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# ASYMPTOTICS OF POLYNOMIAL SOLUTIONS OF A CLASS OF GENERALIZED LAMÉ DIFFERENTIAL EQUATIONS*

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**Abstract.** In this paper we study the asymptotic behavior of sequences of Heine-Stieltjes and Van Vleck polynomials for a class of generalized Lamé differential equations connected with certain equilibrium problems on the unit circle.

**Key words.** Heine-Stieltjes polynomials, Van Vleck polynomials, zeros, asymptotics.

**AMS subject classifications.** 33C45.

## 1. Introduction.

We consider a class of generalized Lamé differential equations

$$A(z)y''(z) + B(z)y'(z) + C(z)y(z) = 0\,, \tag{1.1}$$

where $A$ is a polynomial of degree $p+1$, $p \in \mathbb{N}$, and $\deg B \leq p$. If there exists a polynomial $C$ for which (1.1) admits a polynomial solution $y = \varphi$, then $C$ is called a Van Vleck polynomial, and $\varphi$ is known as a Heine-Stieltjes polynomial. In the simplest case, $p = 1$, (1.1) is the usual hypergeometric equation (and Heine-Stieltjes polynomials are just hypergeometric polynomials), and for $p = 2$ we obtain the so-called Heun equation.

In his well known work, Stieltjes [8] considered the particular setting when $A$ has only simple and real zeros $a_0, \dots, a_p$, and for $j = 1, \dots, p$ the residue $r_j$ of the rational function $B/A$ at $a_j$ is positive. In this situation Stieltjes showed that there exist exactly $\sigma(N) = \binom{N+p-1}{N}$ possible Van Vleck polynomials $C$ for which (1.1) admits a unique (up to normalization) Heine-Stieltjes polynomial $y = \varphi$ of degree $N$. (In a previous paper, Heine [2] proved that in general $\sigma(N)$ is an upper bound for the number of possible polynomials $C$). Stieltjes showed that the collection of admissible pairs of polynomials $(C, \varphi)$ is given by all possible ways the $N$ zeros of $\varphi$ can be distributed in the open intervals defined by the $p+1$ zeros of $A$. The location of the zeros is determined by the equilibrium position of unit charges moving under the action of charges of weight $r_j$ fixed at the zeros $a_j$ of $A$.

There are not so many results on the Heine-Stieltjes polynomials when some of the restrictions imposed by Stieltjes are dropped. In fact, even the existence of these polynomials is an open problem. In a recent paper Grinshpan [1] extended Stieltjes' construction to a class of generalized Lamé equations with coefficients symmetric with respect to the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, so that the zeros of the Heine-Stieltjes polynomials lie on $\mathbb{T}$. More precisely, let in (1.1) $p = 2M$, $M \in \mathbb{N}$, and $A$ be of the form

$$A(z) = z\prod_{j=1}^{M}(z - z_j)(z\overline{z_j} - 1)\,. \tag{1.2}$$

Using ideas from [1] it can be proved that if the residues of $B/A$ satisfy certain conditions (see below), then for each $N \in \mathbb{N}$ there exists a Van Vleck polynomial $C$ for which (1.1) admits a unique, up to normalization, Heine-Stieltjes polynomial of degree $N$ with zeros lying on $\mathbb{T}$.

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In the classical situation described by Stieltjes, the asymptotic behavior of the Van Vleck and Heine-Stieltjes polynomials, based on the electrostatic interpretation of their zeros, has been described recently in [4] in terms of the equilibrium measures in presence of an external field; see also the bibliography therein for some additional historical background. In this paper we perform the asymptotic analysis of the configuration described in [1] in the spirit of [4].

Let $\Lambda$ be an infinite subset of $\mathbb{N}$; for each $N \in \Lambda$ define $M = M(N) \in \mathbb{N}$ and consider a 1-parametric family of generalized Lamé differential equations of the type (1.1),

$$A_N(z)y''(z) + B_N(z)y'(z) + C_N(z)y(z) = 0\,, \quad N \in \Lambda\,, \tag{1.3}$$

where $A_N$ are of the form (1.2), and $C_N$ are Van Vleck polynomials. In other words, for each $N \in \Lambda$ there exists a polynomial solution $y = \varphi_N$ of (1.3) of degree $N$.

