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ASYMPTOTICS FOR EXTREMAL POLYNOMIALS WITH VARYING MEASURES*

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Abstract. In this paper, we give strong asymptotics of extremal polynomials with respect to varying measures of the form $d\sigma_n = \frac{d\sigma}{|Y_n|^p}$, where σ is a positive measure on a closed analytic Jordan curve C, and $\{Y_n\}$ is a sequence of polynomials such that for each n, Y_n has exactly degree n and all its zeros $(\alpha_{n,i})$, $i = 1, 2, \ldots$, lie in the exterior of C.

Key words. Rational Approximation, Orthogonal Polynomials, Varying Measures.

AMS subject classifications. 30E10, 41A20, 42C05.

1. Introduction. The first person that worked with varying measures was Gonchar in 1978, in [2]. He used them for studying the speed of the best rational approximation on an interval $E \subset \boldsymbol{R}$ of Cauchy transforms with finite positive measures and whose supports lie in an interval $F \subset \boldsymbol{R}$, with E and F verifying $E \cap F = \emptyset$. The varying measures have also been used for other works. Gonchar and López-Lagomasino in [3] studied general results of convergence of multipoint Padé approximants for Cauchy transform. Moreover, in [5] López-Lagomasino presented the varying orthogonal polynomials in a way that joins orthogonal polynomials respect to measures with bounded and unbounded support and he proved results for orthogonal polynomials respect to measures with unbounded support using orthogonal polynomials respect to varying measures on $|z| = 1$. In [6] he gave a Szegő theorem for polynomials orthogonal with respect to varying measures. We want to extend this result for L^p-extremal polynomials with respect to varying measures on closed analytic Jordan curves C. This main theorem will be given in this section and proved in Section 3. In Section 2 we set some auxiliary results for proving the main theorem. Finally, in Section 4 we give a theorem of density. We begin by introducing some notations.
Let C be a closed analytic Jordan curve with length l in the z-plane in whose interior α lies. Let $\sigma(s)$ be a positive measure on $[0, l]$. We denote by $L^p(C, \sigma)$ the space of measurable and complex functions on C, such that

$$\|f\|_{p,\sigma} = \left\{ \frac{1}{2\pi} \int_C |f(\zeta)|^p \mathrm{d}\sigma(s) \right\}^{1/p} < \infty, \quad \zeta = \zeta(s),$$

with $\zeta = \zeta(s)$ a parametrization of C. When $C = \{z : |z| = 1\}$, as usual, we write $L^p(\sigma)$. Let B denote the interior of C. Let

$$x = \varphi(z) = \alpha + z + b_2 z^2 + \cdots,\ |z| < 1,\ \alpha \in B$$

be the conformal transform which maps $\boldsymbol{D} = \{z : |z| < 1\}$ onto B, such that $\alpha = \varphi(0)$ and $\varphi'(0) > 0$. From Caratheodory theorem φ can be continuously extended to a function which is an one-to-one mapping from $\{|z| = 1\}$ onto C. Let $z = \gamma(x)$ denote the inverse

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function of φ. Then the measure σ induces an image measure μ on $|z| = 1$ by $\mu(E) = \sigma(\zeta^{-1}(\gamma^{-1}(E))) = \sigma((\gamma \circ \zeta)^{-1}(E))$, thus.

$$\sigma'(s)|d\zeta| = \sigma'(s)|\varphi'(e^{i\theta})|d\theta = \mu'(\theta)d\theta,$$

with σ' and μ' absolutely continuous parts of σ and μ, respectively.
Let K be the unbounded component of the complement of C and $z = \phi(x)$ a function which maps K onto $\boldsymbol{E} = \{z : |z| > 1\}$ so that the points at infinity correspond to each other and $\phi'(\infty) > 0$. Let C_R denote generically the curve $|\phi(x)| = R > 1$ in K.

We define the varying measure

$$d\sigma_n = \frac{d\sigma}{|Y_n|^p}, \tag{1.1}$$

where $\{Y_n\}$, $n \in \mathbf{N}$ is a sequence of polynomials such that for each n, Y_n has exactly degree n, all its zeros $(\alpha_{n,i})$, $i = 1, 2, \ldots$, lie in the unbounded component of the complement of a C_A, $A > 1$, and $Y_n(\alpha) = 1$.

