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AN ELECTROSTATIC INTERPRETATION OF THE ZEROS OF THE FREUD-TYPE ORTHOGONAL POLYNOMIALS*

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Abstract. Polynomials orthogonal with respect to a perturbation of the Freud weight function by the addition of a mass point at zero are considered. These polynomials, called Freud-type orthogonal polynomials, satisfy a second order linear differential equation with varying polynomial coefficients. It plays an important role in the electrostatic interpretation for the distribution of zeros of the corresponding orthogonal polynomials.

Key words. Freud weights, orthogonal polynomials, zeros, potential theory, semiclassical linear functional.

AMS subject classifications. Primary 33C45, secondary 42C05.

1. Introduction. The aim of this paper is to give an electrostatic interpretation for the distribution of zeros of orthogonal polynomials based on the original ideas of Stieltjes, see [9] as well as [11] and [12], in the case of the zeros of Freud-type orthogonal polynomials. Nevertheless, the technics and methods for such a purpose are quite different from those used by Stieltjes simply because the Freud-type polynomials are semiclassical and the electrostatic interpretation makes sense only in the presence of a varying external field. In this contribution we consider a varying external field which is created by charges of constant magnitude but position varying with the number n of other movable charges. Ismail [6] proved for absolutely continuous measures that zeros of general orthogonal polynomials, under some integrability conditions for their weight functions, are the solution of an electrostatic equilibrium problem of n movable unit charges in the presence of an external potential. In particular, as an example, he studied electrostatics of the zeros of the orthogonal polynomial sequence with respect to Freud weight function $w(x) = e^{-x^4}$ supported on the whole real line $\mathbb{R}$. The case of measures with mass points outside or on the boundary of the support of the measure has been analyzed in [3], [4], and [7]. It is a natural question to ask about the location of zeros -from the point of view of the electrostatic interpretation- for polynomials orthogonal with respect to a perturbation of the linear functional $\mathcal{L}$ associated with the Freud weight. We introduce such a kind of perturbation based on the addition of a Dirac linear functional supported at zero. This case has not been considered in [7].

First of all, we obtain the relation between both orthogonal polynomial sequences as well as the expression of the recursion coefficients of the corresponding orthogonal polynomials. The main role is played by the second order linear differential equation which these polynomials satisfy since this yields the electrostatic interpretation. In section 3 it will be obtained through two operators associated with the orthogonal polynomials. In the next section we derive the corresponding differential equation.

2. Preliminaries. Let $\mathcal{U}$ be the linear functional

$$\langle \mathcal{U}, q(x) \rangle = \int_{-\infty}^{\infty} q(x) e^{-x^4} dx + \lambda q(0), \qquad q(x) \in \mathbb{P},$$

where $\lambda \in \mathbb{R}^+$, $\mathbb{P} := \mathbb{R}[x]$ is the linear space of polynomials with real coefficients and $\mathbb{P}_n$ is the linear subspace of polynomials of degree at most n.

A polynomial sequence $\{\tilde{p}_n(x)\}_{n\geq 0}$, where $\deg(\tilde{p}_n(x)) = n$ for $n \geq 0$, is said to be orthogonal with respect to the linear functional $\mathcal{U}$ if and only if

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$$\left\langle \mathcal{U}, \tilde{p}_n(x)\tilde{p}_m(x) \right\rangle = \tilde{k}_n \delta_{n,m},$$

where $\tilde{k}_n > 0$, for $n, m \geq 0$ and, as usual,

$$\delta_{n,m} = \begin{cases} 0 & \text{for} \quad n \neq m, \\ 1 & \text{for} \quad n = m. \end{cases}$$

Such a family of polynomials is said to be a Freud-type sequence.

Let $\mathcal{L}$ be the Freud linear functional defined by

$$\left\langle \mathcal{L}, q(x) \right\rangle = \int_{-\infty}^{\infty} q(x) e^{-x^4} dx, \qquad q(x) \in \mathbb{P}.$$