We study the asymptotic behavior of the sequences $\{\varphi_N\}$ and $\{C_N\}$ as $N \to \infty$, $N \in \Lambda$, under the following additional assumptions:

$$S(N) := -\frac{1}{2} \sum_{\substack{A_N(\xi)=0,\\ \xi\neq 0}} \operatorname{res} \frac{B_N}{A_N}(\xi) > 0, \quad N \in \Lambda, \qquad \text{and} \quad \lim_{N\in\Lambda} \frac{S(N)}{N} = \lambda \in [0,\infty) \tag{1.4}$$

(implicitly, we assume the existence of this limit).

The outline of the paper is as follows. In Sect. 2 we state the problem in detail; the minimum energy problem on the unit circle associated with the Lamé equation above is briefly described. The asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials is analyzed in Sect 3. Finally, in Sect 4, our results are illustrated with some simple examples.

**2. An equilibrium problem on the unit circle.** We are given subsequences $N \in \Lambda$, $M = M(N) \in \mathbb{N}$, and the corresponding Lamé differential equations (1.3), with the following symmetry conditions satisfied for each $N \in \Lambda$:

$$A_N(z) = zQ_N(z)\,, \qquad B_N(z) = Q_N(z) - zR_N(z)\,,$$

where

- $Q_N$ is a polynomial of degree $2M$ with simple zeros and such that $Q_N^*(z) = z^{2M}\overline{Q_N(1/\overline{z})} = Q_N(z)$; in particular, $Q_N(0) \neq 0$ and $Q_N(z) \neq 0$ for $|z| = 1$.
- $R_N$ is a polynomial of degree $2M-1$ such that if $Q_N(\xi) = Q_N(\xi^*) = 0$ , with $\xi^* = 1/\overline{\xi}$, we have

$$\operatorname*{res}_{z=\xi} \frac{R_N}{Q_N}(z) = \operatorname*{res}_{z=\xi^*} \frac{R_N}{Q_N}(z) > 0\,.$$

Under these conditions, we know from [1] that there exists a polynomial $C_N$ of degree $\leq 2M-1$ such that (1.3) admits a polynomial solution $y = \varphi_N$ of degree $N$. Its zeros belong to $\mathbb{T}$ and define the unique equilibrium configuration in the external field generated by positive charges fixed at the zeros of $Q_N$.

Let us denote by $z_{Nk}$, $k = 1, \ldots, M$, the zeros of $Q_N$ that lie in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (recall that we assume also that $|z_{Nk}| > 0$), and by $z_{Nk}^* = 1/\overline{z_{Nk}}$ their symmetric counterparts in $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$.

At each $z_{Nk}$ we place a negative charge of value $-w_{Nk}$, where

$$w_{Nk} = \operatorname*{res}_{z=z_{Nk}} \frac{R_N}{Q_N}(z) > 0\,, \quad k = 1, \ldots, M\,, \tag{2.1}$$

along with $N$ movable unit positive charges $\lambda_{Nk}$, $k = 1, \dots, N$ on $\mathbb{T}$. The problem consists in minimizing the total (logarithmic) energy of the system. This setting extends the one studied in [1], where $M = N$ and $w_{Nk} = 1$ $(k = 1, \dots, M)$.

The total energy of the system described is given by

$$E(\lambda_{N1}, \dots, \lambda_{NN}) = -\sum_{k<l} \log|\lambda_{Nk} - \lambda_{Nl}| + \sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl} \log|\lambda_{Nk} - z_{Nl}|$$

and it is easy to verify that the equilibrium configuration is attained at points $\lambda_{Nk} \in \mathbb{T}$, $k = 1, \dots, N$, if and only if the resultant electrostatic force at each of them is normal to $\mathbb{T}$, i.e.

$$\operatorname{Im}\left\{\lambda_{Nk}\left[\sum_{l\neq k} \frac{1}{\lambda_{Nk} - \lambda_{Nl}} - \sum_{l=1}^{M} \frac{w_{Nl}}{\lambda_{Nk} - z_{Nl}}\right]\right\} = 0, \quad k = 1, \dots, N\,.$$