We want to study the asymptotic behavior of polynomials that solve the next extremal problem

$$\rho_{n,p} = \inf_{Q_n(\alpha)=1} \|Q_n\|_{p,\sigma_n} = \inf_{Q_n(\alpha)=1} \left\{ \frac{1}{2\pi} \int_C \left| \frac{Q_n(\zeta)}{Y_n(\zeta)} \right|^p d\sigma(s) \right\}^{1/p}. \tag{1.2}$$

We denote by $\{P_{n,p}\}$ a sequence of the extremal polynomials, i. e.,

$$\|P_{n,p}\|_{p,\sigma_n} = \min_{Q_n(\alpha)=1} \|Q_n\|_{p,\sigma_n}. \tag{1.3}$$

$H^p(C,\sigma)$ is defined as the $L^p(C,\sigma)$ closure of the polynomials in the variable $\zeta \in C$. $L^p_s(C,\sigma) = \{f \in L^p(C,\sigma) : f = 0, \sigma' - a.e.\}$ and $L^p_a(C,\sigma) = \{f \in L^p(C,\sigma) : f = 0, \sigma_s - a.e.\}$, with σ_s the singular part of σ. Similarly, we define $H^p_s(C,\sigma)$ and $H^p_a(C,\sigma)$. Furthermore, we denote by H^p the classical Hardy space in $\{|z| < 1\}$ and $H^p(\mu)$ the $L^p(\mu)$ closure of the polynomials in $e^{i\theta}$.
We suppose that σ satisfies the Szegő condition, i. e., $\int_C \log \sigma'(s)|\gamma'(\zeta)||d\zeta| > -\infty$ and this is the same as, $\log \mu' \in L^1$. We denote by $D_p(\mu, z)$ the Szegő function

$$D_p(\mu, z) = \exp\left\{ \frac{1}{2p\pi} \int_0^{2\pi} \frac{\zeta + z}{\zeta - z} \log \mu'(\theta) \mathrm{d}\theta \right\}, \ \zeta = e^{i\theta},\ z \in \boldsymbol{D}$$

and

$$\Delta_p(\sigma, x) = D_p(\mu, \gamma(x)). \tag{1.4}$$

The function $\Delta_p(\sigma, x)$ satisfies the following properties:
1) $\Delta_p(\sigma, x)$ is regular in B, more precisely, $\Delta_p(\sigma, x) \in H^p(C,\sigma)$.
2) $\Delta_p(\sigma, x) \neq 0$ in B, and $\Delta_p(\sigma, \alpha) = D_p(\mu, 0) > 0$,
3) $|\gamma'(\zeta)||\Delta_p(\sigma, \zeta)|^p = \sigma'(s)$ almost everywhere in C, with $\zeta \in C$.
The main result of this paper that will be proved in Section 3 is the following theorem:

THEOREM 1.1. *For $0 < p < \infty$, the following statements are equivalent*
i) *σ satisfies the Szegő condition.*

ii) *The next limit exists and is positive*

$$\lim_{n\to\infty} \rho_{n,p} > 0. \tag{1.5}$$

iii) *There exists a function $S(x)$ regular in B with $S(\alpha) = 1$ and $\|S\|_{p,\sigma} < \infty$, such that*

$$\lim_{n\to\infty} \left\| \frac{P_{n,p}(x)}{Y_n(x)} - S(x) \right\|_{p,\sigma} = 0. \tag{1.6}$$

iv) *There exists a function $T(x)$ regular in B such that*

$$\lim_{n\to\infty} \frac{\frac{P_{n,p}(x)}{Y_n(x)}}{\left\| \frac{P_{n,p}}{Y_n} \right\|_{p,\sigma}} = T(x) \tag{1.7}$$

holds uniformly on each compact subset of B.

