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SZEGŐ QUADRATURE AND FREQUENCY ANALYSIS*

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Abstract. A series of papers have treated the frequency analysis problem by studying the zeros of orthogonal polynomials on the unit circle with respect to measures determined by observations of the signal. In the recent paper [3], a different approach was used, where properties of Szegő quadrature formulas associated with the zeros of para-orthogonal polynomials with respect to the same measures were used to determine the frequencies and amplitudes in the signal. In this paper we carry this approach further, and obtain more conclusive results.

Key words. Szegő polynomials, Szegő quadrature formula, frequency analysis problem.

AMS subject classifications. 41A55, 33C45.

1. Introduction.

We are concerned with signals of the form:

$$x(m) = \sum_{j=1}^{I} \left(\alpha_j e^{i\omega_j m} + \alpha_{-j} e^{i\omega_{-j} m} \right),$$

where I is a positive integer, the constants ω_j are the frequencies which satisfy $\omega_{-j} = -\omega_j$, $(\omega_j \in (0, \pi))$, the complex numbers α_j are the amplitudes, where $\alpha_{-j} = \overline{\alpha}_j$, and m is discrete time.

The classical frequency analysis problem consists in determining the frequencies from the $N-$truncated signal $x_N(m)$ defined by:

$$x_N(m) = \begin{cases} \sum_{j=1}^{I} \left(\alpha_j e^{i\omega_j m} + \alpha_{-j} e^{i\omega_{-j} m} \right), & 0 \le m \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Once the frequencies are known, the amplitudes can be computed by solving a system of linear equations.

There is a method developed by Jones, Njåstad and Saff ([6]) for solving this problem based on orthogonal polynomials on the unit circle. The zeros of such polynomials will approach the frequency points $\beta_j = e^{i\omega_j},\ j = \pm 1, \ldots, \pm I$.

For simplicity, let us number the frequency points from 1 to n_0, i.e., $\beta_j,\ j = 1, \ldots, n_0$. (Then $n_0 = 2I$).

The autocorrelation coefficients are the starting point of this method:

$$\mu_k^{(N)} = \sum_{m=0}^{N-1} x_N(m) x_N(m+k).$$

It can verified ([6]) that these quantities are the moments with respect to the distribution function

$$d\psi_N(\theta) = \left| X_N\left(e^{i\theta}\right)\right|^2 d\theta,\ \theta \in [-\pi, \pi],$$

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where $X_N(z) := \sum_{m=0}^{N-1} x_N(m)z^{-m}$, i.e., one can write

$$\mu_k^{(N)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} d\psi_N(\theta).$$

Let $\{\phi_n^{(N)}(z)\}_n$ be the sequence of monic orthogonal polynomials with respect to $\frac{d\psi_N(\theta)}{N}$. Then, for $n \geq n_0$ we have the following result ([9], [8]):

THEOREM 1.1. *Let $\{N_k\}_k$ be an arbitrary subsequence of the sequence of natural numbers. Then, there exists a subsequence $\{N_{k(\nu)}\}$ such that*

$$\lim_{\nu\to\infty} \phi_n^{(N_{k(\nu)})}(z) = Q_{n-n_0}(z)\prod_{j=1}^{n_0}(z-\beta_j),$$

uniformly on compact subsets of $\mathbb{C}$, where the polynomial Q_{n-n_0} of degree $n-n_0$ depends upon the subsequence $\{N_{k(\nu)}\}$.

It follows, by Hurwitz's Theorem, that n_0 zeros of $\phi_n^{(N_{k(\nu)})}$ will approach the frequency points $\{\beta_j\}_{j=1}^{n_0}$ and the $(n-n_0)$ remaining zeros will approach the zeros $\{z_j\}_{j=1}^{n-n_0}$ of Q_{n-n_0}, which are sometimes called the "uninteresting zeros". By Theorem 1.1, it can be extracted that, for $n \geq n_0$, there exists a number $K_n \in (0,1)$, independent of the subsequence, such that $|z_j| < K_n$, for all $j = 1,\dots,n-n_0$. Therefore, we can conclude that such zeros stay away from the unit circle, which is very useful in the process of determining the unknown frequencies.

However, nothing more is known about the value of the number K_n and this is one reason why we chose a different approach to treat the frequency analysis problem, based now on the so-called para-orthogonal polynomials instead of orthogonal polynomials (see also the recent paper [1]).

