[-\alpha],\beta)\, dt \;=0\,, \quad q \in \mathbb{P}_{n-[-\alpha]-1}\,, \tag{6.20}$$

and

$$\int_{\Gamma_\infty} q(t)\, P_n^{(\alpha,\beta)}(t) w(t;\alpha,\beta)\, dt \;= 0\,, \qquad q \in \mathbb{P}_{[-\alpha]+[-\beta]-n-1}\,, \tag{6.21}$$

and these orthogonality conditions characterize the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n up to a constant factor.

Proof. The relations (6.19)–(6.21) were already established in (4.1)–(4.3) from Theorem 4.1.

To show that these conditions characterize the Jacobi polynomial, we assume that p_n is a polynomial of degree $\le n$, that satisfies (6.19)–(6.21). Passing to the contour Γ, we then get that

$$\int_{\Gamma} q(t) p_n(t) w(t;\alpha,\beta) dt = 0 \tag{6.22}$$

for every polynomial q satisfying either $q(t) = (t+1)^{[-\beta]} r(t)$ with $r \in \mathbb{P}_{n-[-\beta]-1}$, or $q(t) = (t-1)^{[-\alpha]} s(t)$ with $s \in \mathbb{P}_{n-[-\alpha]-1}$, or $q(t) \in \mathbb{P}_{[-\alpha]+[-\beta]-n-1}$.

In the rest of the proof we will show that every polynomial of degree $\le n-1$ can be written in the form

$$(t+1)^{[-\beta]} r(t) + (t-1)^{[-\alpha]} s(t) + q(t)$$

with $r \in \mathbb{P}_{n-[-\beta]-1}$, $s \in \mathbb{P}_{n-[-\alpha]-1}$ and $q \in \mathbb{P}_{[-\alpha]+[-\beta]-n-1]}$. Having that, we get that (6.22) holds for every $q \in \mathbb{P}_{n-1}$, and this characterizes the Jacobi polynomial up to a constant factor by Theorem 3.2.

So we want to prove that the linear mapping

$$\begin{aligned} &\mathbb{P}_{n-[-\beta]-1} \times \mathbb{P}_{n-[-\alpha]-1} \times \mathbb{P}_{[-\alpha]+[-\beta]-n-1} \to \mathbb{P}_{n-1} : \\ &(r,s,q) \mapsto (t+1)^{[-\beta]} r(t) + (t-1)^{[-\alpha]} s(t) + q(t) \end{aligned} \tag{6.23}$$

is surjective. Since it is a mapping between n-dimensional vector spaces, it suffices to show that (6.23) is injective.

Assume that (r,s,q) belongs to the kernel of (6.23). We are going to count the possible sign changes among the coefficients of $(t+1)^{[-\beta]} r(t) + q(t) = -(t-1)^{[-\alpha]} s(t)$ with respect to the standard monomial basis. Here we use the following lemma which can be found, for instance, in [28, Section V, Ch. 1, problems 4 and 32].

LEMMA 6.5. *If*

$$p(z) = \sum_{j=0}^{n} a_j z^j \quad \textit{and} \quad (z+1)p(z) = \sum_{j=0}^{n+1} b_j z^j\,,$$

then the number of sign changes among the b_j's is not more than the number of sign changes among the a_j's.

Applying the lemma $[-\beta]$ times, we get that the number of sign changes among the coefficients of $(t+1)^{[-\beta]} r(t)$ is at most the number of sign changes among the coefficients of $r(t)$, which is at most $n-[-\beta]-1$, since $r \in \mathbb{P}_{n-[-\beta]-1}$. When we add a polynomial of degree $\le k-1$, the number of sign changes among the coefficients can increase with at most k. Hence, $-(t-1)^{[-\alpha]} s(t)$ has at most $n-[-\beta]-1+[-\alpha]+[-\beta]-n = [-\alpha]-1$ sign changes among its coefficients. Then by the Descartes rule of signs, see e.g. [28, Section