As it was shown in [1], for an equilibrium configuration $\{\lambda_{N1} \dots, \lambda_{NN}\}$ there exists a polynomial $C_N$ such that

$$\varphi_N(z) = \prod_{j=1}^{N} (z - \lambda_{Nj})$$

is a polynomial solution of (1.3). Moreover, given the polynomial $C_N$, the polynomial solution of (1.3) is unique (up to a constant factor). Indeed, let us write (1.3) in the following "self-adjoint" form:

$$\left(\frac{z}{P_N(z)} y'\right)' + \frac{C_N}{P_N(z)\, Q_N(z)} y = 0\,, \tag{2.2}$$

where $P_N(z) = \prod_{k=1}^{M} (z - z_{Nk})^{w_{Nk}} (z - z_{Nk}^*)^{w_{Nk}}$ is a function satisfying that $P_N'/P_N = R_N/Q_N$. Thus, if $u$ and $v$ are two polynomial solutions of (2.2), it yields

$$\left(\frac{z}{P_N(z)} u'\right)' v - \left(\frac{z}{P_N(z)} v'\right)' u = 0\,,$$

which implies

$$\left(\frac{z}{P_N(z)} (u'v - uv')\right)' = 0\,,$$

and, hence, we obtain that $z(u'v - uv') = \text{const} \cdot P_N$. Since $P_N(0) \neq 0$, we conclude that $u$ and $v$ are linearly dependent. Moreover, this argument shows that any other solution of the differential equation cannot be regular at the origin.

**3. Asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials.** We start by recalling some facts from the logarithmic potential theory (for basic definitions, see e.g. [5, Section I.1] or [7, Appendix]).

If $\mu$ is a finite and compactly supported Borel measure on the complex plane $\mathbb{C}$, we denote by $\operatorname{supp}(\mu)$ its support, by

$$V(\mu; z) = \int \log\frac{1}{|z - t|}\, d\mu(t)$$

its logarithmic potential, and by

$$I(\mu) = \iint \log \frac{1}{|z-t|}\, d\mu(t)\, d\mu(z)$$

its logarithmic energy (in fact, doubled).

A function $w : \mathbb{T} \to \mathbb{R}_+$ is an *admissible weight* on $\mathbb{T}$ if $w$ is upper-semicontinuous and the set $\{z \in \mathbb{T} : w(z) > 0\}$ has positive logarithmic capacity . The (admissible) *external field* $\psi$ on $\mathbb{T}$ is defined by

$$w(z) = e^{-\psi(z)}, \quad z \in \mathbb{T},$$

and the weighted energy $I_\psi(\mu)$ of a Borel measure $\mu$ on $\mathbb{T}$, by

$$I_\psi(\mu) = I(\mu) + 2\int \psi\, d\mu\,.$$

We now consider the problem of minimization of the weighted energy $I_\psi(\mu)$ in the class $\mathcal{M}$ of positive Borel unit measures on $\mathbb{T}$. The following lemma is a direct consequence of well-known results (see [5, Theorem I.1.3]):

LEMMA 3.1. *If $\psi$ is an admissible weight then there exists a unique $\mu = \mu(\psi) \in \mathcal{M}$* (the equilibrium measure) *such that*

$$I_\psi(\mu) \le I_\psi(\nu)\,, \quad \textit{for every } \nu \in \mathcal{M}\,.$$

*Moreover, $\mu$ is characterized by the following property:*

$$\min_{z\in\mathbb{T}} \Big( V(\mu; z) + \psi(z) \Big) = V(\mu; z) + \psi(z)\,, \quad z \in \operatorname{supp}(\mu)\,.$$

In particular, if $\psi = 0$, we have that $\mu$ is the normalized Lebesgue measure on $\mathbb{T}$ (called also *Robin distribution* on $\mathbb{T}$).

Recall also that given a unit Borel measure $\nu$ on $\mathbb{C} \setminus \mathbb{T}$, the balayage measure $\widehat{\nu}$ of $\nu$ onto $\mathbb{T}$ is the unit measure supported on $\mathbb{T}$ such that

$$V(\nu; z) - V(\widehat{\nu}; z) \equiv \text{const}\,, \quad z \in \mathbb{T}\,,$$

(cf. [5, II.4]).

Back to our problem, for each $N$ and $M = M(N)$, we know that given $Q_N$ and $R_N$ as above, there exists a polynomial $C_N$ of degree $\le 2M-1$ for which the differential equation (1.3) admits a unique (up to normalization) polynomial solution $y = \varphi_N$ of degree $N$, whose zeros are in the equilibrium position described above. We have assumed also the existence of the limit in (1.4).

