**Electronic Transactions on Numerical Analysis.**
**Volume 19, pp. 105-112, 2005.**

**ISSN 1068-9613.**

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**

# THREE CASES OF NORMALITY OF HESSENBERG'S MATRIX RELATED WITH ATOMIC COMPLEX DISTRIBUTIONS*

VENANCIO TOMEO† AND EMILIO TORRANO‡

**Abstract.** In this work we prove that Hessenberg's infinite matrix, associated with an hermitian OPS that generalizes the Jacobi matrix, is normal under the assumption that the OPS is generated from a discrete infinite bounded distribution of non-aligned points in the complex plane with some geometrical restrictions. This matrix is also normal if we consider a real bounded distribution with a finite amount of atomic complex points. In this case we still have normality with infinite points, but an additional condition is required. Some other interesting properties of that matrix are obtained.

**Key words.** orthogonal polynomials, Hessenberg's matrix, normal operator.

**AMS subject classifications.** 33C45.

## 1. Introduction.

Let $\mu(x)$ be a positive and finite Borel measure with real support. It is well known that there is an associated OPS, $\{P_n(x)\}_{n=0}^{\infty}$, that satisfies a three-term recurrence relation, with coefficients $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$.

These coefficients are the entries of a Jacobi tridiagonal matrix $J$. The spectrum of this infinite matrix, considered as an operator $J : \ell^2 \longrightarrow \ell^2$, permits us to study the properties of the measure;, see for instance [5] and [6].

Recently the interest to extend the results of the real case to Borel measures, supported in some bounded set of the complex plane, has increased; see ([10]). The role of the Jacobi tridiagonal matrix now is played by the upper Hessenberg matrix $D$, which is the expression of $S_\mu$ in the NOPS $\{\widehat{P}_n(z)\}_{n=0}^{\infty}$. Here $S_\mu$ is the multiplication operator by $z$ in $\overline{\Pi}$, the closure of the polynomials in $L^2_\mu$.

An important result of A. Atzmon for the unit disk (see [1]), which was extended in [12] to a bounded set of the complex plane, says that a matrix $M = (c_{j,k})_{j,k=0}^{\infty}$, which is HPD, is a moment matrix, i.e., exists $\Omega \subset \mathbb{C}$ and $\mu : \Omega \longrightarrow \mathbb{R}_+$, with $c_{j,k} = \int_\Omega z^j \overline{z}^k d\mu(z)$, if and only if, the operator $D : \ell^2 \to \ell^2$ is subnormal.

In the sequel we will study three distributions such as this operator $D$ is the minimal normal extension of itself. This paper extends the results of [11], related to the discrete finite bounded case.

In section 2 we have introduced lemmas and definitions, and in section 3 we will study the matrix D and some properties of it. In section 4 we are going to develop the discrete infinite bounded case. Finally, in sections 5 and 6 we will study the normality of infinite matrices related to real bounded distributions with a finite or infinite set of complex points.

## 2. Lemmas and Definitions.

LEMMA 2.1. *(page 40 [9]) Let $\mu(z)$ be a positive and finite Borel measure with bounded support $\Omega \subset \mathbb{C}$. Let $z_0$ be an arbitrary complex number. Then $\min \int_\Omega |Q_n(z)|^2 d\mu(z) = 1/K_n(z_0, z_0)$, where the minimum is computed as $Q_n(z)$ ranges over all complex polynomials of degree at most $n$ with the constraint $Q_n(z_0) = 1$. The minimum is attained for $Q_n(z) = K_n(z_0, z)/K_n(z_0, z_0)$, with $K_n(z, w) = \sum_{k=0}^{n} \widehat{P}_k(z)\overline{\widehat{P}_n(w)}$, where $\{\widehat{P}_k(z)\}$ is*

*Received November 30, 2002. Accepted for publication May 10, 2003. Communicated by F. Marcellán.
†Dpto. Algebra E.U. Estadística UCM.
‡Dpto. Mat. Aplicada Fac. Informática UPM.