Consider $\{p_n(x)\}_{n\geq 0}$ the monic orthogonal polynomial sequence with respect to $\mathcal{L}$. Then

$$\left\langle \mathcal{L}, p_n(x)p_m(x) \right\rangle = k_n \delta_{n,m},$$

where $k_n > 0$ for $n \geq 0$. The Freud linear functional is symmetric (see [2]). Thus $\{p_n(x)\}_{n\geq 0}$ satisfies the three-term recurrence relation (TTRR)

$$x p_n(x) = p_{n+1}(x) + a_n p_{n-1}(x), \qquad n \geq 0,$$

where by convention $p_{-1}(x) = 0$ and $a_n \in \mathbb{R}_+$ is

$$a_n = \frac{\left\langle \mathcal{L}, p_n^2(x) \right\rangle}{\left\langle \mathcal{L}, p_{n-1}^2(x) \right\rangle}, \qquad n \geq 1.$$

Moreover, a_n satisfies the nonlinear recurrence relation, (see [8])

$$n = 4a_n(a_{n+1} + a_n + a_{n-1}), \qquad n \geq 1,$$

with initial conditions $a_0 = 0$, $a_1 = \dfrac{\Gamma(3/4)}{\Gamma(1/4)}$, and

$$a_n = \left(\frac{1}{2\sqrt{3}}\right)\sqrt{n}\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

Both linear functionals are closely connected. Indeed

$$\left\langle \mathcal{U}, q(x) \right\rangle = \left\langle \mathcal{L}, q(x) \right\rangle + \lambda q(0), \qquad q(x) \in \mathbb{P}.$$

Let $\{\widehat{p}_n(x)\}_{n\geq 0}$ be the monic polynomial sequence orthogonal with respect to $\mathcal{U}$. Now we derive the three-term recurrence relation that the polynomial sequence $\{\widehat{p}_n(x)\}_{n=0}^{\infty}$ satisfies. Since $\mathcal{U}$ is symmetric, the recurrence formula satisfied by $\{\widehat{p}_n(x)\}_{n\geq 0}$ is

$$x\widehat{p}_n(x) = \widehat{p}_{n+1}(x) + \widehat{a}_n \widehat{p}_{n-1}(x), \quad n \geq 0.$$

We denote

$$K_n(x, y) = \sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{\left\langle \mathcal{L}, p_j^2(x) \right\rangle},$$

then by the confluent formula (see [2])

$$K_{2n}(0,0) = \frac{p'_{2n+1}(0)p_{2n}(0)}{\langle \mathcal{L}, p_{2n}^2(x) \rangle},$$

and using the asymptotic expressions for $p_{2n}(0)$ and $p'_{2n+1}(0)$ (see [8])

$$K_{2n}(0,0) = \frac{2\sqrt[4]{8}}{3} n^{3/4} \big(A + o(1)\big)^2,$$

where $A = \sqrt[8]{12}/\sqrt{\pi}$ is a constant independent of n, determined by Nevai in [8]. Thus

$$K_{2n}(0,0) = \mathcal{O}(n^{3/4}).$$

PROPOSITION 2.1. *For* $n \geq 0$,

$$\begin{aligned} &(i) \qquad \widehat{p}_{2n+1}(x) = p_{2n+1}(x), \\ &(ii) \qquad \widehat{p}_{2n}(x) = p_{2n}(x) - \lambda \frac{p_{2n}(0)K_{2n-2}(x,0)}{1+\lambda K_{2n-2}(0,0)}. \end{aligned}$$

Proof. If we consider the Fourier expansion

$$\widehat{p}_n(x) = p_n(x) + \sum_{k=0}^{n-1} a_{n,k} p_k(x), \qquad n \geq 0,$$

then

$$a_{n,k} = \frac{\langle \mathcal{L}, \widehat{p}_n(x) p_k(x) \rangle}{\langle \mathcal{L}, p_k^2(x) \rangle} = -\frac{\lambda \widehat{p}_n(0) p_k(0)}{\langle \mathcal{L}, p_k^2(x) \rangle}, \qquad 1 \leq k \leq n-1.$$

Thus

$$\widehat{p}_n(x) = p_n(x) - \lambda \widehat{p}_n(0) K_{n-1}(x,0), \qquad n \geq 1.$$

The evaluation at zero of the last expression yields

$$\widehat{p}_n(0) = \frac{p_n(0)}{1+\lambda K_{n-1}(0,0)}.$$

Therefore this completes the proof. □

The coefficients $\widehat{a}_n$ satisfy the following recursive expressions (see [1])