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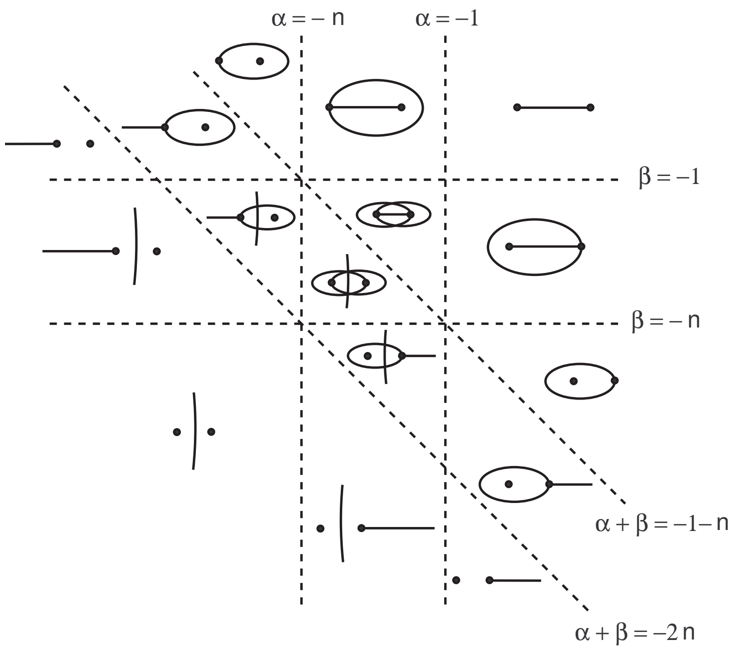

FIG. 6.1. *Contours with orthogonality conditions (black dots denote -1 and 1).*

V, Ch. 1, problem 36], $-(t-1)^{[-\alpha]}s(t)$ has at most $[-\alpha]-1$ positive real zeros, counted according to their multiplicities, unless it is identically zero. So we conclude that $s \equiv 0$. Then also $r \equiv 0$ and $q \equiv 0$. This shows that (6.23) is indeed injective, which completes the proof of Theorem 6.4. □

To summarize, the paths of orthogonality, corresponding to different cases studied, are represented schematically in Fig. 6.1.

7. Zeros. Finally, it is interesting to discuss the implication of the orthogonality relations derived to the location of the zeros of the Jacobi polynomials.

Standard (hermitian) orthogonality conditions with respect to a positive measure on the real line yield a lower bound (and in some situation, the exact number) of different zeros of the orthogonal polynomial on the convex hull of the support of the measure of orthogonality. For instance, by a classical argument, if two of the three conditions in (5.1) are fulfilled, orthogonality (5.3) ensures that $P_n^{(\alpha,\beta)}$ has exactly n real and simple zeros, all located in the interval Δ given by (5.4).

In the situation of Theorem 6.1, that is when (α,β) or $(\beta,\alpha) \in (-1,+\infty)\times(-n,-1)$, quasi-orthogonality (6.2) implies that $P_n^{(\alpha,\beta)}$ has *at least* $n-k$ different zeros on $(-1,1)$, where $k=[-\alpha]$ if $-n<\alpha<-1$, and $k=[-\beta]$ if $-n<\beta<-1$; see [8].

Finally, in the situation of Theorem 6.3, $\alpha,\beta \in (-n,-1)$ and $\alpha+\beta+n>-1$, we have that $P_n^{(\alpha,\beta)}$ has at least $n-[-\alpha]-[-\beta]$ different zeros on $(-1,1)$.

In the other situations we have no quasi-orthogonality on the real line, and the information on the real zeros is void.

It is interesting to compare these values with the well-known Hilbert-Klein formulas [29, Theorem 6.72] that give us the exact number of zeros of $P_n^{(\alpha,\beta)}$ on the real line. It is sufficient to restrict our attention to $[-1,1]$, since the other two intervals of $\mathbb{R}$ can be analyzed similarly.

Following [29], let

$$E(u) = \begin{cases} 0, & \text{if } u \le 0, \\ u-1, & \text{if } u \in \mathbb{N}, \\ [u], & \text{otherwise,} \end{cases}$$

and let $\mathrm{even}(u)$ (resp., $\mathrm{odd}(u)$) be the even (resp., odd) value from the set $\{u, u+1\}$, when $u \in \mathbb{N}$. Then the number of zeros of $P_n^{(\alpha,\beta))}$ in $(-1,1)$ is given by

$$N = \begin{cases} \mathrm{even}\left(E\left(\frac{|2n+\alpha+\beta+1|-|\alpha|-|\beta|+1}{2}\right)\right), & \text{if } \kappa_n(\alpha,\beta) > 0, \\ \mathrm{odd}\left(E\left(\frac{|2n+\alpha+\beta+1|-|\alpha|-|\beta|+1}{2}\right)\right), & \text{if } \kappa_n(\alpha,\beta) < 0, \end{cases} \tag{7.1}$$

where $\kappa_n(\alpha,\beta) = (-1)^n(\alpha+1)\cdots(\alpha+n)(\beta+1)\cdots(\beta+n)$.