2. Preliminary results. A para-orthogonal polynomial with respect to the measure $\frac{d\psi_N(\theta)}{N}$ is a polynomial of the form:

$$B_n^{(N)}(z,\tau) = \phi_n^{(N)}(z) + \tau\phi_n^{(N)^*}(z),\ \tau \in \mathbb{T},$$

where $\phi_n^{(N)^*}(z) = z^n\overline{\phi_n^{(N)}(1/\overline{z})}$ is the reciprocal polynomial. It is known, ([5]) that its zeros are simple and lie on $\mathbb{T}$. The following convergence result, similar to Theorem 1.1, holds (see [7]):

THEOREM 2.1. *Let $\{N_k\}_k$ be an arbitrary subsequence of the sequence of natural numbers. Then, there exists a subsequence $\{N_{k(\nu)}\}$ such that, for all $n > n_0$,*

$$\lim_{\nu\to\infty} B_n^{(N_{k(\nu)})}(z,\tau) = W_{n-n_0}(z,\tau)\prod_{j=1}^{n_0}(z-\beta_j) = B_n(z,\tau),\ \text{(Limit polynomial)},$$

uniformly on compact subsets of $\mathbb{C}$, where the polynomial W_{n-n_0} of degree $n-n_0$ depends upon the subsequence $\{N_{k(\nu)}\}$, and can be expressed, in terms of the polynomial Q_{n-n_0} given in Theorem 1.1, as

$$W_{n-n_0}(z,\tau) = Q_{n-n_0}(z) + \tau Q^*_{n-n_0}(z),\ (|\tau| = 1).$$

Thus, it follows that some of the zeros $\{z_j^{N_{k(\nu)}}\}_{j=1}^{n}$ of $B_n^{(N_{k(\nu)})}$ will converge to the frequency points $\{\beta_j\}_{j=1}^{n_0}$ and the rest converge to the zeros $\{\beta_j\}_{j=n_0+1}^{n}$ of W_{n-n_0}. The question now is how to distinguish the frequency points from the zeros of W_{n-n_0}. Note that a frequency point may also be a zero of W_{n-n_0}.

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For this purpose, the corresponding weights $\{\lambda_j^{(N)}\}_{j=1}^n$ in the Szegő quadrature formula ([5]) with respect to $\frac{d\psi_N(\theta)}{N}$, were used in [3]. These quadrature weights will also provide useful estimates of the modulus of the squared amplitudes $\{|\alpha_j|^2\}_{j=1}^{n_0}$ in the signal as can be seen in the following result (see [3] for details).

THEOREM 2.2. *Let the situation be as in Theorem 2.1, and assume that the limit polynomial B_n has at most double zeros. Then, the following statements are true:*

(i) Let β_j be a frequency point which is a simple zero of the limit polynomial B_n (i. e., $W_{n-n_0}(\beta_j,\tau) \neq 0$), and let $\lim_{\nu\to\infty} z_j^{N_{k(\nu)}} = \beta_j$. Then, for the corresponding weights one has

$$\lim_{\nu\to\infty} \lambda_j^{N_{k(\nu)}} = |\alpha_j|^2 .$$

(ii) Let β_j be a frequency point which is a double zero of the limit polynomial B_n (i. e., $W_{n-n_0}(\beta_j,\tau) = 0,\ W'_{n-n_0}(\beta_j,\tau) \neq 0$), then, by Theorem 2.1, let $\{z_{j_1}^{N_{k(\nu)}}\}_\nu$, and $\{z_{j_2}^{N_{k(\nu)}}\}_\nu$ be two distinct sequences such that $\lim_{\nu\to\infty} z_{j_m}^{N_{k(\nu)}} = \beta_j,\ m = 1, 2$. Then, for the corresponding sequences of weights one has

$$\lim_{\nu\to\infty} \left(\lambda_{j_1}^{N_{k(\nu)}} + \lambda_{j_2}^{N_{k(\nu)}}\right) = |\alpha_j|^2 . \tag{2.1}$$

(iii) Let β_j be a zero of the limit polynomial B_n which is not a frequency point, and let $\lim_{\nu\to\infty} z_j^{N_{k(\nu)}} = \beta_j$. Then, for the weights one has

$$\lim_{\nu\to\infty} \lambda_j^{N_{k(\nu)}} = 0.$$

Thus, for zeros tending to zeros of W_{n-n_0} which are not frequency points, the corresponding weights will tend to zero. Observe that this result does not depend on the multiplicities of the zeros. However, in the next Section, we will prove that W_{n-n_0} has simple zeros and consequently, a frequency point will be a zero of the limit polynomial B_n of multiplicity 2 at most.