If as above $z_{Nk}$ denote the $M$ zeros of $Q_N$ which lie in $\mathbb{D}$, $w_{Nk}$ are given by (2.1), and $S = S(N) = w_{N1} + \cdots + w_{NM}$, the expression

$$\nu_N = \frac{1}{S}\sum_{k=1}^{M} w_{Nk}\delta_{z_{Nk}}$$

defines a unit positive measure with support in $\mathbb{D}$. By a weak compactness argument, there exists an infinite subsequence of $\Lambda$ (that for simplicity we will denote by $\Lambda$ again) such that $\nu_N$ is convergent along $\Lambda$ in the weak-* topology to a unit measure $\nu$, supported in $\overline{\mathbb{D}}$:

$$\nu_N \longrightarrow \nu\,, \quad N \in \Lambda\,. \tag{3.1}$$

We can construct the symmetric measure $\nu^*$, supported in $\overline{\mathbb{E}}$, by taking $d\nu^*(t) = d\nu\left(1/\overline{t}\right)$.

In the same way, to each Heine-Stieltjes polynomial $\varphi_N$ with the zeros $\lambda_{Nk} \in \mathbb{T}$, $k = 1, \dots, N$, we can associate the unit counting measure

$$\mu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_{Nk}} .$$

THEOREM 3.2. *In the conditions above,*

*(i)* $\mu_N \longrightarrow \mu$, $N \in \Lambda$, *where* $\mu = \mu(\psi)$ *is the equilibrium measure of* $\mathbb{T}$ *in the external field* $\psi = -\lambda V(\nu; \cdot)$.
*Furthermore, for the monic Heine-Stieltjes polynomials* $\{\varphi_N\}_{N\in\Lambda}$ *it holds locally uniformly in* $\mathbb{C} \setminus \mathbb{T}$ *that*

$$\lim_{N\in\Lambda} |\varphi_N(z)|^{1/N} = \exp\left(-V(\mu; z)\right) . \tag{3.2}$$

*(ii) Assume additionally that there exists a compact set* $\Delta \subset \mathbb{D} \setminus \{0\}$ *such that*

$$z_{Nk} \in \Delta \quad \text{for } k = 1, \dots, N, \text{ and for all } N \in \Lambda \text{ sufficiently large.} \tag{3.3}$$

*Let* $\widehat{\nu}$ *denote the balayage of* $\nu$ *from* $\mathbb{D}$ *onto* $\mathbb{T}$, *and* $\omega_{\mathbb{T}}$ *stands for the normalized Lebesgue measure (Robin distribution) of* $\mathbb{T}$. *If* $\lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ *is a positive measure on* $\mathbb{T}$ *(for instance, if* $\lambda \in [0, 1]$*), then*

$$\mu_N \longrightarrow \lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}\,, \quad N \in \Lambda\,, \tag{3.4}$$

*and for the monic Heine-Stieltjes polynomials* $\{\varphi_N\}_{N\in\Lambda}$ *it holds*

$$\lim_{N\in\Lambda} |\varphi_N(z)|^{1/N} = \begin{cases} \exp\left(-\lambda\left(V(\nu^*; z) - C\right)\right), & z \in \mathbb{D}, \\ |z|^{1-\lambda} \exp\left(-\lambda V(\nu; z)\right), & z \in \mathbb{E}, \end{cases} \tag{3.5}$$

*locally uniformly in* $\mathbb{C} \setminus \mathbb{T}$, *where* $C = \int \log|t|\, d\nu(t)$.

*Proof*. By [1], for each $N \in \Lambda$ the zeros $\lambda_{Nk}$, $k = 1, \dots, N$, of the Heine-Stieltjes polynomial $\varphi_N$, endowed with a $+1$ charge each, are in the equilibrium position in the external field created by $M$ *negative* charges, of weight $-w_{Nk}$, placed at $z_{Nk}$, $k = 1, \dots, M$, where $w_{Nk}$ are defined in (2.1). This yields the fact that $\mu_N \to \mu$, $N \in \Lambda$ by standard arguments (see e.g. [4]).

Indeed, the doubled total (discrete) energy of the configuration described above is

$$\begin{aligned} K_N = 2E\left(\lambda_{N1}, \dots, \lambda_{NN}\right) &= -\sum_{k\neq l} \log|\lambda_{Nk} - \lambda_{Nl}| + 2\sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl} \log|\lambda_{Nk} - z_{Nl}|\,, \\ &= N^2 I\left(\mu_N\right) + 2S \sum_{k=1}^{N} \psi_N\left(\lambda_{Nk}\right)\,, \end{aligned}$$

where

$$\psi_N(t) = \frac{1}{S} \sum_{l=1}^{M} w_{Nl} \log|t - z_{Nl}| = -V(\nu_N; t)\,.$$