It is straightforward to check that N is positive only in one of the following cases:

- $\alpha, \beta > -1$. Then $N = n$, as ensured by (5.3).
- (α,β) or $(\beta,\alpha) \in (-1,+\infty) \times (-n,-1)$. For instance, if $\beta > -1$ and $-n < \alpha < -1$, then

$$E\left(\frac{|2n+\alpha+\beta+1|-|\alpha|-|\beta|+1}{2}\right) = [n+\alpha+1] = n - [-\alpha]\,,$$

  since for $\alpha < 0$, $[\alpha] + [-\alpha] = -1$. In other words, this value matches the lower bound on the number of zeros N, predicted by the quasi-orthogonality relation on $[-1,1]$ given in Theorem 6.1.
- If $\alpha, \beta \in (-n,-1)$ and $\alpha + \beta + n > -1$,

$$E\left(\frac{|2n+\alpha+\beta+1|-|\alpha|-|\beta|+1}{2}\right) = \begin{cases} [n+\alpha+\beta+1] = n - [-\alpha-\beta]\,, & \text{if } \alpha+\beta \notin \mathbb{Z}\,, \\ n+\alpha+\beta, & \text{otherwise.} \end{cases}$$

  Observe that for $\tilde\alpha = \alpha + [-\alpha]$, $\tilde\beta = \beta + [-\beta]$,

$$\begin{aligned}(-1)^{n-[-\alpha]-[-\beta]}\kappa_n(\alpha,\beta) =& (-\alpha-1)\cdots(-\tilde\alpha)(\tilde\alpha+1)\cdots(\alpha+n)\times \\ & (-\beta-1)\cdots(-\tilde\beta)(\tilde\beta+1)\cdots(\beta+n) > 0\,.\end{aligned}$$

  Since $n - [-\alpha-\beta] \le n - [-\alpha] - [-\beta] \le n - [-\alpha-\beta] + 1$, we get from (7.1) that $N = n - [-\alpha] - [-\beta]$, which coincides again with the lower bound on the number of zeros, predicted by Theorem 6.3.

Thus, quasi-orthogonality sheds a new light on the Hilbert-Klein formulas, explaining the lower bound on the number of zeros on $\mathbb{R}$.

This connection seems to go beyond the real zeros. In the example in Fig. 7.1 the number of zeros on $[-1,1]$, Γ_{-1}, and Γ_1 matches the number of orthogonality conditions on the corresponding curves, given by Theorem 6.3. We cannot prove this experimental fact, but we expect to be able to establish this in an asymptotic sense by means of the techniques from [23]–[26].

Note added in proof: After submission of this manuscript we were indeed able to use the Riemann-Hilbert problem to derive strong asymptotics of the Jacobi polynomials which

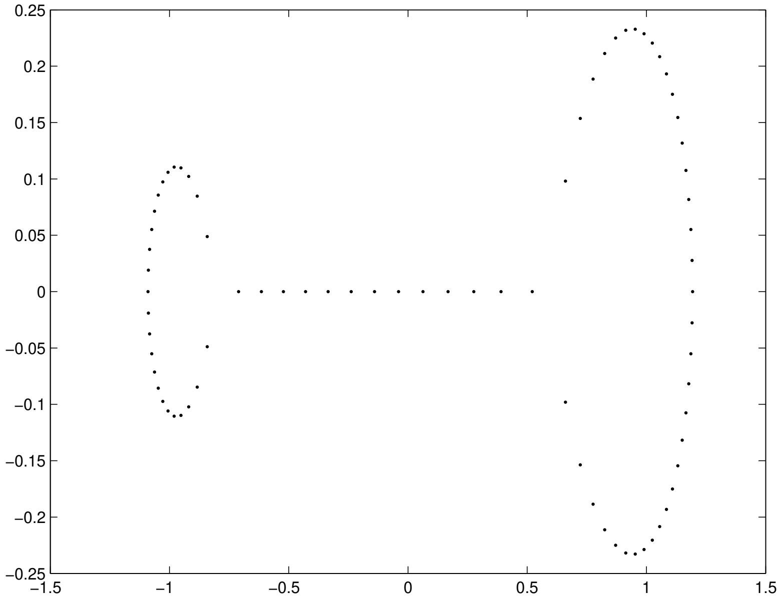

FIG. 7.1. *Zeros of* $P_{75}^{(-37.4,-25.1)}$. *We have* $13 = n - [-\alpha] - [-\beta]$ *zeros on* $(-1, 1)$, $25 = [-\beta]$ *zeros aligned on a curve on the left* (Γ_1), *and* $37 = [-\alpha]$ *zeros aligned on a curve on the right* (Γ_{-1}).

explains the observed behavior of the zeros in an asymptotic sense. More precisely, we studied Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ with varying parameters α_n and β_n such that $\alpha_n/n \to A$ and $\beta_n/n \to B$ as $n \to \infty$. In [31] the case $A, B > -1$ with $A + B < -1$ was considered, which corresponds to the multiple orthogonality on three contours, as given in Theorem 6.4. In [32] the case $A < 0 < B$ with $A + B > -1$ was treated in detail.

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