**ETNA**
Kent State University
etna@mcs.kent.edu



*the normalized OPS associated to $\mu$.*

DEFINITION 2.2. *Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear transformations from the Hilbert space $\mathcal{H}$ into $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, then $T$ is normal if and only if $T^H T = TT^H$.*

DEFINITION 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T$ is quasi normal if and only if $(T^H T)T = T(T^H T)$.*

DEFINITION 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T$ is subnormal if and only if $T$ has a normal extension.*

DEFINITION 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T$ is hyponormal if and only if $T^H T - TT^H \geq 0$.*

DEFINITION 2.6. *If $T \in \mathcal{B}(\mathcal{H})$, the point spectrum of $T$, $\sigma_p(T)$, is defined by $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq \{0\}\}$.*

LEMMA 2.7. *(problems 195 and 203 of [7]) Assume $\|T\| < +\infty$. Then $T$ normal $\Rightarrow$ $T$ quasi normal $\Rightarrow T$ subnormal $\Rightarrow T$ hyponormal.*

LEMMA 2.8. *(problem 207 of [7]) If $A$ is hyponormal, $A = B + iC$, with $B$ and $C$ Hermitian and $C$ compact, then $A$ is normal.*

### 3. The infinite Hessenberg matrix $D$.

Given an infinite Hermitian positive definite matrix (HPD) $M = (c_{ij})_{i,j=0}^{\infty}$, coming from a measure or not, we call $M'$ the matrix obtained eliminating from matrix $M$ its first column. $M_n$ and $M'_n$ are the corresponding sections of order $n$ of $M$ and $M'$ respectively, i.e., the restrictions to their first $n$ rows and $n$ columns.

¿From $M$, an infinite Hessenberg matrix $D = (d_{ij})_{i,j=0}^{\infty}$ can be constructed such that its sections of order $n$ satisfy

$$D_n = T_n^{-1} M'_n T_n^{-H} = T_n^H F_n T_n^{-H},$$

where $M_n = T_n T_n^H$ is the Cholesky decomposition of $M_n$, and $F_n$ is the Frobenius matrix associated to $P_n(z)$, where $\{P_n(z)\}$ is the O.P.S. associated to $M$, with

$$P_n(z) = \begin{vmatrix} c_{00} & c_{10} & c_{20} & \cdots & c_{n0} \\ c_{01} & c_{11} & c_{21} & \cdots & c_{n1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1} & c_{1,n-1} & c_{2,n-1} & \cdots & c_{n,n-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}.$$

The triangularity of the matrix $T_n$, implies that

$$D = T^{-1} M' T^{-H} = T^H S_R T^{-H}, \tag{3.1}$$

where $S_R$ is the infinite matrix of the shift-right operator in $\ell^2$. We must be careful, because $T^{-1}$, $T^H$ and $T^{-H}$ are infinite triangular matrices but they do not define necessarily operators in $\ell^2$. In the sequel $\widetilde{P}_n(z)$ will be the monic polynomial and $\widehat{P}_n(z)$ will be the normalized polynomial.

LEMMA 3.1. *([1], [12])*

*If $M$ is an infinite and HPD (Hermitian positive definite) matrix, and $\|D\| < +\infty$, then $M$ is a moment matrix (i.e., the Gram matrix associated with the moments of a positive and finite Borel measure supported in some set of the complex plane) if and only if $D$ is subnormal.*

PROPOSITION 3.2. *If $z_{nk}$ is a root of $P_n(z)$ then*

$$z_{nk} \begin{pmatrix} \widehat{P}_0(z_{nk}) \\ \widehat{P}_1(z_{nk}) \\ \vdots \\ \widehat{P}_{n-1}(z_{nk}) \end{pmatrix} = D_n^t \begin{pmatrix} \widehat{P}_0(z_{nk}) \\ \widehat{P}_1(z_{nk}) \\ \vdots \\ \widehat{P}_{n-1}(z_{nk}) \end{pmatrix}.$$

*Proof.* We expand $\widetilde{P}_n(z) = |I_n z - D_n|$, and since $\|\widetilde{P}_n(z)\| = d_{21}d_{32}\ldots d_{n+1,n}$, it follows that

$$z\widehat{P}_{n-1}(z) = d_{1n}\widehat{P}_0(z) + d_{2n}\widehat{P}_1(z) + d_{3n}\widehat{P}_2(z) + \ldots + d_{n,n}\widehat{P}_{n-1}(z) + d_{n+1,n}\widehat{P}_n(z). \tag{3.2}$$

Taking $n = 1$, $n = 2$, …, $n = n$, in (3.2), row by row we have

$$z\begin{pmatrix} \widehat{P}_0(z) \\ \widehat{P}_1(z) \\ \widehat{P}_2(z) \\ \vdots \\ \widehat{P}_{n-1}(z) \end{pmatrix} = \begin{pmatrix} d_{11} & d_{21} & 0 & \ldots & 0 & 0 \\ d_{12} & d_{22} & d_{32} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{1,n-2} & d_{2,n-2} & d_{3,n-2} & \ldots & d_{n-1,n-2} & 0 \\ d_{1,n-1} & d_{2,n-1} & d_{3,n-1} & \ldots & d_{n-1,n-1} & d_{n,n-1} \\ d_{1,n} & d_{2,n} & d_{3,n} & \ldots & d_{n-1,n} & d_{n,n} \end{pmatrix}\begin{pmatrix} \widehat{P}_0(z) \\ \widehat{P}_1(z) \\ \widehat{P}_2(z) \\ \vdots \\ \widehat{P}_{n-1}(z) \end{pmatrix}$$

$$+ d_{n+1,n}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\widehat{P}_n(z). \tag{3.3}$$