Moreover, if i) holds $\lim_{n\to\infty} \rho_{n,p} = \Delta_p(\sigma,\alpha)$, $S(x) = \dfrac{\Delta_p(\sigma,\alpha)}{\Delta_p(\sigma,x)}$, and $T(x) = \dfrac{1}{\Delta_p(\sigma,x)}$.

2. Auxiliary Results. Before we can prove the theorems in the following sections, we need to establish several auxiliary results.

Let F be a closed limited point set and let $f(x)$ be a function continuous on F. Let F_n be the set of functions of form

$$\pi_n(x) = \frac{b_{n,0}x^n + b_{n,1}x^{n-1} + \cdots + b_{n,n}}{(x-\alpha_{n,1})(x-\alpha_{n,2})\cdots(x-\alpha_{n,n})}. \tag{2.1}$$

Pick $r_n(f) \in F_n$ such that it is the best approximation to $f(x)$ on F in the sense of Tchebycheff, i. e.

$$\|f - r_n(f)\| = \min\{\|f - \pi_n\| : \pi_n \in F_n\}$$

with $\|\cdot\|$ the supremun norm on F.

THEOREM 2.1. *(see [8], p. 253). Let C be a closed analytic Jordan curve and let the points $\{\alpha_{n,k},\ k = 1, 2, \ldots, n;\ n = 0, 1, 2, \ldots\}$ be given with no limit points interior to C_A. Let f(x) be analytic on and within C_T, then there exists a sequence $r_n(x)$ of functions of form (2.1) such that*

$$\lim_{n\to\infty} r_n(x) = f(x), \text{uniformly for } x \text{ on each closed subset of } C_R,$$

where $R = \dfrac{A^2T + T + 2A}{2AT + A^2 + 1}$.

The Keldysh theorem is another auxiliary theorem to prove Theorem 1.1. This theorem can be seen in [4]. Before introducing it we need the next result which can be found in [1].

If $f \in H^p(\mu)$, then there exist unique functions $\tilde{f}$, f_s such that

$$f = K_p\tilde{f} + f_s, \quad \tilde{f} \in H^p, \text{ and } f_s \in L^p_s(\mu), \tag{2.2}$$

with

$$K_p(\mu, z) = \begin{cases} \frac{D_p(\mu,0)}{D_p(\mu,z)}, & \text{if } z \in (S_a \cup \{z : |z| < 1\}), \\ 0, & \text{if } z \in S_s, \end{cases} \tag{2.3}$$

where S_a and S_s are a disjoint decomposition of the unit circle such that μ' and μ_s live on these sets respectively

THEOREM 2.2. *Assume that μ satisfies the Szegő condition and $\{f_n\} \subset H^p(\mu)$, $0 < p < \infty$, such that*

i) $\lim_{n\to\infty} \tilde{f}_n(0) = 1$;

ii) $\lim_{n\to\infty} \|f_n\|_{p,\mu} = D_p(\mu, 0)$.

Then

a) $\lim_{n\to\infty} \tilde{f}_n(z) = 1$ *holds uniformly on each compact subset of* $\boldsymbol{D}$.

b) $\lim_{n\to\infty} \|f_n - K_p(\mu, z)\|_{p,\mu} = 0$.

An extension of this theorem is given in [1].

3. Proof of Theorem 1.1. Before proving Theorem 1.1, we are going to prove an intermediate result.

THEOREM 3.1. *For* $0 < p < \infty$

$$\lim_{n\to\infty} \left\| \frac{P_{n,p}}{Y_n} \right\|_{p,\sigma} = \Delta_p(\sigma, \alpha).$$

where 0 *replaces* $\Delta_p(\sigma, \alpha)$ *if* σ *does not satisfy the Szegő condition.*