3. Main result.

THEOREM 3.1. *Let the situation be as in Theorem 2.1. Then, the polynomial $W_{n-n_0}^{(N_{k(\nu)})}$ is the $(n-n_0)th$ para-orthogonal polynomial with respect to some positive measure μ on $\mathbb{T}$.*

Proof. Let $\{\phi_n^{(N)}\}$ be the sequence of monic orthogonal polynomials with respect to $\frac{d\psi_N(\theta)}{N}$. By Theorem 1.1

$$\lim_{\nu\to\infty} \phi_n^{(N_{k(\nu)})}(z) = Q_{n-n_0}(z) \prod_{j=1}^{n_0} (z-\beta_j).$$

On the other hand, by Theorem 2.1, we know that the polynomials W_{n-n_0} and Q_{n-n_0} are related by

$$W_{n-n_0}(z,\tau) = Q_{n-n_0}(z) + \tau Q^*_{n-n_0}(z),\ (|\tau| = 1).$$

Hence, it is sufficient to show that the polynomial Q_{n-n_0} is orthogonal with respect to some positive measure on $\mathbb{T}$.

For simplicity, let

$$\rho_{n_0}(z) = \prod_{j=1}^{n_0} (z-\beta_j) \tag{3.1}$$

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and $\rho_n(z) = Q_{n-n_0}(z)\rho_{n_0}(z)$, for $n > n_0$.

Let $\delta_n^{(N_{k(\nu)})} = \phi_n^{(N_{k(\nu)})}(0)$. Then, taking into account that the frequency points $\{\beta_j\}_{j=1}^{n_0}$ appear in complex conjugate pairs, one has

$$\lim_{\nu\to\infty} \delta_n^{(N_{k(\nu)})} = Q_{n-n_0}(0)\prod_{j=1}^{n_0}\beta_j = Q_{n-n_0}(0) := \delta_{n-n_0}.$$

Since the zeros $\{z_j\}_{j=n_0+1}^{n}$ of Q_{n-n_0} satisfy $|z_j| < 1$, it follows that $|\delta_{n-n_0}| < 1$.

From the recursion formula for orthogonal polynomials (see [11]), we have

$$\phi_{n+1}^{(N_{k(\nu)})}(z) = z\phi_n^{(N_{k(\nu)})}(z) + \delta_{n+1}^{(N_{k(\nu)})}\phi_n^{(N_{k(\nu)})^*}(z).$$

By letting $\nu \to \infty$, we find

$$\rho_{n+1}(z) = z\rho_n(z) + \delta_{n-n_0+1}\rho_n^*(z). \tag{3.2}$$

(A similar argument is used in [10]). Note that $\rho_n^*(z) = \rho_{n_0}^*(z)Q_{n-n_0}^*(z) = \rho_{n_0}(z)Q_{n-n_0}^*(z)$.

By cancellation of ρ_{n_0} in (3.2) one has

$$Q_{n+1-n_0}(z) = zQ_{n-n_0}(z) + \delta_{n-n_0+1}Q_{n-n_0}^*(z).$$

With $p = n - n_0 \geq 0$, the last may be written as

$$Q_{p+1}(z) = zQ_p(z) + \delta_{p+1}Q_p^*(z),\ p = 0, 1, \ldots.$$

Here $|\delta_p| < 1$.

Therefore, by a Favard-type theorem (see [5], [4]), there exists a positive measure μ on $\mathbb{T}$ such that $\{Q_p\}_p$ are the corresponding (monic) orthogonal polynomials. ∎

Thus, all the zeros of the limit polynomial B_n that are not frequency points, are simple and the corresponding weights tend to zero. The frequency points are either simple or double zeros of the limit polynomial B_n. If a frequency point β_j is a simple zero, the weights will tend to the squared modulus $|\alpha_j|^2$ of the corresponding amplitude and in the double zero case, it is now the sum of the corresponding weights that tends to $|\alpha_j|^2$.

Therefore, in all the cases, the behaviour of the weights allows us to know which zeros tend to the frequency points and which zeros do not. Furthermore, the weights will also determine the modulus of the amplitudes.

4. Related questions.

From Theorem 3.1 we can ask ourselves the following two questions:

4.1. Orthogonality of the polynomials Q_{n-n_0}.

Note that in the proof of Theorem 3.1, we showed only the existence of a measure μ for which the polynomial W_{n-n_0} is para-orthogonal. Thus, since this polynomial depends upon the subsequence chosen, how can we determine such a measure in each case?

For simplicity, we now write the signal as

$$x(m) = \sum_{j=1}^{n_0}\alpha_j e^{i\omega_j m}, \tag{4.1}$$

so that $\alpha_{I+k} = \alpha_{-k}$ and $\omega_{I+k} = \omega_{-k}$ for $1 \leq k \leq I$. Then $X_N(z)$ can be written as

$$X_N(z) = \sum_{m=0}^{N-1} x_N(m)z^{-m} = \sum_{j=1}^{n_0}\alpha_j\sum_{m=0}^{N-1}e^{i\omega_j m}z^{-m} = z^{-(N-1)}\sum_{j=1}^{n_0}\alpha_j\frac{z^N - e^{iN\omega_j}}{z - e^{i\omega_j}}.$$