PROPOSITION 2.2. *The relation between the recursion coefficients is*

$$\widehat{a}_{2n+1}\widehat{a}_{2n} = a_{2n+1}a_{2n}, \qquad n \geq 0,$$

$$\widehat{a}_{2n} + \widehat{a}_{2n-1} = a_{2n} + a_{2n-1}, \qquad n \geq 1.$$

From the expression (see [1])

$$\frac{\widehat{a}_{2n}}{a_{2n}} = \frac{1+\lambda K_{2n}(0,0)}{1+\lambda K_{2n-2}(0,0)},$$

we get easily

$$\frac{\widehat{a}_{2n}}{a_{2n}} = 1 + \mathcal{O}\left(n^{-1}\right),$$

or equivalently

$$\frac{\widehat{a}_{2n}}{a_{2n}} \sim 1 + \frac{3}{4n}. \tag{2.1}$$

The notation $x_n \sim y_n$ means that x_n behaves as y_n when $n \to \infty$, more precisely, $\lim_{n\to\infty} x_n/y_n = 1$. This notation provides information about the multiplying factor involved in the leading term of the asymptotic behavior.

For the odd coefficients we deduce a similar asymptotic property from the previous theorem

$$\frac{\widehat{a}_{2n-1}}{a_{2n-1}} \sim 1 - \frac{3}{4n}. \tag{2.2}$$

3. Differential recurrence relation. We are interested in finding a second order linear differential equation satisfied by the orthogonal polynomials $\widehat{p}_n$. To obtain it we will use the fact that the linear functional $\mathcal{U}$ is semiclassical, that is, it satisfies the following distributional Pearson equation

$$D(x^2\mathcal{U}) = (2x - 4x^5)\mathcal{U},$$

where D denotes the distributional derivative.

3.1. Lowering operator. First, we will obtain a differential difference equation for the monic orthogonal polynomial sequence $\widehat{p}_n(x)$.

THEOREM 3.1. $\widehat{p}_n(x)$ *satisfies the following differential recurrence relation*

$$x^2\widehat{p}_n'(x) = A(x,n)\widehat{p}_{n-1}(x) - B(x,n)\widehat{p}_n(x), \qquad n \geq 1, \tag{3.1}$$

where

$$\begin{aligned} A(x,n) &= \widehat{a}_n\Big[4x^4 + 4(\widehat{a}_{n+1} + \widehat{a}_n)x^2 + C_{n+1} + C_n\Big], \\ B(x,n) &= 4\widehat{a}_n x^3 + C_n x, \end{aligned}$$

with

$$C_n = -n + 4\widehat{a}_n(\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1}), \quad n \geq 1.$$

Proof. Consider the Fourier expansion

$$x^2\widehat{p}_n'(x) = n\widehat{p}_{n+1}(x) + \sum_{j=0}^{n} \lambda_{n,j}\widehat{p}_j(x),$$

where

$$\lambda_{n,j} = \frac{\big\langle \mathcal{U}, x^2\widehat{p}_n'(x)\widehat{p}_j(x)\big\rangle}{\big\langle \mathcal{U}, \widehat{p}_j^2(x)\big\rangle}.$$

We compute the numerator of the above expression:

$$\begin{aligned} \big\langle \mathcal{U}, x^2\widehat{p}_n'(x)\widehat{p}_j(x)\big\rangle = {}& -\big\langle \mathcal{U}, 2x\widehat{p}_n(x)\widehat{p}_j(x)\big\rangle + 4\big\langle \mathcal{U}, x^5\widehat{p}_n(x)\widehat{p}_j(x)\big\rangle \\ & -\big\langle \mathcal{U}, x^2\widehat{p}_n(x)\widehat{p}_j'(x)\big\rangle. \end{aligned}$$

For $j < n-5$, we get

$$\left\langle \mathcal{U}, x^2 \widehat{p}_n'(x) \widehat{p}_j(x) \right\rangle = 0.$$

Thus if $j \leq n-6$ then

$$\lambda_{n,j} = 0.$$

Now we compute the coefficients $\lambda_{n,j}$ for $j \geq n-5$. Taking into account $\mathcal{U}$ is a symmetric linear functional

$$\lambda_{n,j} = 0, \qquad \text{for} \quad j = n, n-2, n-4.$$

Furthermore

$$\begin{aligned}
\lambda_{n,n-5} &= 4\widehat{a}_n \widehat{a}_{n-1} \widehat{a}_{n-2} \widehat{a}_{n-3} \widehat{a}_{n-4}, \\
\lambda_{n,n-3} &= 4(\widehat{a}_n \widehat{a}_{n-1} \widehat{a}_{n-2}) \big[\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1} + \widehat{a}_{n-2} + \widehat{a}_{n-3}\big], \\
\lambda_{n,n-1} &= \widehat{a}_n \Big[-(n+1) + 4\big[\widehat{a}_{n+1}(\widehat{a}_{n+2} + \widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1}) \\
&\qquad + \widehat{a}_n(\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1}) + \widehat{a}_{n-1}(\widehat{a}_n + \widehat{a}_{n-1} + \widehat{a}_{n-2})\big]\Big].
\end{aligned}$$