We take $z = z_{nk}$, one of the roots of $P_n(z)$. It follows that

$$z_{nk}\begin{pmatrix} \widehat{P}_0(z_{nk}) \\ \widehat{P}_1(z_{nk}) \\ \widehat{P}_2(z_{nk}) \\ \vdots \\ \widehat{P}_{n-1}(z_{nk}) \end{pmatrix} = \begin{pmatrix} d_{11} & d_{21} & 0 & \ldots & 0 & 0 \\ d_{12} & d_{22} & d_{32} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{1,n-2} & d_{2,n-2} & d_{3,n-2} & \ldots & d_{n-1,n-2} & 0 \\ d_{1,n-1} & d_{2,n-1} & d_{3,n-1} & \ldots & d_{n-1,n-1} & d_{n,n-1} \\ d_{1,n} & d_{2,n} & d_{3,n} & \ldots & d_{n-1,n} & d_{n,n} \end{pmatrix}\begin{pmatrix} \widehat{P}_0(z_{nk}) \\ \widehat{P}_1(z_{nk}) \\ \widehat{P}_2(z_{nk}) \\ \vdots \\ \widehat{P}_{n-1}(z_{nk}) \end{pmatrix};$$

in other words $z_{nk}\overline{v}_n = D_n^t\overline{v}_n$. Hence $z_{nk}$ is an eigenvalue of $D_n^t$ and its eigenvector is

$$\overline{v}_n = (\widehat{P}_0(z_{nk}), \widehat{P}_1(z_{nk}), \ldots, \widehat{P}_{n-1}(z_{nk}))^t.$$

□

REMARK 3.3. *It is obvious that the eigenvalues of $D_n$ and $D_n^t$ are the same. Complex conjugation in (3.3) with $z = z_{nk}$, yields*

$$\overline{z}_{nk}\begin{pmatrix} \overline{\widehat{P}_0(z_{nk})} \\ \overline{\widehat{P}_1(z_{nk})} \\ \vdots \\ \overline{\widehat{P}_{n-1}(z_{nk})} \end{pmatrix} = D_n^H\begin{pmatrix} \overline{\widehat{P}_0(z_{nk})} \\ \overline{\widehat{P}_1(z_{nk})} \\ \vdots \\ \overline{\widehat{P}_{n-1}(z_{nk})} \end{pmatrix}.$$

*The eigenvectors of $D_n^H$ and $D_n$ are conjugate complex vectors, but we can't say anything about $D_n^t$. We also have that*

$$\sum_{k=0}^{n} |\widehat{P}_k(z_{nk})|^2 = K_n(z_{nk}, z_{nk}) = K_{n-1}(z_{nk}, z_{nk}) = \|\overline{v}_n\|^2.$$

ETNA
Kent State University
etna@mcs.kent.edu



*Note that the norm squared of the eigenvector associated to $z_{nk}$ is just the evaluation of the $n$-th kernel polynomial in the root. In the tridiagonal case, we know that the Christoffel constant $p_{nk}$, associated to $z_{nk}$, is $p_{nk} = 1/K_n(z_{nk}, z_{nk})$.*

PROPOSITION 3.4. *Let $D^t$ be a bounded operator, and take $\lambda \in \mathbb{C}$. Then $\lim_{n\to\infty} K_n(\lambda,\lambda) < +\infty$, if and only if, $\lambda \in \sigma_p(D^t)$.*

*Proof.* $\Rightarrow$) Let $\lambda \in \mathbb{C}$ be such that $\lim_n K_n(\lambda,\lambda) < +\infty$. Take $z = \lambda$ in (3.3). The sequence of kernel polynomials $K_n(\lambda,\lambda) = \sum_{k=0}^{n} \widehat{P}_k(\lambda)\overline{\widehat{P}_k(\lambda)}$ converges when $n$ tends to infinity and a consequence, $\lim_{n\to\infty} \widehat{P}_n(\lambda) = 0$. The boundedness of $D^t$ and $D$ implies that all its entries are bounded by $\|D\|$. In particular $|d_{n+1,n}| \le \|D\|$. The vector in the right member of (3.3) converges to the null vector. On the other hand $K_n(\lambda,\lambda)$ is convergent and hence $(\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \widehat{P}_2(\lambda), \ldots)^t \in \ell^2$, $\overline{v} = (\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \widehat{P}_2(\lambda), \ldots)^t$, and on taking limits we conclude that $\lambda\overline{v} = D^t\overline{v}$, where $\lambda \in \sigma_p(D^t)$.