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If we put $\beta_j = e^{i\omega_j}$, we can write $X_N(z) = z\sum_{j=1}^{n_0} \frac{\alpha_j}{z-\beta_j} - z^{-(N-1)}\sum_{j=1}^{n_0} \frac{\alpha_j \beta_j^N}{z-\beta_j}$. If we introduce the polynomials L_{n_0-1} and $M_{n_0-1,N}$ of degree at most $n_0 - 1$ defined by the formulas

$$\sum_{j=1}^{n_0} \frac{\alpha_j}{z-\beta_j} = \frac{L_{n_0-1}(z)}{\rho_{n_0}(z)}, \qquad \sum_{j=1}^{n_0} \frac{\alpha_j \beta_j^N}{z-\beta_j} = \frac{M_{n_0-1,N}(z)}{\rho_{n_0}(z)},$$

then

$$X_N(z) = \frac{zL_{n_0-1}(z) - z^{-(N-1)}M_{n_0-1,N}(z)}{\rho_{n_0}(z)}. \tag{4.2}$$

Observe that we now have

$$\begin{aligned} |\rho_{n_0}(e^{i\theta})|^2|X_N(e^{i\theta})|^2 &= |L_{n_0-1}(e^{i\theta}) - e^{-iN\theta}M_{n_0-1,N}(e^{i\theta})|^2 \\ &= |L_{n_0-1}(e^{i\theta})|^2 + |M_{n_0-1,N}(e^{i\theta})|^2 \\ &\quad -e^{iN\theta}L_{n_0-1}\overline{M_{n_0-1,N}} - e^{-iN\theta}\overline{L_{n_0-1}}M_{n_0-1,N}. \end{aligned} \tag{4.3}$$

The polynomials L_{n_0-1} are independent of the number of samples N, but the polynomials $M_{n_0-1,N}$ depend on this N. But clearly $|M_{n_0-1,N}|$ remains bounded on compact sets of the complex plane, hence every subsequence $\{N_k\}$ has a subsequence $\{N_{k(\nu)}\}$ such that $M_{n_0-1,N_{k(\nu)}}(e^{i\theta})$ converges uniformly to $M_{n_0-1}(e^{i\theta})$ for all $\theta \in [0, 2\pi]$, where M_{n_0-1} is now a polynomial independent of N, but it may depend on the subsequence $\{N_{k(\nu)}\}$.

LEMMA 4.1. *Suppose $\{N_{k(\nu)}\}$ is a subsequence of integers, such that*

$$\lim_{\nu\to\infty} |M_{n_0-1,N_{k(\nu)}}(e^{i\theta}) - M_{n_0-1}(e^{i\theta})| = 0, \text{ uniformly on } [0, 2\pi]. \tag{4.4}$$

Then the measures $|\rho_{n_0}(e^{i\theta})|^2|X_{N_{k(\nu)}}(e^{i\theta})|^2\,d\theta$ converge weakly to the measure $(|L_{n_0-1}(e^{i\theta})|^2 + |M_{n_0-1}(e^{i\theta})|^2)\,d\theta$.

Proof. If we use (4.4) then we already see that we have

$$\lim_{\nu\to\infty} \left(|L_{n_0-1}(e^{i\theta})|^2 + |M_{n_0-1,N_{k(\nu)}}(e^{i\theta})|^2\right) = |L_{n_0-1}(e^{i\theta})|^2 + |M_{n_0-1}(e^{i\theta})|^2.$$

Furthermore, by the Riemann-Lebesgue lemma we know that $\displaystyle\lim_{N\to\infty}\int_0^{2\pi} f(\theta)e^{\pm iN\theta}\,d\theta = 0$, for every $f \in L^1$. This means, also using (4.4), that for every continuous f on the unit circle, we have

$$\lim_{\nu\to\infty}\int_0^{2\pi} f(\theta)e^{iN_{k(\nu)}\theta}L_{n_0-1}(e^{i\theta})\overline{M_{n_0-1,N_{k(\nu)}}(e^{i\theta})}\,d\theta = 0,$$

and similarly

$$\lim_{\nu\to\infty}\int_0^{2\pi} f(\theta)e^{-iN_{k(\nu)}\theta}\overline{L_{n_0-1}(e^{i\theta})}M_{n_0-1,N_{k(\nu)}}(e^{i\theta})\,d\theta = 0.$$

This gives the required weak convergence. ☐

We are now in position to prove the following

THEOREM 4.2. *Let $\{N_k\}$ be a subsequence such that*

$$\lim_{k\to\infty} \phi_n^{(N_k)}(z) = Q_{n-n_0}(z)\rho_{n_0}(z). \tag{4.5}$$

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Then, the polynomial Q_{n-n_0} *is orthogonal with respect to the measure* $(|L_{n_0-1}(e^{i\theta})|^2 + |M_{n_0-1}(e^{i\theta})|^2)\, d\theta$.