We compute the polynomials involved in the differential recurrence relation by using the TTRR in terms of $\widehat{p}_{n-1}(x)$ and $\widehat{p}_n(x)$. So the equation can be reduced to

$$\begin{aligned}
x^2 \widehat{p}_n'(x) = \widehat{a}_n \Big[& 4x^4 + 4(\widehat{a}_{n+1} + \widehat{a}_n) x^2 - (2n+1) \\
& + 4\big[\widehat{a}_{n+1}(\widehat{a}_{n+2} + \widehat{a}_{n+1} + \widehat{a}_n) + \widehat{a}_n\,(\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1})\,\big]\Big] \widehat{p}_{n-1}(x) \\
& - \Big[4\widehat{a}_n x^3 - \big[n - 4\widehat{a}_n(\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1})\big] x \Big] \widehat{p}_n(x).
\end{aligned}$$

If $A(x,n)$ denotes the coefficient of $\widehat{p}_{n-1}(x)$ and $B(x,n)$ denotes the coefficient of $\widehat{p}_n(x)$, i.e.

$$x^2 \widehat{p}_n'(x) = A(x,n) \widehat{p}_{n-1}(x) - B(x,n) \widehat{p}_n(x),$$

and

$$C_n = -n + 4\widehat{a}_n(\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1}), \qquad n \geq 1,$$

then the differential recurrence relation becomes

$$\begin{aligned}
x^2 \widehat{p}_n'(x) &= \widehat{a}_n \Big[4x^4 + 4(\widehat{a}_{n+1} + \widehat{a}_n) x^2 + C_{n+1} + C_n \Big] \widehat{p}_{n-1}(x) \\
&\quad - \Big[4\widehat{a}_n x^3 + C_n x \Big] \widehat{p}_n(x), \qquad\qquad n \geq 1.
\end{aligned}$$

Thus our statement follows. □

Let us denote $L_1(x,n)$ the lowering operator

$$L_1(x,n) = \Big[x^2 \frac{d}{dx} + B(x,n) \Big]. \tag{3.2}$$

The statement of the previous theorem leads

$$L_1(x,n) \widehat{p}_n(x) = A(x,n) \widehat{p}_{n-1}(x).$$

Hence, the following properties for the coefficients $A(x,n)$, $B(x,n)$, and C_n can be immediately proved:

LEMMA 3.2. *For $n \geq 1$, $A(x,n)$ and $B(x,n)$ satisfy*

$$\begin{aligned}
&(i) \qquad \frac{1}{x}\Big[B(x,n) + B(x,n+1) \Big] = \frac{A(x,n)}{\widehat{a}_n} - 4x^4, \\
&(ii) \quad x\Big[B(x,n+1) - B(x,n) \Big] = A(x,n+1) - \frac{\widehat{a}_n}{\widehat{a}_{n-1}} A(x,n-1) + x^2.
\end{aligned}$$

THEOREM 3.3. *For* $n \geq 1$,

(i) $C_{2n} + C_{2n-1} = 0$,
(ii) $\widehat{a}_{2n}\big(C_{2n+1} + C_{2n}\big) = \widehat{a}_{2n-1}\big(C_{2n-1} + C_{2n-2}\big)$.