$\Leftarrow$) To prove the converse, if $\lambda \in \sigma_p(D^t)$, there exists an $\overline{v} \in \ell^2$ such that $D^t\overline{v} = \lambda\overline{v}$. We have

$$\lambda \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix} = D^t \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix},$$

since

$$\begin{aligned}
\lambda v_0 &= d_{11}v_0 + d_{21}v_1 \\
\lambda v_1 &= d_{12}v_0 + d_{22}v_1 + d_{32}v_2 \\
\lambda v_2 &= d_{13}v_0 + d_{23}v_1 + d_{33}v_2 + d_{43}v_3 \\
&\vdots \\
\lambda v_{n-1} &= d_{1n}v_0 + d_{2n}v_1 + \ldots + d_{n,n}v_{n-1} + d_{n+1,n}v_n, \qquad (3.4)\\
&\vdots
\end{aligned}$$

with $d_{n+1,n} = \sqrt{\frac{|M_{n+1}|\,|M_{n-1}|}{|M_n|^2}} > 0, \forall n \in \mathbb{N}$.

We have only two possibilities: either $v_0 = 0$ or $v_0 \neq 0$. If $v_0 = 0$ then $v_1 = 0$, and if $v_0 = v_1 = 0$, then $v_2 = 0, \ldots$. Consequently $\overline{v} = 0$ and $\lambda$ is not an eigenvalue. Hence $v_0 \neq 0$. We prove $v_n = \widehat{P}_n(\lambda)v_0$ by induction.

The result is true for $v_1$. Since $d_{2,1} > 0$, $v_1$, we have $v_1 = \frac{\lambda - d_{11}}{d_{21}}\, v_0$, but $\lambda - d_{11} = \widetilde{P}_1(\lambda)$, and $d_{21} = \frac{1}{\gamma_1} = \|\widetilde{P}_1(z)\|$; hence, $v_1 = \widehat{P}_1(\lambda)v_0$. Suppose that $v_k = v_0\widehat{P}_k(\lambda)$, $\forall k \le n-1$. We need to prove that $v_n = \widehat{P}_n(\lambda)v_0$. Consider $v_n$ in (3.4). By the induction hypothesis we have[1]

$$\begin{aligned}
d_{n+1,n}v_n &= (\lambda - d_{nn})v_{n-1} - d_{n-1,n}v_{n-2} - \ldots - d_{1n}v_0 \\
&= (\lambda - d_{nn})\widehat{P}_{n-1}(\lambda)v_0 - d_{n-1,n}\widehat{P}_{n-2}(\lambda)v_0 - \ldots - d_{1n}\widehat{P}_0(\lambda)v_0 \\
&= \left[(\lambda - d_{nn})\widehat{P}_{n-1}(\lambda) - d_{n-1,n}\widehat{P}_{n-2}(\lambda) - \ldots - d_{1n}\widehat{P}_0(\lambda)\right] v_0,
\end{aligned}$$

[1] In the sequel we will suppose that the matrix $M$ is normalized; in other words if $c_{00} \neq 1$, we divide the matrix by $c_{00}$. Obviously we have $\widehat{P}_0(\lambda) = 1$.

the last parenthesis is substituted by using (3.2), then $d_{n+1,n}v_n = d_{n+1,n}\widehat{P}_n(\lambda)v_0$. Hence $v_n = \widehat{P}_n(\lambda)v_0$, and therefore we have that the eigenvector of $\lambda$ is proportional to $(\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \widehat{P}_2(\lambda), \ldots, \widehat{P}_n(\lambda), \ldots)^t \in \ell^2$. □

PROPOSITION 3.5. *Let $\mu(z)$ be a positive and finite measure of Borel with bounded support $\Omega \subset \mathbb{C}$. Let $\lambda$ be an atomic point of this measure. Then $\lambda \in \sigma_p(D^t)$ and $\overline{\lambda} \in \sigma_p(D^H)$.*

*Proof.* Assume that the measure $\mu$ has a weight $p_0$ in $\lambda$, that is $\mu(\lambda) = p_0$. For any polynomial $Q_n(z)$ such that $Q_n(\lambda) = 1$, we have that

$$\|Q_n(z)\|^2 = \int_\Omega |Q_n(z)|^2 d\mu(z) \geq |Q_n(\lambda)|^2 p_0 = p_0.$$