Proof. For every monic polynomial q_{n-n_0} of degree $n - n_0$ we have

$$\int_0^{2\pi} |\phi_n^{(N)}(e^{i\theta})|^2 |X_N(e^{i\theta})|^2\, d\theta \le \int_0^{2\pi} |q_{n-n_0}(e^{i\theta})|^2 |\rho_{n_0}(e^{i\theta})|^2 |X_N(e^{i\theta})|^2\, d\theta, \tag{4.6}$$

where ρ_{n_0} is given as in (3.1). This is a consequence of the extremal property of monic orthogonal polynomials.

We will use the identity: $\int |f(\theta)|\, d\mu(\theta) = 2\int f^+(\theta)\, d\mu(\theta) - \int f(\theta)\, d\mu(\theta)$, where f is a real and μ-measurable function, μ a positive measure on the real line (in our case, we will use measures on $[0, 2\pi]$), and $f^+(t) = \max\{0, f(t)\}$. If we choose $f(\theta) = |\rho_{n_0}(e^{i\theta}) Q_{n-n_0}(e^{i\theta})|^2 - |\phi_n^{(N_k)}(e^{i\theta})|^2$ and $d\mu(\theta) = |X_{N_k}(e^{i\theta})|^2\, d\theta$, then this gives

$$\begin{aligned}\int_0^{2\pi} \left| |\phi_n^{(N_k)}|^2 - |\rho_{n_0}|^2 |Q_{n-n_0}|^2 \right| |X_{N_k}|^2\, d\theta &= 2\int_0^{2\pi} \left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right)^+ |X_{N_k}|^2\, d\theta \\ &\quad - \int_0^{2\pi} \left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right) |X_{N_k}|^2\, d\theta \\ &\le 2\int_0^{2\pi} \left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right)^+ |X_{N_k}|^2\, d\theta,\end{aligned}$$

where the inequality follows from (4.6), with $q_{n-n_0} \equiv Q_{n-n_0}$.

For every $\theta \in [0, 2\pi]$ we have

$$0 \le \left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right)^+ \le |\rho_{n_0}|^2 |Q_{n-n_0}|^2,$$

and furthermore, it follows from (4.2) that

$$|\rho_{n_0}|^2 |X_{N_k}|^2 = |L_{n_0-1} - e^{-iN_k\theta} M_{n_0-1,N_k}|^2 \le 2(|L_{n_0-1}|^2 + |M_{n_0-1,N_k}|^2). \tag{4.7}$$

Recall that $|M_{n_0-1,N}|$ is bounded on the unit circle, independent of N.

On the other hand, by (4.5) and (4.7), one can conclude that

$$\lim_{k\to\infty} \left(|\phi_n^{(N_k)}|^2 - |\rho_{n_0}|^2 |Q_{n-n_0}|^2| \right)^+ |X_{N_k}|^2 = 0, \text{ except at the frequency points.}$$

We are therefore allowed to use Lebesgue's dominated convergence theorem to the sequence $\left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right)^+ |X_{N_k}|^2$ to conclude that

$$\lim_{k\to\infty} \int_0^{2\pi} \left(|\rho_{n_0}|^2 |Q_{n-n_0}|^2 - |\phi_n^{(N_k)}|^2 \right)^+ |X_{N_k}|^2\, d\theta = 0,$$

and therefore also

$$\lim_{k\to\infty} \int_0^{2\pi} \left| |\phi_n^{(N_k)}|^2 - |\rho_{n_0}|^2 |Q_{n-n_0}|^2 \right| |X_{N_k}|^2\, d\theta = 0. \tag{4.8}$$

Now we return to (4.6) and use the result from (4.8) to conclude that

$$\lim_{k\to\infty} \int_0^{2\pi} |\rho_{n_0}|^2 |Q_{n-n_0}|^2 |X_{N_k}|^2\, d\theta \le \lim_{k\to\infty} \int_0^{2\pi} |\rho_{n_0}|^2 |q_{n-n_0}|^2 |X_{N_k}|^2\, d\theta.$$

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In particular, the sequence $\{N_k\}$ has a subsequence $\{N_{k(\nu)}\}$ for which (4.4) holds. Now use Lemma 4.1, then we find

$$\int_0^{2\pi} |Q_{n-n_0}|^2(|L_{n_0-1}|^2+|M_{n_0-1}|^2)\,d\theta \le \int_0^{2\pi} |q_{n-n_0}|^2(|L_{n_0-1}|^2+|M_{n_0-1}|^2)\,d\theta,$$

and this holds for every monic polynomial q_{n-n_0} of degree $n-n_0$. But there is only one monic polynomial Q_{n-n_0} that satisfies this inequality for all q_{n-n_0} and that is the monic orthogonal polynomial for the measure with density $|L_{n_0-1}|^2+|M_{n_0-1}|^2$. This is what we wanted to prove. □