From the previous theorem we get

$$\begin{aligned}\big(C_{2n+1} - C_{2n-1}\big) &= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}\big(C_{2n-1} - C_{2n-3}\big)\\ &= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}\frac{\widehat{a}_{2n-3}}{\widehat{a}_{2n-2}}\big(C_{2n-3} - C_{2n-5}\big)\\ &\vdots\\ &= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}\frac{\widehat{a}_{2n-3}}{\widehat{a}_{2n-2}}\cdots\frac{\widehat{a}_{3}}{\widehat{a}_{4}}\big(C_3 - C_1\big), \qquad n \geq 1.\end{aligned}$$

It is easy to check that $C_3 - C_1 < 0$. Thus we deduce that

$$C_{2n+1} - C_{2n-1} < 0, \quad n \geq 1.$$

THEOREM 3.4. *For* $n \geq 1$
(i) $C_{2n} = 1 + \dfrac{3}{8n} + \mathcal{O}\Big(\dfrac{1}{n^{5/2}}\Big)$,
(ii) $C_{2n-1} = -1 - \dfrac{3}{8n} + \mathcal{O}\Big(\dfrac{1}{n^{5/2}}\Big)$.

Proof. Taking into account the definition of the coefficients C_n and the properties (2.1) and (2.2). □

The coefficient $A(x,n)$ plays a very important role in the study of the electrostatic interpretation. So we study the behavior of its roots since they will provide us the location of some fixed charges in the last section.

COROLLARY 3.5. *For* $n \geq 1$, $A(x,n)$ *has two real roots and two simple conjugate complex roots.*

$$r_{1,2}(n) = \pm\sqrt{\frac{c(n)}{2\left(a(n) + \sqrt{a(n)^2 + c(n)}\right)}},$$

$$r_{3,4}(n) = \pm i\sqrt{\frac{a(n) + \sqrt{a(n)^2 + c(n)}}{2}},$$

where $c(n) = -\left(C_{n+1} + C_n\right)$ *and* $a(n) = \widehat{a}_{n+1} + \widehat{a}_n$. *Moreover, the real roots tend to the origin and the complex roots tend to infinity when* $n \to \infty$. *In particular, in the odd case* $A(x, 2n+1)$ *has a double root at zero,* $s_1 = 0$, $s_2 = 0$, *and two simple conjugate complex roots*

$$s_3(n) = \sqrt{a(n)}i,$$
$$s_4(n) = -\sqrt{a(n)}i.$$

Proof. It follows after the computation of the roots of $A(x,n)$ and applying the asymptotic behavior of the coefficients $\widehat{a}_n$, (2.1), (2.2), and Theorem 3.4 for C_n. □

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3.2. Raising operator . Using the TTRR and substituting it at the lowering operator then the raising operator for the orthogonal polynomials $\widehat{p}_n$ is deduced.

THEOREM 3.6. *$\widehat{p}_n(x)$ satisfies*

$$\left[x^2\frac{d}{dx} - \frac{A(x,n)}{\widehat{a}_n}x + B(x,n)\right]\widehat{p}_n(x) = \frac{-A(x,n)}{\widehat{a}_n}\widehat{p}_{n+1}(x).$$

Proof. From the TTRR we get

$$\widehat{p}_{n-1}(x) = \frac{x}{\widehat{a}_n}\widehat{p}_n(x) - \frac{1}{\widehat{a}_n}\widehat{p}_{n+1}(x).$$

Substituting in the expression of the lowering operator

$$x^2\widehat{p}_n'(x) = A(x,n)\left[\frac{x}{\widehat{a}_n}\widehat{p}_n(x) - \frac{1}{\widehat{a}_n}\widehat{p}_{n+1}(x)\right] - B(x,n)\widehat{p}_n(x).$$

Thus our statement follows. □

We will denote $L_2(x,n)$ the raising operator, i.e.

$$L_2(x,n) = \left[x^2\frac{d}{dx} - \frac{A(x,n)}{\widehat{a}_n}x + B(x,n)\right].$$

4. Second-order linear differential equation. Combining raising and lowering operators we obtain a second order linear differential equation which is the key in order to give the electrostatic interpretation for the zero distribution of $\widehat{p}_n$. It is worth pointing out the role of the lowering operator since the coefficient $A(x,n)$ is involved in the second order linear differential equation.

THEOREM 4.1. *$\widehat{p}_n(x)$ satisfies the following second order differential equation*

$$M(x,n)\widehat{p}_n''(x) + N(x,n)\widehat{p}_n'(x) + R(x,n)\widehat{p}_n(x) = 0, \qquad n \geq 0,$$

where

$$\begin{aligned} M(x,n) &= x^4 A(x,n), \\ N(x,n) &= -x^4 A'(x,n) + A(x,n)\left[2x^3 - 4x^7\right], \end{aligned}$$

and $R(x,n)$ can be explicitly given in terms of $A(x,n)$ and $B(x,n)$ (see (4.3)).