Proof. Let μ be the image measure of σ on $|z| = 1$ by $\gamma \circ \zeta$. From Szegő, [7] p. 297, we know that if $T_{n,2}(z)$, are the extremal polynomials such that

$$\|T_{n,2}\|_{2,\mu} = \min\{\|Q_n\|_{2,\mu} : Q_n \text{ monic of degree } n\},$$

then

$$\lim_{n\to\infty} \|T_{n,2}\|^2_{2,\mu} = \lim_{n\to\infty} \|T^*_{n,2}\|^2_{2,\mu} = D_2(\mu, 0)^2, \tag{3.1}$$

with $D_2(\mu, 0) = 0$ if $\log \mu'(\theta)$ is not integrable and $T^*_{n,2}(z) = z^n \bar{T}_{n,2}(\frac{1}{\bar{z}})$. Let $\delta_n(x) = (T^*_{n,2}(\gamma(x)))^{2/p}$ which is analytic in $\overline{B}$ since the zeros of $T^*_{n,2}$ lie in $\{|z| > 1\}$, then from Theorem 2.1, there exists a sequence $\left\{\frac{R_{m_n}}{Y_{m_n}}\right\}$ such that

$$\lim_{n\to\infty} \sup_{\zeta\in C} \left| \frac{R_{m_n}(\zeta)}{Y_{m_n}(\zeta)} - \delta_n(\zeta) \right| = 0,$$

and the convergence is uniform in $\overline{B}$. In particular, there is convergence in $x = \alpha$ and as $\delta_n(\alpha) = (T^*_{n,2}(0))^{2/p} = 1 = Y_{m_n}(\alpha)$ we have $\lim_{n\to\infty} R_{m_n}(\alpha) = 1$. Hence

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_C \left| \frac{R_{m_n}(\zeta)}{Y_{m_n}(\zeta)} \right|^p \mathrm{d}\sigma(s) = \lim_{n\to\infty} \frac{1}{2\pi} \int_C |\delta_n(\zeta)|^p \mathrm{d}\sigma(s) =$$

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |T^*_{n,2}(e^{i\theta})|^2 \mathrm{d}\mu(\theta) = D_2(\mu, 0)^2 = D_p(\mu, 0)^p,$$

therefore

$$\limsup_{n\to\infty} \left\| \frac{P_{n,p}}{Y_n} \right\|_{p,\sigma} \leq D_p(\mu, \gamma(\alpha)) = \Delta_p(\sigma, \alpha). \tag{3.2}$$

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On the other hand, using Jensen inequality

$$\frac{1}{2\pi}\int_C \left|\frac{P_{n,p}(\zeta)}{Y_n(\zeta)}\right|^p \mathrm{d}\sigma(s) = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_{n,p}(\varphi(e^{i\theta}))}{Y_n(\varphi(e^{i\theta}))}\right|^p \mathrm{d}\mu(\theta) \geq$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_{n,p}(\varphi(e^{i\theta}))}{Y_n(\varphi(e^{i\theta}))}\right|^p \mu'(\theta)\mathrm{d}\theta \geq$$

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{P_{n,p}(\varphi(e^{i\theta}))}{Y_n(\varphi(e^{i\theta}))}\right|^p \mathrm{d}\theta\right\} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\mu'(\theta)\mathrm{d}\theta\right\} \geq$$

$$\left|\frac{P_{n,p}(\varphi(0))}{Y_n(\varphi(0))}\right|^p D_p(\mu,0)^p = D_p(\mu,0)^p.$$

Hence, with this and (3.2) we obtain

$$\lim_{n\to\infty}\left\|\frac{P_{n,p}}{Y_n}\right\|_{p,\sigma} = D_p(\mu,\gamma(\alpha)).$$

□

Proof of Theorem 1.1:

Proof. $i) \Leftrightarrow ii)$. It is done in Theorem 3.1.
$i) \Rightarrow iii)$. We consider the function

$$h_n(z) = \frac{P_{n,p}(\varphi(z))D_p(\mu,z)}{Y_n(\varphi(z))D_p(\mu,0)}, \tag{3.3}$$

that is regular in $\boldsymbol{D}$ and $h_n(0)=1$. From $i) \iff ii)$ and $|D_p(\mu,e^{i\theta})|^p = \mu'(\theta)$, we have