REMARK 4.3. *Every subsequence for which (4.4) holds, will then produce a measure with respect to which Q_{n-n_0} is orthogonal. But, by the Favard-type theorem in Section 3, the measure in each case must be the same.*

Let us consider now some examples:

Example 1: Consider the signal $x_N(m)=e^{im\pi/4}+e^{-im\pi/4}=2\cos(\pi m/4)$, $0\le m\le N-1$. In this case

$$\frac{1}{z-e^{i\pi/4}}+\frac{1}{z-e^{-i\pi/4}}=\frac{2z-\sqrt{2}}{z^2-\sqrt{2}z+1},$$

so that $L_1(z)=2z-\sqrt{2}$. We also have

$$\frac{e^{i\pi N/4}}{z-e^{i\pi/4}}+\frac{e^{-i\pi N/4}}{z-e^{-i\pi/4}}=\frac{2z\cos(\pi N/4)-2\cos(\pi(N-1)/4)}{z^2-\sqrt{2}z+1},$$

so that $M_{1,N}(z)=2z\cos(\pi N/4)-2\cos(\pi(N-1)/4)$.

For $N_k=4k$ we see that $M_{1,4k}(z)=(-1)^k(2z-\sqrt{2})$. This sequence of functions is not convergent but one can extract two subsequences, say $\{M_{1,8\nu}(z)=2z-\sqrt{2}\}$ and $\{M_{1,8\nu+4}(z)=-(2z-\sqrt{2})\}$, that converge to $M_1^{(1)}(z)=2z-\sqrt{2}$ and $M_1^{(2)}(z)=-(2z-\sqrt{2})$, respectively. But those limiting functions must produce the same measure, (observe that $\left|M_1^{(1)}(z)\right|^2=\left|M_1^{(2)}(z)\right|^2$).

In this case we have $|M_{1,4k}(z)|^2=|L_1(z)|^2$. Hence the polynomials $Q^{(1)}_{n-n_0}$, which appear as limits of the subsequence $N_k=4k$, are orthogonal polynomials for the measure with density $|L_1(e^{i\theta})|^2=6-4\sqrt{2}\cos\theta$. This can be checked for $Q_2^{(1)}(z)=z^2+\frac{3\sqrt{2}}{7}z+\frac{2}{7}$.

If we take $N_k=4k+2$, then $M_{1,4k+2}(z)=(-1)^{k+1}\sqrt{2}$. Now, we would have the subsequences $\{M_{1,8\nu+2}(z)=-\sqrt{2}\}$ and $\{M_{1,8\nu+6}(z)=\sqrt{2}\}$, which converge to $M_1^{(1)}(z)=-\sqrt{2}$ and $M_1^{(2)}(z)=\sqrt{2}$, respectively.

In this case, we can write $|L_1(e^{i\theta})|^2+|M_{1,4k+2}(e^{i\theta})|^2=8-4\sqrt{2}\cos\theta$, and the polynomials $Q^{(2)}_{n-n_0}$ which appear as limits for this subsequence are orthogonal for this density. This can be checked for $Q_2^{(2)}(z)=z^2+\frac{2\sqrt{2}}{7}z+\frac{1}{7}$.

When $N_k=4k+1$ the limits are the same as those for the subsequence $N_k=4k$. When $N_k=4k+3$ the limits are the same as in the case for $N_k=4k+2$.

Example 2: Consider now the signal $x_N(m)=e^{im\pi/3}+e^{-im\pi/3}=2\cos(\pi m/3)$, $0\le m\le N-1$. In this case

$$\frac{1}{z-e^{i\pi/3}}+\frac{1}{z-e^{-i\pi/3}}=\frac{2z-1}{z^2-z+1},$$

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so that $L_1(z) = 2z - 1$ and $|L_1(e^{i\theta})|^2 = 5 - 4\cos\theta$. We also have

$$\frac{e^{i\pi N/3}}{z - e^{i\pi/3}} + \frac{e^{-i\pi N/3}}{z - e^{-i\pi/3}} = \frac{2z\cos(\pi N/3) - 2\cos(\pi(N-1)/3)}{z^2 - z + 1},$$

so that $M_{1,N}(z) = 2z\cos(\pi N/3) - 2\cos(\pi(N - 1/3)$.