Proof. Applying the lowering operator (3.2) to $\widehat{p}_n(x)$

$$\left[x^2\frac{d}{dx} + B(x,n)\right]\widehat{p}_n(x) = A(x,n)\widehat{p}_{n-1}(x), \tag{4.1}$$

and the raising operator, see Theorem 3.6, for $n-1$ to $\widehat{p}_{n-1}(x)$ one gets

$$\left[x^2\frac{d}{dx} - \frac{A(x,n-1)}{\widehat{a}_{n-1}}x + B(x,n-1)\right]\widehat{p}_{n-1}(x) = \frac{-A(x,n-1)}{\widehat{a}_{n-1}}\widehat{p}_n(x).$$

From (i) in Lemma 3.2 we get

$$\left[x^2\frac{d}{dx}-4x^5-B(x,n)\right]\widehat{p}_{n-1}(x)=\frac{-A(x,n-1)}{\widehat{a}_{n-1}}\widehat{p}_n(x). \tag{4.2}$$

Thus from (4.1)

$$\frac{1}{A(x,n)}\left[x^2\widehat{p}_n'(x)+B(x,n)\widehat{p}_n(x)\right]=\widehat{p}_{n-1}(x).$$

Applying (4.2) in both hand sides of the previous expression, we get

$$\begin{aligned}&x^2\Big[-\frac{A'(x,n)}{A^2(x,n)}\big(x^2\widehat{p}_n'(x)+B(x,n)\widehat{p}_n(x)\big)\\ &+\frac{1}{A(x,n)}\big(x^2\widehat{p}_n''(x)+(2x+B(x,n))\widehat{p}_n'(x)+B(x,n)\widehat{p}_n(x)\big)\Big]\\ &-\frac{4x^5+B(x,n)}{A(x,n)}\big(x^2\widehat{p}_n'(x)+B(x,n)\widehat{p}_n(x)\big)=-\frac{A(x,n-1)}{\widehat{a}_{n-1}}\widehat{p}_n(x).\end{aligned}$$

Our statement follows from

$$R(x,n)=A(x,n)\left[B'(x,n)x^2-4x^5+\frac{A(x,n-1)A(x,n)}{\widehat{a}_{n-1}}\right]-A'(x,n)B(x,n)x^2. \tag{4.3}$$

□

5. Electrostatic interpretation. In this section we propose an electrostatic model in the presence of a varying external potential from the second order linear differential equation deduced in the previous section. We will study the asymptotic behavior of the position of the movable constant charges involved in the external field. As we have shown in section 2 odd Freud-type orthogonal polynomials coincide with the Freud polynomials of odd degree. Thus the electrostatic interpretation for these polynomials comes from the electrostatics for Freud polynomials. We denote $\{x_{n,k}\}_{1\le k\le n}$ the zeros of $\widehat{p}_n(x)$. Evaluating the second-order differential equation at $x_{n,k}$

$$M(x_{n,k},n)\widehat{p}_n''(x_{n,k})+N(x_{n,k},n)\widehat{p}_n'(x_{n,k})=0,\quad 1\le k\le n.$$

Then

$$\begin{aligned}\frac{\widehat{p}_n''(x_{n,k})}{\widehat{p}_n'(x_{n,k})}&=-\frac{N(x_{n,k},n)}{M(x_{n,k},n)}\\ &=\frac{A'(x_{n,k},n)}{A(x_{n,k},n)}-\frac{2}{x_{n,k}}+4x_{n,k}^3,\quad 1\le k\le n.\end{aligned} \tag{5.1}$$

We must point out that $A(x_{2n,k},2n)\neq 0$. Otherwise, from (3.1) we get $\widehat{p}_{2n}'(x_{2n,k})=0$, which is a contradiction because the zeros of the polynomials are simple. In the odd case, apparently it can be thought that we divide by zero because zero is a root of $\widehat{p}_{2n+1}(x)$ but the equation (5.1) can be modified so that such a pathology does not appear.