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} |h_n(e^{i\theta})|^p \mathrm{d}\theta\right\} = 1, \tag{3.4}$$

then, applying Theorem 2.2

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} |h_n(e^{i\theta})-1|^p \mathrm{d}\theta\right\} = 0,$$

hence

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_{n,p}(\varphi(e^{i\theta}))D_p(\mu,e^{i\theta})}{Y_n(\varphi(e^{i\theta}))D_p(\mu,0)} - 1\right|^p \mathrm{d}\theta\right\} = 0$$

and then

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_{n,p}(\varphi(e^{i\theta}))}{Y_n(\varphi(e^{i\theta}))} - \frac{D_p(\mu,0)}{D_p(\mu,e^{i\theta})}\right|^p \mu'(\theta)\mathrm{d}\theta\right\} = 0.$$

Therefore, using (3.4) and Theorem 3.1, we have

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_{n,p}(\varphi(e^{i\theta}))}{Y_n(\varphi(e^{i\theta}))} - \frac{D_p(\mu,0)}{D_p(\mu,e^{i\theta})}\right|^p \mathrm{d}\mu(\theta)\right\} = 0,$$

and this is the same as *iii)*, where $S(x) = \dfrac{\Delta_p(\sigma,\alpha)}{\Delta_p(\sigma,x)}$.
$iii) \Rightarrow i)$. It follows from the relation

$$\liminf_{n\to\infty} \left\| \frac{P_{n,p}}{Y_n} - \psi \right\|_p = 0, \tag{3.5}$$

where $\psi(\zeta)$ is such that $\|\psi\|_p \neq 0$.
In fact, from (3.5) it follows that there exists a subsequence $\{n_\nu\}$ such that

$$\lim_{\nu\to\infty} \left\| \frac{P_{n_\nu,p}}{Y_{n_\nu}} - \psi \right\|_p = 0.$$

If *i)* does not hold, from *ii)*

$$\lim_{\nu\to\infty} \left\| \frac{P_{n_\nu,p}}{Y_{n_\nu}} \right\|_p = 0,$$

and we obtain $\|\psi\|_p = 0$, that it is a contradiction.
$iii) \Rightarrow iv)$. The sequence of functions $\{h_n\}$ as in (3.3) satisfies the hypothesis of Theorem 2.2, hence $\lim\limits_{n\to\infty} h_n(z) = 1$, holds uniformly on each compact subset of B. Now, since *i)* is equivalent to *iii)*, from Theorem 3.1 we have

$$\lim_{n\to\infty} \frac{\frac{P_{n,p}(x)}{Y_n(x)}}{\left\| \frac{P_{n,p}(x)}{Y_n(x)} \right\|_p} = \frac{1}{D_p(\mu,\gamma(x))}.$$

$iv) \Rightarrow i)$. From *iv)* and Theorem 3.1 we have

$$\lim_{n\to\infty} \frac{\frac{P_{n,p}(\alpha)}{Y_n(\alpha)}}{\left\| \frac{P_{n,p}(x)}{Y_n(x)} \right\|_p} = T(\alpha) = \frac{1}{\Delta_p(\sigma,\alpha)} < \infty,$$

but this is true if and only if *i)* holds.
□

4. Density Theorem.

In this section we give a density theorem that may be seen as an "application" of the main theorem.
We introduce the notation: $R_{n,k} = \{\frac{h}{Y_n} : \text{ degree } h \leq n-k\}$.

THEOREM 4.1. *Let σ_a be a measure in C absolutely continuous with respect to Lebesgue measure, and that satisfies the Szegő condition, then the following statements are equivalent:*
i) *For each $j \in \mathbf{Z}_+$*

$$\lim_{n\to\infty} \left\| \frac{P_{n,n-j,p}}{Y_n} \right\|_{p,\sigma_a} = \Delta_p(\sigma_a,\alpha),$$

where $P_{n,n-j,p}$ denotes an extremal polynomial, with $P_{n,n-j,p}(\alpha) = 1$, i.e.

$$\left\| \frac{P_{n,n-j,p}}{Y_n} \right\|_{p,\sigma_a} = \min \left\{ \left\| \frac{Q_{n-j}}{Y_n} \right\|_{p,\sigma_a} : Q_{n-j} \in \Pi_{n-j},\, Q_{n-j}(\alpha) = 1 \right\}.$$

ii) *For each $k \in \mathbf{Z}_+$, $R_{n,k}$ is dense in $H^p(\sigma_a)$.*

Proof.