This easily follows from the definition of the Lebesgue-Stieltjes integral. By lemma 2.1 we have

$$\min \|Q_n(z)\|^2 = \min \int_\Omega |Q_n(z)|^2 d\mu(z) = \frac{1}{K_n(\lambda,\lambda)} \geq p_0,$$

hence, $K_n(\lambda,\lambda) \leq 1/p_0$. This inequality is true in the limit, so that

$$0 \leq \lim_{n\to\infty} K_n(\lambda,\lambda) = \sum_{k=0}^{\infty} \widehat{P}_k(\lambda)\overline{\widehat{P}_k(\lambda)} \leq \frac{1}{p_0}.$$

Hence the sequence $(\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \widehat{P}_2(\lambda), \widehat{P}_3(\lambda), \ldots)$, and its conjugate are in $\ell^2$. By proposition 3.4, $\lambda \in \sigma_p(D^t)$ and $\overline{\lambda} \in \sigma_p(D^H)$. □

## 4. Discrete infinite bounded case.

DEFINITION 4.1. *We speak of the discrete infinite bounded case when we assume a discrete set of bounded complex points $Z = \{z_k\}_{k=1}^{\infty} \subset \mathbb{C}$ with weights $\{p_k\}_{k=1}^{\infty} \subset \mathbb{R}_+^*$, such that $\sum_k^\infty p_k < +\infty$.*

For that distribution we have the moment matrix $M = (c_{ij})_{ij=0}^{\infty}$, where $c_{jk} = \sum_{n=1}^{\infty} z_n^j \overline{z}_n^k p_n$. Let $D$ be the associated Hessenberg matrix. Obviously $\mathrm{supp}(\mu) = \overline{Z}$.

As usual (see [3] page 114), $N_\mu$ will be the operator multiplication by $z$ in $L_\mu^2$. We know that $N_\mu$ is the minimal normal extension of $S_\mu$. Assuming that all the operators that are in the context are bounded, it is easy to prove that $S_\mu$ is unitarily equivalent to the infinite Hessenberg matrix $D$, considered as an operator in $\ell^2$, and $N_\mu$ is unitarily equivalent to operator $N$, which is the minimal normal extension of $D$. The next elegant proof is due to Prof. Raquel Gonzalo.

PROPOSITION 4.2. *If $\mathbb{C} \setminus \overline{Z}$ is a connected set and the interior of $\overline{Z}$ is empty, then the infinite Hessenberg matrix $D$ is a normal operator in $\ell^2$.*

*Proof.* The set $K = \overline{Z}$ is compact. As usual we call $C(K)$ to the space of all continuous functions with support $K$. The set $K$ satisfies the hypothesis of Mergelyan theorem (see [3] page 363), and in consequence $\forall f \in C(K)$ and $\forall \epsilon > 0$ , there exists a polynomial $P(z)$ such that $|f(z) - P(z)| < \epsilon$. This implies that $\int_{\mathrm{supp}(\mu)} |f(z) - P(z)|^2 d\mu(z) < \epsilon$. Clearly $C(K) = \overline{\Pi}$. As we know that $C(K)$ is dense in $L_\mu^2(K)$, we conclude that $\overline{\Pi} = L_\mu^2(K)$. Therefore we are in a *complete case*. It follows that $S_\mu = N_\mu$, and also $D = N$, in consequence $D$ is a normal operator. □

THEOREM 4.3. *With the previous hypothesis about $Z$ and if $Z' \cap Z = \emptyset$, then*

$$D = U^H (\delta_{ij} z_i)_{i,j=1}^{\infty} U, \quad \textit{and} \quad U^H U = U U^H = I.$$

*Here $U = VT^{-H}$, where $T$ is the Cholesky factor in the decomposition $M = TT^H$, and $V$ is the Vandermonde matrix of the atoms*

$$V = \begin{pmatrix} \sqrt{p_1} & \sqrt{p_1}z_1 & \sqrt{p_1}z_1^2 & \cdots \\ \sqrt{p_2} & \sqrt{p_2}z_2 & \sqrt{p_2}z_2^2 & \cdots \\ \sqrt{p_3} & \sqrt{p_3}z_3 & \sqrt{p_3}z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (\sqrt{p_i}z_i^{j-1})_{i,j=1}^{\infty}.$$