Applying the following property (see [7] and [12])

$$\frac{\widehat{p}_n''(x_{n,k})}{\widehat{p}_n'(x_{n,k})}=-2\sum_{j=1,j\neq k}^{n}\frac{1}{x_{n,j}-x_{n,k}},$$

the system (5.1) becomes

$$\sum_{j=1, j\neq k}^{n} \frac{1}{x_{n,j}-x_{n,k}} + \frac{A'(x_{n,k},n)}{2A(x_{n,k},n)} - \frac{1}{x_{n,k}} + 2x_{n,k}^3 = 0, \qquad 1 \le k \le n. \tag{5.2}$$

The total external potential $V(x)$ is the sum of two kinds of potentials, one independent of the number n of charges $v(x) = x^4/2$, and the other one depending on n (varying external potential) $\frac{1}{2}\ln|\frac{A(x,n)}{x^2}|$. They correspond to the long and short range interactions, respectively (see [7] for more details). Thus

$$\begin{aligned} V(x) &= \frac{x^4}{2} + \frac{1}{2}\ln|\frac{A(x,n)}{x}| \\ &= \frac{x^4}{2} + \frac{1}{2}\ln|x-r_1(n)| + \frac{1}{2}\ln|x-r_2(n)| \\ &+ \frac{1}{2}\ln|x-r_3(n)| + \frac{1}{2}\ln|x-r_4(n)| - \ln|x|, \qquad x \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{5.3}$$

We consider the potential energy at x of a point charge q located at t is $-q\ln|x-t|$.

Let introduce the following electrostatic model:

> *Consider the system of n movable positive unit charges at n distinct points $\{x_{n,i}\}_{i=1}^n$ of the real line in the presence of the total external potential $V(x)$.*

Notice that from (5.3) we can deduce that the roots of $A(x,n)$ give us the position (depending on n) of four fixed charges. Then the external field is generated by a fixed charge $+1$ at the origin –due to a perturbation of the weight function– plus four fixed charges of magnitude $-1/2$; two of them located at the real positions $r_1(n)$, $r_2(n)$, and the remaining ones at the complex positions $r_3(n)$ and $r_4(n)$ (see Corollary 3.5). When n tends to infinity $r_1(n)$ and $r_2(n)$ tend to the origin, consequently, they will be cancelled with the charge $+1$ at the origin, recovering the classical electrostatic interpretation in the limit. In the odd case, we know $A(x,2n-1)$ is reduced because of Theorem 3.3. From Corollary 3.5 we get the external field is generated by two fixed charges at complex points, $s_3(n)$, $s_4(n)$. It is worthy of pointing out the charge -1 from the varying external potential at the origin and the charge $+1$ because of the mass point at the origin are cancelled each other. For that reason the zero of $\widehat{p}_n(x)$ will be located at the origin. Again this case coincides with the classical electrostatic interpretation.

We denote $\mathbf{x} = (x_{n,1}, x_{n,2}, ..., x_{n,n})$. The total energy of the system is

$$E(\mathbf{x}) = \sum_{k=1}^{n} V(x_{n,k}) - \sum_{1\le j<k\le n} \ln|x_{n,j}-x_{n,k}|. \tag{5.4}$$

Notice that (5.2) is the derivative of the above energy function. This means the zeros of the Freud-type orthogonal polynomials are critical points of the energy function. These critical points could represent local or global equilibria. From the physical point of view, these kind of possibilities could correspond to the stable or unstable equilibrium situations. We understand the equilibrium as the zero gradient of the total energy of the system. By stable equilibrium we mean the existence of a global minimum of the total energy. Nevertheless, the study of the stability for the equilibrium configuration (global minimum), if any, requires a deeper discussion, which we will omit here. Despite this fact, we partially solve this question focusing in the study of the local minima of the energy function (5.4). For this purpose, we consider the Hessian matrix

$$H = (h_{i,j}), \quad h_{i,j} = \frac{\partial^2 E}{\partial x_i \partial x_j},$$

and deduce when $E(\mathbf{x})$ has local minima at the zeros of the polynomials $\widehat{p}_n(x)$. Indeed, taking into account

$$h_{k,l}=\begin{cases} \dfrac{1}{2}\dfrac{\partial}{\partial x_{n,k}}\left(\dfrac{A'(x_{n,k},n)}{A(x_{n,k},n)}\right)+\dfrac{1}{x_{n,k}^2}+6x_{n,k}^2+\displaystyle\sum_{j=1,j\neq k}^{n}\dfrac{1}{(x_{n,j}-x_{n,k})^2}, & \text{if } k=l, \\ -\dfrac{1}{(x_{n,l}-x_{n,k})^2}, & \text{if } k\neq l, \end{cases}$$