$ii) \Longrightarrow i)$ Here we use the same technique as in the proof of Theorem 3.1. From Szegő, [7], we know that the result is true for $T_{n-k,2}$, then

$$\lim_{n\to\infty} \|T^*_{n-k,2}\|_2 = D_2(\mu,0)^2.$$

Let $\delta_{n-k}(x) = (T^*_{n-k,2}(\gamma(x)))^{2/p}$ which is analytic in $\overline{B}$ since the zeros of $T^*_{n-k,2}$ lie in $\{|z| > 1\}$, then from *ii)* there exists a sequence of polynomials $\{R_{m_n-k}\}$ of degree $m_n - k$ such that

$$\limsup_{\zeta\in C} \left| \frac{R_{m_n-k}(\zeta)}{Y_{m_n}(\zeta)} - \delta_{n-k}(\zeta) \right| = 0$$

and the convergence is uniform in $\overline{B}$. Then

a) $\lim_{n\to\infty} R_{m_n-k}(\alpha) = 1$;

b) $\lim_{n\to\infty} \left\| \frac{R_{m_n-k}}{Y_{m_n}} \right\|^p_{p,\sigma_a} = \Delta_p(\sigma_a,\alpha)^p.$

Given $\Lambda \subset \mathbf{N}$ an index sequence, from *ii)* we observe that the sequence $\{m_n\}$ can be chosen in Λ, so *i)* follows from a) and b).

$i) \Longrightarrow ii)$**.** Set $i,\ j \in \mathbf{Z}_+$, using *i)* and Theorem 2.2 we have

$$\frac{P_{n,n-(i+j),p}(\zeta)}{Y_n(\zeta)} \longrightarrow \frac{\Delta_p(\sigma_a,\alpha)}{\Delta_p(\sigma_a,\zeta)},$$

in $L^p(C,\sigma_a)$, thus

$$\frac{\zeta^i P_{n,n-(i+j),p}(\zeta)}{Y_n(\zeta)} \longrightarrow \zeta^i \frac{\Delta_p(\sigma_a,\alpha)}{\Delta_p(\sigma_a,\zeta)}$$

in $H^p(C,\sigma_a)$. Because of $H^p(C,\sigma_a) = H^p(C)\cdot \frac{\Delta_p(\sigma_a,\alpha)}{\Delta_p(\sigma_a,\cdot)}$ and $H^p(C)$ is the closure of the polynomials in $L^p(C)$, $R_{n,j}$ satisfies *ii)*.

□

REFERENCES

[1] M. BELLO HERNÁNDEZ, F. MARCELLÁN, J. MÍNGUEZ CENICEROS, *Pseudo-uniform convexity in H^p and some extremal problems on Sobolev spaces*, Complex Variables Theory and Application, 48 (2003), pp. 429-440.

[2] A. A. GONCHAR, *On the speed of rational approximation of some analytic functions*, Math. USSR Sbornik., 34 (1978), No. 2, pp. 147-163.

[3] A. A. GONCHAR, G. LÓPEZ LAGOMASINO, *On Markov's theorem for multipoint Padé approximants*, Math. USSR Sbornik., 34 (1978), No. 4, pp. 512-524.

[4] M. V. KELDYSH, *Selected papers*, Academic Press, Moscow, 1985. [In Russian]

[5] G. LÓPEZ LAGOMASINO, *Relative Asymptotics for polynomials orthogonal on the real axis*, Math. USSR Sbornik., 65 (1988), No. 2, pp. 505-529.

[6] G. LÓPEZ, *Szegő theorem for polynomials orthogonal with respect to varying measures*, in Orthogonal polynomials and their applications (Segovia, 1986), pp. 255–260, Lecture Notes in Math., 1329, Springer, Berlin, 1988.

[7] G. SZEGŐ, *Orthogonal Polynomials*, 4th ed., American Math. Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, Rhode Island, 1975.

[8] J. L. WALSH, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 3th ed., American Math. Society Colloquium Publications, Vol. XX, Amer. Math. Society, Providence 6, Rhode Island, 1960.