When $N_k = 3k+1$ then $M_{1,3k+1}(z) = (-1)^k(z-2)$, so that $|M_{1,3k+1}(e^{i\theta})|^2 = 5 - 4\cos\theta = |L_1(e^{i\theta})|^2$. The polynomials $Q^{(1)}_{n-n_0}$ for this subsequence are orthogonal polynomials for the measure with density $5 - 4\cos\theta$. This can indeed be verified for $Q^{(1)}_1(z) = z + \frac{2}{5}$, or for $Q^{(1)}_2(z) = z^2 + \frac{10}{21}z + \frac{4}{21}$. The subsequence $N = 3k$ gives the same polynomials.

When $N_k = 3k+2$ then $M_{1,3k+2}(z) = (-1)^{k+1}(z+1)$. This means that $|M_{1,3k+2}(e^{i\theta})|^2 = 2 + 2\cos\theta$. The polynomials $Q^{(2)}_{n-n_0}$ for this subsequence are thus orthogonal polynomials for the measure with density $|L_1|^2 + |M_1|^2 = 7 - 2\cos\theta$. This can indeed be checked for $Q^{(2)}_1(z) = z + \frac{1}{7}$ and for $Q^{(2)}_2(z) = z^2 + \frac{7}{48}z + \frac{1}{48}$.

4.2. Weights at a double zero. Suppose that a frequency point β_j, for some $j = 1,\dots,n_0$, is a zero of the limit polynomial B_n of multiplicity 2. As we stated before, there exist two sequences $\{z^{N_{k(\nu)}}_{j_1}\}_\nu$, and $\{z^{N_{k(\nu)}}_{j_2}\}_\nu$ of zeros of $B^{(N_{k(\nu)})}_n$ such that $\lim_{\nu\to\infty} z^{N_{k(\nu)}}_{j_m} = \beta_j,\ m = 1,2$. And the corresponding sequences of weights $\{\lambda^{N_{k(\nu)}}_{j_1}\}_\nu$, and $\{\lambda^{N_{k(\nu)}}_{j_2}\}_\nu$ satisfy (2.1).

The question is: What happens to $\lambda^{N_{k(\nu)}}_{j_1}$ and $\lambda^{N_{k(\nu)}}_{j_2}$ as $\nu\to\infty$?

Example 1: Consider the signal $x_N(m) = e^{im\pi/2} + e^{-im\pi/2},\ 0 \le m \le N-1$. Then, the amplitudes are $\alpha_1 = \alpha_2 = 1$, and the frequency points are $\beta_1 = e^{i\pi/2} = i$, $\beta_2 = e^{-i\pi/2} = -i$, $(\rho_2(z) = z^2+1)$.

Here we find that $\lim_{k\to\infty} B^{2k+1}_5(z,\tau) = (z^3+\tau)(z^2+1)$.

Therefore, in this case $W_3(z,\tau) = z^3+\tau$, and when $\tau = -i$ the frequency point $\beta_2 = -i$ is a double zero.

For $k = 2500, 25000, 250000$ we have computed the corresponding zeros of $B^{2k+1}_5(z,\tau)$ which are displayed in the following table:

$k = 2500$	$k = 25000$	$k = 250000$
$\underline{-0.0115455 - 0.9999333i}$	$\underline{-0.00365143 - 0.99999333i}$	$\underline{-0.00115469 - 0.999999333i}$
$\underline{0.0115455 - 0.9999333i}$	$\underline{0.00365143 - 0.99999333i}$	$\underline{0.00115469 - 0.999999333i}$
$-0.86600616 + 0.50003332i$	$-0.86602347 + 0.50000333i$	$-0.86602521 + 0.50000033i$
$0.86600616 + 0.50003332i$	$0.86602347 + 0.50000333i$	$0.86602521 + 0.50000033i$
i	i	i

From these results we can say that the two first numbers in each column would be the two sequences of points that approach $\beta_2 = -i$.

For the corresponding weights in the Szegő quadrature formula we obtained the following results:

$k = 2500$	$k = 25000$	$k = 250000$
$\underline{0.50007221}$	$\underline{0.50000722}$	$\underline{0.50000072}$
$\underline{0.50007221}$	$\underline{0.50000722}$	$\underline{0.50000072}$
0.00017774	0.00001777	0.00000177
0.00017774	0.00001777	0.00000177
0.9999	0.99999	0.999999

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The entries of this table indicate that $\lim_{k\to\infty} \lambda_{j_1}^{2k+1} = \lim_{k\to\infty} \lambda_{j_2}^{2k+1} = 0.5$

Example 2: Let us consider now the signal $x(m) = e^{-i\pi m/4} + e^{i\pi m/4}$. In this case, the amplitudes are $\alpha_1 = \alpha_2 = 1$, and the frequency points are $\beta_1 = e^{i\pi/4}$, $\beta_2 = e^{-i\pi/4}$, $(\rho_2(z) = z^2 - \sqrt{2}z + 1)$.