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Now, if $\{r_{Nk}\}_{k=1}^{N}$ is any collection of $N$ pairwise distinct points in $\mathbb{T}$, by the equilibrium conditions,

$$K_N \leq -\sum_{k\neq l} \log|r_{Nk}-r_{Nl}| + 2\sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl}\log|r_{Nk}-z_{Nl}| \;;$$

integrating with respect to $d\mu\left(r_{N1}\right)\ldots d\mu\left(r_{NN}\right)$, we obtain that

$$K_N \leq N\left(N-1\right)I\left(\mu\right) + 2S\sum_{k=1}^{N}\int \psi_N\left(r_{Nk}\right)d\mu\left(r_{Nk}\right)\,,$$

which yields

$$K_N \leq N\left(N-1\right)I\left(\mu\right) + 2NS\int \psi_N d\mu\,. \tag{3.6}$$

On the other hand, if we take $\varepsilon > 0$ and set $K_\varepsilon\left(x,y\right) = \min\left\{-\log|x-y|\,,-\log\varepsilon\right\}$, it is easy to see that

$$\begin{aligned} &N^2\iint K_\varepsilon\left(x,y\right)d\mu_N\left(x\right)d\mu_N\left(y\right) + 2NS\int \psi_N d\mu_N \\ &\qquad\qquad = N^2 I\left(\mu_N\right) + 2NS\int \psi_N d\mu_N - N\log\varepsilon = K_N - N\log\varepsilon\,. \end{aligned}$$

The sequence $\{\mu_N\}$ is weakly compact; thus, eventually passing to a subsequence, we may assume that $\mu_N \to \sigma$, $N\in\Lambda$, where $\sigma$ is a unit measure supported on $\mathbb{T}$. Then dividing (3.6) through by $N^2$ and taking limits when $N\to\infty$, $N\in\Lambda$, we obtain that

$$\iint K_\varepsilon\left(x,y\right)d\sigma\left(x\right)d\sigma\left(y\right) + 2\lambda\int\psi\,d\sigma \leq I\left(\mu\right) + 2\lambda\int\psi\,d\mu\,, \qquad \psi\left(s\right) = -V(\nu;s)\,.$$

Since $\varepsilon > 0$ is arbitrary, it yields

$$I_{\lambda\psi}\left(\sigma\right) = I\left(\sigma\right) + 2\lambda\int\psi\,d\sigma \leq I\left(\mu\right) + 2\lambda\int\psi\,d\mu = I_{\lambda\psi}\left(\mu\right)\,.$$

The uniqueness of the equilibrium measure in an external field implies that $\sigma = \mu$.

Formula (3.2) is a direct consequence of the identity

$$\log|\varphi_N(z)| = -N\,V(\mu_N;z)\,, \quad z\in\mathbb{C}\setminus\mathbb{T}\,,$$

valid for monic polynomials $\varphi_N$, and the fact that all the zeros of $\varphi_N$ belong to $\mathbb{T}$.

Now, assume condition (3.3) satisfied; then $\operatorname{supp}(\nu)\cap\mathbb{T}=\emptyset$, $0\notin\operatorname{supp}\nu$, and we can construct the balayage measure $\widehat{\nu}$ of $\nu$ onto $\mathbb{T}$.

Formula (3.4) is a well-known fact; see e. g. [5, Lemma IV.4.4]. It is sufficient to observe that $\mu = \lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a unit measure and

$$V(\mu;z) + \psi(z) = \lambda V(\widehat{\nu};z) + (1-\lambda)\,V(\omega_{\mathbb{T}};z) - \lambda V(\nu;z) \equiv \text{const}\,, \quad z\in\mathbb{T}\,.$$

By Lemma 3.1, the expression in the right hand side of (3.4) is the equilibrium measure in the external field $\psi$, and the statement follows from *(i)*.

Having (3.4) we can be more specific in (3.2). Indeed, let

$$g_{\mathbb{D}}(z;t) = \log\left|\frac{1-\overline{t}z}{z-t}\right|$$

denote the Green function of $\mathbb{D}$; the Green potential of $\nu$ is given by

$$G(\mathbb{D},\nu;z) = \int g_{\mathbb{D}}(z;t)\,d\nu(t) = \int \log\left|\frac{1-\overline{t}z}{z-t}\right| d\nu(t) = V(\nu;z) - V(\nu^*;z) + C,$$

with $C = \int \log|t|\,d\nu(t)$, and $\nu^*$ the measure symmetric to $\nu$ with respect to $\mathbb{T}$.