*Proof.* We call $L = (\delta_{ij}z_i)_{i,j=1}^{\infty}$. It is clear that $M' = V^H LV$. From $D = T^{-1}M'T^{-H}$ it follows that $D = T^{-1}V^H LVT^{-H}$. We have that the elements of the ith. column of the infinite matrix $T^{-H}$, are the coefficients of $\widehat{P}_{i-1}(z)$ in the basis $\{z^k\}$. Therefore $U = VT^H = \left(\sqrt{p_i}\,\widehat{P}_{j-1}(z_i)\right)_{i,j=1}^{\infty}$. Now we calculate $U^H U$ and have $(U^H U)_{i,j} = \sum_{k=1}^{\infty} \overline{\widehat{P}_i(z_k)}\widehat{P}_j(z_k)p_k = \delta_{ij}$, because the orthogonality of the NOPS on the set $Z = \{z_1, z_2, \ldots\}$. On the other hand the product $UU^H$ is

$$UU^H = (\sqrt{p_i}\sqrt{p_j}\sum_{k=0}^{\infty}\widehat{P}_k(z_i)\overline{\widehat{P}_k(z_j)})_{i,j=1}^{\infty}.$$

To prove the statement we need also that $(UU^H)_{ij} = \delta_{ij}$. For that we introduce the bounded functionals $L_i : \overline{\Pi} \to \overline{\Pi}$ such that $L_i(f) = f(z_i)$. Recall that the inner product in $\Pi$ is $\langle P(z), Q(z)\rangle = \sum_{k=1}^{\infty} P(z_k)\overline{Q(z_k)}p_k$, and it is extended to $\overline{\Pi}$ as usual. Obviously $\|L_i\| \le 1/p_i$. It is clear that the $n$-kernel $K_n(z, z_i) = \sum_{k=0}^{n} \overline{\widehat{P}_k(z)}\widehat{P}_k(z_i)$, with $n > j$, has the reproducing property, that is $\langle Q_j(z), K_n(z, z_i)\rangle = Q_j(z_i)$. The function $K(z, z_i) = \lim_n K_n(z, z_i)$ defined on $Z = \{z_1, z_2, \ldots\}$, has the same property. With the additional hypothesis, as the points of $Z$ are isolated, $\chi_{z_i}(z)/p_i \in C(K)$. Where $\chi_{z_i}(z_j) = \delta_{ij}$. From the previous proposition we have that $\chi_{z_i}(z)/p_i \in C(K) = \overline{\Pi} = L^2_\mu(K)$. Hence $\langle f(z), K(z, z_i)\rangle = \langle f(z), \frac{\chi_{z_i}(z)}{p_i}\rangle = f(z_i)$, $\forall f \in \overline{\Pi} = L^2_\mu$, then $\chi_{z_i}(z) = K(z, z_i)$, a.e. in $L^2_\mu$. In particular $\chi_{z_i}(z) = K(z, z_i)$ at the points with positive measure, i.e., $K(z, z_i) = \chi_{z_i}(z)$ on $Z$. In consequence $K(z_j, z_i) = \delta_{ij}$, therefore $UU^H = I$. □

## 5. Real bounded distribution with a finite set of complex points.

THEOREM 5.1. *Consider a bounded and real distribution supported in a finite set to a bounded set of complex points $\{z_k\}_{k=1}^N$, such that $\Im(z_k) \neq 0$, $k = 1, \ldots, N$, with weights $\{p_k\}_{k=1}^N$. Then the infinite matrix $D = T^{-1}M'T^{-H}$ is normal.*

*Proof.* Let $H$ be the Hankel matrix associated to the real distribution, i.e., the moment matrix. Let $\{z_k\}_{k=1}^N$ be the complex points with weights $\{p_k\}_{k=1}^N$. The moment matrix is $M = H + L$, where $L = (l_{i,j})_{i,j=0}^{\infty}$ with[2] $l_{i,j} = \sum_{k=1}^{N} \overline{z}_k^j z_k^i p_k$. Actually $L$ is an infinite matrix, but with rank $N$. We have $D_n = T_n^{-1}M_n'T_n^H$, and at the same time we have

$$D_n = \left(\frac{D_n + D_n^H}{2}\right) + \left(\frac{D_n - D_n^H}{2i}\right) i.$$

Since $M$ is a moment matrix $D$ is subnormal by Lemma 3.1, then $D$ is hyponormal. By Lemma 2.8 we only need to prove that $D - D^H$ is a compact operator.

We calculate $D_n - D_n^H$. We have $D_n^H = T_n^{-1}(M_n')^H T_n^{-H}$, so that

$$\begin{aligned} D_n - D_n^H &= T_n^{-1}M_n'T_n^{-H} - T_n^{-1}[M_n']^H T_n^{-H} \\ &= T_n^{-1}\left(M_n' - [M_n']^H\right)T_n^{-H}. \end{aligned}$$

[2]The notation for moment matrices is contrary to the usual.