the Hessian matrix is real and symmetric. If the Hessian matrix is strictly diagonally dominant and its diagonal terms are positive, then, by corollary (7.2.3) [5], H is be positive definite. In this case, we will have conditions that guarantee that the equilibrium position of the proposed system will be reached at the zeros of $\widehat{p}_n(x)$. Thus, we need to guarantee the following function to be positive

$$\frac{1}{2}\frac{\partial}{\partial x_{n,k}}\left(\frac{A'(x_{n,k},n)}{A(x_{n,k},n)}\right)+\frac{1}{x_{n,k}^2}+6x_{n,k}^2, \quad 1\leq k\leq n.$$

So we study the function

$$\begin{aligned} f(x,n) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{A'(x,n)}{A(x,n)}\right)+\frac{1}{x^2}+6x^2 \\ &= \frac{96x^{12}+192a(n)x^{10}+\left[96a(n)^2-48c(n)-16\right]x^8}{\left[4x^4+4a(n)x^2-c(n)\right]^2 x^2} \\ &+\frac{\left[16a(n)-48a(n)c(n)\right]x^6+\left[-32c(n)+6c(n)^2\right]x^4}{\left[4x^4+4a(n)x^2-c(n)\right]^2 x^2} \\ &-\frac{12a(n)c(n)x^2-c(n)^2}{\left[4x^4+4a(n)x^2-c(n)\right]^2 x^2}. \end{aligned}$$

For n sufficiently large and using the expressions (2.1), (2.2) and Theorem 3.4 we conclude

$$f(x,n)\sim\frac{6x^6+12a(n)x^4+[6a(n)^2-1]x^2+a(n)}{[x^2+a(n)]^2}.$$

Thus, for every n we can find a λ such that the above Hessian matrix is positive definite. Therefore the electrostatic equilibrium position in the presence of the external field is obtained at the zeros $\{x_{n,i}\}_{i=1}^{n}$ of the Freud-type orthogonal polynomial $\widehat{p}_n(x)$, provided that the particle interaction obeys a logarithmic potential. Otherwise we cannot assert that the total energy achieves its local minimum (hence, yielding a stable electrostatic equilibrium) at the zeros of the orthogonal polynomials. In this sense a further study is needed. For such a purpose an estimate of n is demanded.

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REFERENCES

[1] J. ARVESÚ, J. ATIA, AND F. MARCELLÁN, *On Semiclassical Linear Functionals: The Symmetric Companion*. Communications in the Analytic Theory of Continued Fractions, 10 (2002), pp. 13-29.
[2] T. S. CHIHARA, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publishers, New York, 1978.
[3] F. A. GRÜNBAUM, *Variations on a theme of Heine and Stieltjes: an Electrostatic Interpretation of the Zeros of Certain Polynomials*, J. Comp. Appl. Math., 99 (1998), pp. 189-194.
[4] F. A. GRÜNBAUM, *Electrostatics Interpretation for the Zeros of Certain Polynomials and the Darboux Process*, J. Comp. Appl. Math., 133 (2001), pp. 397-412.
[5] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1992.
[6] M. E. H. ISMAIL, *An Electrostatics Model for Zeros of General Orthogonal Polynomials*, Pacific J. Math., 193 (2000), pp. 355-369.
[7] M. E. H. ISMAIL, *More on Electrostatic Model for Zeros of General Orthogonal Polynomials*, J. Nonlinear Funct. Anal. Opt., 21 (2000), pp. 191-204.
[8] P. NEVAI, *Orthogonal Polynomials Associated with* e^{-x^4}, Conference Proceedings. Can. Math. Soc, vol., 3 (1983), pp. 263-285.
[9] T. J. STIELTJES, *Collected Papers*, G. van Dikj Editor, Springer Verlag, Berlin, 1993.
[10] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc, Providence, RI, 4th edition, 1975.
[11] G. VALENT AND W. VAN ASSCHE, *The impact of Stieltjes Work on Continued Fractions and Orthogonal Polynomials. Additional Material*, J. Comp. Appl. Math. 65 (1995), pp. 419-447.
[12] W. VAN ASSCHE, *The impact of Stieltjes Work on Continued Fractions and Orthogonal Polynomials*, In Collected Papers, G. Van Dijk Editor, Springer Verlag, Berlin, 1993, pp. 5-37.