By [5, Theorem II.5.1],

$$V(\widehat{\nu};z) - V(\nu;z) = \begin{cases} -G(\mathbb{D},\nu;z), & z \in \mathbb{D}, \\ 0, & z \in \mathbb{E}. \end{cases}$$

Thus,

$$V(\widehat{\nu};z) = \begin{cases} V(\nu^*;z) - C, & z \in \mathbb{D}, \\ V(\nu;z), & z \in \mathbb{E}. \end{cases}$$

Taking into account that

$$V(\omega_{\mathbb{T}};z) = \begin{cases} 0, & z \in \mathbb{D}, \\ -\log|z|, & z \in \mathbb{E}, \end{cases}$$

we obtain that

$$V(\mu;z) = \lambda V(\widehat{\nu};z) + (1-\lambda)V(\omega_{\mathbb{T}};z) = \begin{cases} \lambda\left(V(\nu^*;z) - C\right), & z \in \mathbb{D}, \\ \lambda V(\nu;(z) - (1-\lambda)\log|z|, & z \in \mathbb{E}, \end{cases}$$

and along with (3.2) this settles the proof. □

REMARKS: For the case where $\lambda > 1$ and $\lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ is not a positive measure on the whole $\mathbb{T}$, in general it can be said only that the support of $\mu$ is contained in the subset of $\mathbb{T}$ where $\lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ is positive. However, we shall see in the last section that in certain situations more information is available.

Once we know the asymptotic behavior of the Heine-Stieltjes polynomials $\varphi_N$, we can obtain the relative asymptotics for the Van Vleck polynomials $C_N$. For that purpose we rewrite the differential equation for $\varphi_N$ in terms of the logarithmic derivative of its solution, reducing (1.3) to a Riccati equation (see e.g. [3, I.4.9], [6] or [9, Section 86]).

For each $N \in \Lambda$ let

$$h_N(z) = \frac{1}{N}\frac{\varphi_N'(z)}{\varphi_N(z)} = \int \frac{d\mu_N(x)}{z-x}$$

and

$$k_N(z) = \frac{1}{2S}\frac{R_N(z)}{Q_N(z)} = \frac{1}{2S}\frac{P_N'(z)}{P_N(z)} = \frac{1}{2}\int \frac{d\nu_N(x) + d\nu_N^*(x)}{z-x}.$$

By Theorem 3.2,

$$\lim_{N\in\Lambda} h_N(z) = h(z) = \int \frac{d\mu(x)}{z-x}, \tag{3.7}$$

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uniformly in compact subsets of $\mathbb{C}\backslash\mathbb{T}$. Analogously, by (3.3) there exists a compact set $K = \Delta \cup \Delta^* \subset \mathbb{C}\backslash(\mathbb{T}\cup\{0\})$ such that

$$\lim_{N\in\Lambda} k_N(z) = k(z) = \frac{1}{2}\int \frac{d\nu(x) + d\nu^*(x)}{z-x}, \tag{3.8}$$

locally uniformly in $\mathbb{C}\backslash K$.

With these notations, (1.3) can be rewritten as

$$\frac{z}{2S}\left(Nh_N^2(z) + h_N'(z)\right) + \left(\frac{1}{2S} - zk_N(z)\right)h_N(z) + \frac{C_N(z)}{2NSQ_N(z)} = 0\,.$$

Since with assumption (3.3) $\{h_N'\}_{N\in\Lambda}$ is also uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{T}$, taking limits as $N\in\Lambda$ and using (1.4) yields that the limit

$$f(z) = \lim_{N\in\Lambda}\frac{C_N(z)}{N^2Q_N(z)}$$

exists and satisfies the equation

$$\frac{z}{2\lambda}h^2(z) - zh(z)\,k(z) + \frac{1}{2\lambda}f(z) = 0.$$

Thus, we have established the following

THEOREM 3.3. *Under assumption (3.3) the Van Vleck polynomials $\{C_N\}_{N\in\Lambda}$ satisfy the following relative asymptotics*

$$\lim_{N\in\Lambda}\frac{C_N(z)}{N^2Q_N(z)} = zh(z)\left(2\lambda k(z) - h(z)\right), \tag{3.9}$$

*where the convergence is uniform in compact subsets of $\mathbb{C}\backslash\{\mathbb{T}\cup K\}$, and where $h(z)$ and $k(z)$ have been defined in (3.7) and (3.8), respectively.*

In particular, the Hurwitz theorem can be used together with (3.9) to get information on the accumulation points of the zeros of Van Vleck polynomials $C_N$.