For $N = 4k$ we have that

$$\lim_{k\to\infty} B_4^{4k}(z,\tau) = \left(z^2 + \frac{3\sqrt{2}}{7}z + \frac{2}{7} + \tau\left(1 + \frac{3\sqrt{2}}{7}z + \frac{2}{7}z^2\right)\right)(z^2 - \sqrt{2}z + 1).$$

Therefore, when $\tau = -\frac{4}{5} + \frac{3}{5}i$ the frequency point $\beta_2 = e^{-i\pi/4}$ is a double zero.

For $k = 100, 1000, 10000$, we calculated the corresponding zeros of $B_4^{4k}(z,\tau)$. They are displayed in the following table:

$k = 100$	$k = 1000$	$k = 10000$
$\underline{0.6593645 - 0.7518233i}$	$\underline{0.6922122 - 0.7216939i}$	$\underline{0.7024190 - 0.7117636i}$
$\underline{0.7526900 - 0.658374i}$	$\underline{0.7217858 - 0.6921164i}$	$\underline{0.7117729 - 0.7024095i}$
$-0.9570064 + 0.29006671i$	$-0.9567071 + 0.2910522i$	$-0.9566772 + 0.29115068i$
$0.704267 + 0.709934i$	$0.7068238 + 0.7073896i$	$0.7070784 + 0.7071350i$

Observe that the two first numbers in each column would be the two sequences of points that approach $\beta_2 = e^{-i\pi/4}$.

For the weights we obtained the following results:

$k = 100$	$k = 1000$	$k = 10000$
$\underline{0.5011506}$	$\underline{0.5000509}$	$\underline{0.4999848}$
$\underline{0.5017167}$	$\underline{0.5002369}$	$\underline{0.5000439}$
0.0019164	0.0001918	0.0000191
0.9952162	0.9995201	0.9999520

The above table indicates again that $\lim_{k\to\infty} \lambda_{j_1}^{4k} = \lim_{k\to\infty} \lambda_{j_2}^{4k} = 0.5$.

As we already saw, the last two examples indicate that both sequences of weights tend to the same number which coincides with half of the corresponding squared amplitudes, i. e.,

$$\lim_{\nu\to\infty} \lambda_{j_1}^{N_{k(\nu)}} = \lim_{\nu\to\infty} \lambda_{j_2}^{N_{k(\nu)}} = \frac{|\alpha_j|^2}{2} = 0.5. \tag{4.9}$$

On the other hand, observe that in both cases we have chosen real amplitudes and the frequencies as rational multiples of π. It could happen that for this choice, the subsequences $\{2k+1\}$ and $\{4k\}$, respectively, may work in order to obtain this particular result given by (4.9). For this reason we will consider the following example:

Example 3: Let us consider now the signal $x(m) = (1+i)e^{i\pi m/\sqrt{2}} + (1-i)e^{-i\pi m/\sqrt{2}}$. $(n_0 = 2)$, where the amplitudes $\alpha_1 = 1 + i$ and $\alpha_2 = 1 - i$ are complex numbers and the arguments of the frequency points, $\beta_1 = e^{i\pi/\sqrt{2}}$, and $\beta_2 = e^{-i\pi/\sqrt{2}}$, are not rational multiples of π.

We chose the subsequence $N = 2k$ and $n = 3$. In this case it was difficult to obtain an explicit expression of $B_3^{2k}(z,\tau)$, but what we know is that

$$\lim_{k\to\infty} B_3^{2k}(z,\tau) = (a + z + \tau(1 + az))(z^2 - 2\cos\frac{\pi}{\sqrt{2}}z + 1),$$

where $a = \lim_{k\to\infty} \phi_3^{2k}(0)$. If we take $k = 10^9$, we will have $a \approx -0.169002$.

Thus, if we choose $\tau \approx 0.779219 - 0.626751\, i$, then the frequency point $\beta_1 \approx -0.60569 + 0.79569\, i$ is a double zero.

The corresponding zeros and weights in this case are given in the following table:

Zeros	Weights
$\underline{-0.6057280819 + 0.7956717229i}$	$\underline{0.9999976166}$
$\underline{-0.6056716509 + 0.7957146795i}$	$\underline{1.0000023851}$
$-0.6056998665 - 0.7956932019i$	2.0000000001

Observe that now $|\alpha_1|^2 = 2$ and the above table indicates again the following:

$$\lim_{k\to\infty} \lambda_{j_1}^{2k} = \lim_{k\to\infty} \lambda_{j_2}^{2k} = \frac{|\alpha_1|^2}{2} = 1.$$

Thus, the above numerical results suggest (at least under suitable restrictions) that:
Conjecture: $\lim_{\nu\to\infty} \lambda_{j_1}^{N_{k(\nu)}} = \lim_{\nu\to\infty} \lambda_{j_2}^{N_{k(\nu)}} = \frac{|\alpha_j|^2}{2}$.

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