We use $H'$ and $L'$ in the same way as $M'$ for $M$. We can write $M'_n = H'_n + L'_n$. Since $H'_n$ is a real Hankel matrix, $H'_n - [H'_n]^H = 0$, and hence

$$M'_n - [M'_n]^H = L'_n - [L'_n]^H = \begin{pmatrix} l_{10} - \overline{l}_{10} & l_{20} - \overline{l}_{11} & \dots & l_{n0} - \overline{l}_{1,n-1} \\ l_{11} - \overline{l}_{20} & l_{21} - \overline{l}_{21} & \dots & l_{n1} - \overline{l}_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1,n-1} - \overline{l}_{n0} & l_{2,n-1} - \overline{l}_{n1} & \dots & l_{n,n-1} - \overline{l}_{n,n-1} \end{pmatrix}.$$

It is clear[3] that $\forall n > N$ we have

$$\operatorname{rank}\left(M'_n - [M'_n]^H\right) = \operatorname{rank}\left(L'_n - [L'_n]^H\right) = N.$$

The matrix $M$ is a moment matrix and is HPD; hence the matrices $T_n$ and $T_n^H$ exist and are non-singular for every $n$. The matrix $D_n - D_n^H$ is equivalent to $L'_n - [L'_n]^H$, hence, $\operatorname{rank}\left(D_n - D_n^H\right) = N, \forall n > N$. Therefore $\operatorname{rank}(D - D^H) = N$. $(D - D^H)$ is an operator of finite rank in $\ell^2$, and hence is a compact operator. By Lemma 2.8, $D$ is a normal matrix. ☐

## 6. Real bounded distribution with a infinite bounded set of complex points.

THEOREM 6.1. *Consider be a bounded real distribution supported in a bounded set of infinite complex points $\{z_k\}_{k=1}^\infty$ with weights $\{p_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty p_k < +\infty$. If all the accumulation points of $\{z_k\}_{k=1}^\infty$ are in $\mathbb{R}$, then the infinite matrix $D = T^{-1}M'T^{-H}$ is normal.*

*Proof*. Let $H$ be the Hankel matrix of the bounded real distribution. The moment matrix is $M = H + L$, where $L = (l_{i,j})_{i,j=0}^\infty$ with $l_{i,j} = \sum_{k=1}^\infty \overline{z}_k^j z_k^i p_k$. $L$, $H$, and $M$ are infinite positive definite Hermitian matrices.

Since $M = TT^H$, we can write

$$D - D^H = T^{-1}\left(H' + L'\right)T^{-H} - T^{-1}\left(H' + L'\right)^H T^{-H}.$$

Since $H' - [H']^H = 0$, it follows that $D - D^H = T^{-1}\left(L' - [L']^H\right)T^{-H}$. We can reorder the sequence $\{z_k\}_{k=1}^\infty$ in a such way that[4] $\Im(z_k) \geq \Im(z_{k+1})$. We build the infinite matrices

$$V = \begin{pmatrix} \sqrt{p_1} & \sqrt{p_1}z_1 & \sqrt{p_1}z_1^2 & \sqrt{p_1}z_1^3 & \dots \\ \sqrt{p_2} & \sqrt{p_2}z_2 & \sqrt{p_2}z_2^2 & \sqrt{p_2}z_2^3 & \dots \\ \sqrt{p_3} & \sqrt{p_3}z_3 & \sqrt{p_3}z_3^2 & \sqrt{p_3}z_3^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } Z = \begin{pmatrix} z_1 & 0 & 0 & \dots \\ 0 & z_2 & 0 & \dots \\ 0 & 0 & z_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

¿From $L' = V^H Z V$, we have $[L']^H = V^H Z^H V$, and $L = V^H V$. Therefore

$$D - D^H = T^{-1}\left(V^H\left(Z - Z^H\right)V\right)T^{-H} = \left(T^{-1}V^H\right)\left(Z - Z^H\right)\left(VT^{-H}\right). \tag{6.1}$$

All the matrices in the last formula are well-defined and the products exist; hence the associative property holds. In particular, since the factor $T^{-1}$ is a lower triangular infinite matrix, the product $T^{-1}V^H$ exists element by element, (independently of the fact that the rows or columns belongs to $\ell^2$). In the same way $VT^{-H}$ exists. In addition, we are going to prove that $T^{-1}V^H$ and its transposed conjugated matrix $VT^{-H}$ define bounded operators in $\ell^2$.

---

[3]Expressing $L_n$ as a product of two Vandermonde matrices, with $z_1, z_2, \dots, z_n$ and $p_1, p_2, \dots, p_N$, we have $L_n = W_{n,N}^H W_{N,n}$, since $L'_n = W_{n,N}^H Z_N W_{N,n}$ and $[L'_n]^H = W_{n,N}^H Z_N^H W_{N,n}$, $L'_n - [L'_n]^H = W_{n,N}^H(Z_N - Z_N^H)W_{N,n}$. Hence, $\operatorname{rank}\left(L'_n - [L'_n]^H\right) = \operatorname{rank}\left(Z_N - Z_N^H\right) = N$.