**4. Examples.** In this section, some simple examples illustrating the results above will be considered.

EXAMPLE 1. Let $0 < r < 1$ and denote by $\mathbb{U} = \{z\in\mathbb{C} : |z| = r\}$ the circle of radius $r$ centered at the origin. Assume that the zeros of $A_N$ contained in $\mathbb{D}$ are distributed asymptotically uniformly on $U$. Then obviously (3.1) holds for $\nu = \omega_{\mathbb{U}}$, the equilibrium measure of $\mathbb{U}$. Moreover, condition (3.3) is satisfied.

We have

$$V(\nu; z) = \begin{cases} -\log r, & |z| \le r, \\ -\log|z|\,, & |z| > r, \end{cases}$$

and for $\lambda\in[0,\infty)$, we have:

$$V(\omega_{\mathbb{T}}; z) = \lambda V(\nu; z)\,, \quad z\in\mathbb{T}\,.$$

Hence, in this case $\mu = \omega_{\mathbb{T}}$, and by Theorem 3.2 for the Heine-Stieltjes polynomials we have

$$\lim_{N\to\infty}|\varphi_N(z)|^{1/N} = \begin{cases} 1, & z\in\mathbb{D}, \\ |z|\,, & z\in\mathbb{E}, \end{cases}$$

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uniformly on compact subsets of the respective domains.

Furthermore, taking into account (3.7) and (3.8),

$$h(z) = \int \frac{d\mu(x)}{z-x} = \begin{cases} 0, & z \in \mathbb{D}, \\ 1/z, & z \in \mathbb{E}, \end{cases} \quad \text{and} \quad k(z) = \begin{cases} 1/z, & |z| > 1/r, \\ 1/(2z), & r < |z| < 1/r, \\ 0, & |z| < r. \end{cases}$$

Consequently,

$$\lim_N \frac{C_N(z)}{N^2 Q_N(z)} = \begin{cases} 0, & z \in \mathbb{D} \setminus \mathbb{U}, \\ (\lambda-1)/z, & 1 < |z| < 1/r, \\ (2\lambda-1)/z, & |z| > 1/r, \end{cases} \tag{4.1}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\mathbb{T} \cup \mathbb{U} \cup \mathbb{U}^*)$, where $U^* = \{z \in \mathbb{C} : |z| = 1/r\}$.

Using Hurwitz theorem we conclude for instance that if $\lambda \notin \{1/2, 1\}$, the zeros of Van Vleck polynomials can accumulate only in $\overline{\mathbb{D}}$ or on the circle $U^*$. Moreover, if $\lambda = 1$, according to (4.1) they might accumulate also in the gap between $\mathbb{D}$ and $U^*$. In [1] it was considered the case with $S = M = N$ and $Q_N(z) = \left(z^N - r^N\right)\left(1 - r^N z^N\right)$, so that $\varphi_N(z) = z^N - 1$ and $C_N(z) = -N^2 r^N z^{N-1}\left(z^N + 1\right)$. In this situation the limit points of the zeros of Van Vleck polynomials still lie in $\overline{\mathbb{D}}$. It would be interesting to find an example when these zeros leave the unit disc.

EXAMPLE 2. Assume now that all the zeros of $A_N$ inside $\mathbb{D}$ cluster asymptotically on a single point $a$, $|a| < 1$, so that $\nu = \delta_a$.

Let us consider first the case $\lambda \in [0, 1]$, where we know that $\mu = \lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$. By (3.5), the monic Heine-Stieltjes polynomials have the following $n$-th root asymptotics:

$$\lim_N |\varphi_N(z)|^{1/N} = \begin{cases} |1 - \overline{a}z|^{\lambda}, & z \in \mathbb{D} \setminus \{a\}, \\ |z|^{1-\lambda} |z-a|^{\lambda}, & z \in \mathbb{E}, \end{cases}$$

uniformly on compact subsets of the respective domains. Furthermore, by (3.7) and (3.8),

$$h(z) = \begin{cases} \lambda/(z-a^*), & z \in \mathbb{D}, \\ \lambda/(z-a) + (1-\lambda)/z, & z \in \mathbb{E}, \end{cases}$$

and

$$k(z) = \frac{1}{2}\left(\frac{1}{z-a} + \frac{1}{z-a^*}\right), \quad z \neq a, a^*,$$

and

$$2\lambda k(z) - h(z) = \begin{cases} \lambda/(z-a), & z \in \mathbb{D} \setminus \{a\}, \\ \lambda/(z-a^*) - (1-\lambda)/z, & z \in \mathbb{E} \setminus \{a^*\}, \end{cases}$$

where $a^* = \dfrac{1}{\overline{a}}$. Therefore,

$$\lim_{N\to\infty} \frac{C_N(z)}{N^2 Q_N(z)} = \begin{cases} \dfrac{\lambda^2 z}{(z-a)(z-a^*)}, & z \in \mathbb{D} \setminus \{a\}, \\ z\left(\dfrac{\lambda}{z-a} + \dfrac{1-\lambda}{z}\right)\left(\dfrac{\lambda}{z-a^*} - \dfrac{1-\lambda}{z}\right), & z \in \mathbb{E} \setminus \{a^*\}, \end{cases}$$

uniformly on compact subsets of the respective domains.

Using the Hurwitz theorem we see that if $\lambda = 1/2$ or $\frac{1+|a|}{1+2|a|} \leq \lambda \leq 1$, the zeros of Van Vleck polynomials can accumulate only either on $\mathbb{T}$ or at the points $0$, $a$ and $1/\overline{a}$. On the other hand, if $0 < \lambda < \frac{1+|a|}{1+2|a|}$, $\lambda \neq 1/2$, an additional accumulation point might appear at $\xi = \frac{\lambda-1}{2\lambda-1}\,\frac{1}{\overline{a}}$.

Assume now $\lambda > 1$; we have seen that (3.4) is still valid for such values of $\lambda$ for which $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a positive measure on $\mathbb{T}$.

We know (cf. [5, Sect. II.4]) that

$$d\widehat{\delta_a}\,(t) = \frac{1}{2\pi}\,\frac{1-|a|^2}{|t-a|^2}\,d\theta, \quad t = e^{i\theta}\,.$$

Therefore, $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a positive measure on the whole $\mathbb{T}$ if and only if:

$$|t-a|^2 \leq \frac{\lambda}{\lambda-1}\left(1-|a|^2\right), \quad t \in \mathbb{T};$$

in other words, $\mathbb{T}$ must be contained in the disc centered at $a$ with radius $\left[\left(1-|a|^2\right)\lambda/(\lambda-1)\right]^{1/2}$. This condition is satisfied if

$$\lambda \leq \frac{1+|a|}{2\,|a|}\,,$$

and asymptotic formulas above are valid.

On the other hand, if $\lambda > \frac{1+|a|}{2|a|}$, it is easy to see that $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is positive on

$$S_\lambda = \left\{t \in \mathbb{T}:\ \operatorname{Re}\,(\overline{a}t) \geq \frac{(2\lambda-1)\,|a|^2-1}{2\,(\lambda-1)}\right\}.$$

In the general case we can assure only that $\operatorname{supp}(\mu) \subset S_\lambda$, but in our situation we may assert additionally that $\operatorname{supp}\mu$ is an arc contained in $S_\lambda$. Indeed, for $z \in \mathbb{T}$,

$$V^\mu\,(z) = \text{const} + \lambda V^\nu\,(z) = \text{const} - \lambda \log|z-a|\ .$$

Applying the maximum principle for potentials [5, corollary II.3.3] we conclude that the support is a connected subset of $\mathbb{T}$ (that is, an arc) contained in $S_\lambda$. Obviously, this arc is symmetric with respect $z_0 = a/\,|a|$. When $\lambda \to \infty$, the arc $S_\lambda$ tends to the limit arc $S_\infty = \left\{t \in \mathbb{T}/\operatorname{Re}\,(\overline{a}t) \geq |a|^2\right\}$.

The considerations above can be extended to the case of an atomic measure, that is, when

$$\nu = \sum_{i=1}^{p} \alpha_i \delta_{a_i}, \quad \text{with} \quad |a_i| < 1,\ \alpha_i > 0,\ i = 1, \ldots, p \quad \text{and} \quad \sum_{i=1}^{p} \alpha_i = 1\,.$$

In this case, $\operatorname{supp}(\mu) = \mathbb{T}$ if and only if for all $t \in \mathbb{T}$,

$$\sum_{i=1}^{p} \alpha_i \frac{1-|a_i|^2}{|t-a_i|^2} \geq \frac{\lambda-1}{\lambda}\,.$$

Otherwise, $\operatorname{supp}(\mu)$ will consist of the union of $\leq p$ arcs.

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