[4]If the terms were equal, we could establish a criterion from left to right and over to under the real axes.

By hypothesis the diagonal matrix $Z - Z^H = (2i\Im(z_k)\delta_{j,k})_{j,k=1}^{\infty}$ is such that $\lim_k \Im(z_k) = 0$; hence $Z - Z^H$ defines a compact operator in $\ell^2$.

To prove that the matrix $VT^{-H}$ is bounded, we know that in general $\|A^H A\| = \|A\|^2$, and we have that $\|T^{-1}V^H\|^2 = \|T^{-1}V^H VT^{-H}\| = \|T^{-1}LT^{-H}\|$.

At the same time $T^{-1}MT^{-H} = T^{-1}TT^HT = I = T^{-1}(H + L)T^{-H}$; hence, $T^{-1}LT^{-H} = I - T^{-1}HT^{-H}$, and $T^{-1}LT^{-H} \geq 0$. Since $L$ is positive definite, therefore $I - T^{-1}HT^{-H} \geq 0$, and we can write

$$\langle I\overline{x}, \overline{x}\rangle \geq \langle T^{-1}HT^{-H}\overline{x}, \overline{x}\rangle, \qquad \forall \overline{x} \in \ell^2. \tag{6.2}$$

Obviously $(T^{-1}HT^{-H})^H = T^{-1}HT^{-H}$ since $H$ is a Hankel matrix and, hence, a symmetric matrix. We know that if $A = A^H$, then $\sup_{\|\overline{x}\|=1} |\langle A\overline{x}, \overline{x}\rangle| = \|A\|$. In our case, dividing by $\|\overline{x}\|^2$ in (6.2), and taking the supremum, we have $\|T^{-1}HT^{-H}\| \leq 1$. Finally

$$\|T^{-1}LT^{-H}\| = \|I - T^{-1}HT^{-H}\| \leq \|I\| + \|T^{-1}HT^{-H}\| \leq 2,$$

hence $\|T^{-1}V^H\| = \|VT^{-H}\| \leq \sqrt{2}$. The matrices at both sides of $Z - Z^H$ in (6.1) define bounded operators in $\ell^2$.

We know (see page 158 in [8] ) that the product of a bounded operator by a compact one is compact. Applying twice this property in (6.1), it follows that $D - D^H$ is a compact operator. Since $M$ is a moment matrix, $D$ is subnormal, and hence, $D$ is a hyponormal operator. From lemma 2.8 $D$ is normal. □

REMARK 6.2. *Theorems 5.1 and 6.1 still hold if the support of the real distribution does not lie on the real line but on a straight line of the complex plane. Obviously, in Theorem 6.1 it is required that the accumulation points should be on this complex line.*

REFERENCES

[1] A. ATZMON, *A moment problem for positive measures on the unit disc*, Pacific J. Math., 59 (1975), pp. 317-325.
[2] J. BRAM, *Subnormal Operators*, Duke Math. J., 22 (1955), pp. 75-94.
[3] JOHN B. CONWAY, *Subnormal Operators*, Pitman Publishing Inc., London, 1981.
[4] JOHN B. CONWAY, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1985.
[5] J. DOMBROWSKI, *Spectral Properties of Real Parts of Weighted Shift Operators*, Indiana Univ. Math. J., 29 (1980), pp. 249-259.
[6] J. DOMBROWSKI, *Orthogonal Polynomials and Functional Analysis*, P. Nevai, ed., Orthogonal Polynomials, Kluwer Academic Publishers, 1990, pp. 147-161.
[7] PAUL R. HALMOS, *A Hilbert Space Problem Book*, Second Edition, Springer-Verlag, New York, 1980.
[8] T. KATO, *Perturbation Theory for Linear Operators*, Springer, New York, 1976.
[9] J.A. SHOHAT AND J.D. TAMARKIN, *The Problem of Moments*, Mathematical Surveys, vol. 1, Providence, R.I., 1943; reprinted 1970.
[10] HERBERT STAHL AND VILMOS TOTIK, *General Orthogonal Polynomials*, Cambridge University Press, Canada, 1992.
[11] E. TORRANO, *Interpretación matricial de los polinomios ortogonales en el caso complejo*, Tesis Doctoral, Universidad de Santander, 1987.
[12] E. TORRANO AND R. GUADALUPE, *On the moment problem in the bounded case*, J. Comp. Appl. Math., 49 (1993), pp. 263-269.