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LOCALIZED POLYNOMIAL BASES ON THE SPHERE*

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Abstract. The subject of many areas of investigation, such as meteorology or crystallography, is the reconstruction of a continuous signal on the 2-sphere from scattered data. A classical approximation method is *polynomial interpolation*. Let V_n denote the space of polynomials of degree at most n on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. As it is well known, the so-called *spherical harmonics* form an orthonormal basis of the space V_n. Since these functions exhibit a poor localization behavior, it is natural to ask for better localized bases. Given $\{\xi_i\}_{i=1,\ldots,(n+1)^2} \subset \mathbb{S}^2$, we consider the spherical polynomials

$$\varphi_i^n(\xi) := \sum_{l=0}^{n} \frac{2l+1}{4\pi} P_l(\xi_i \cdot \xi),$$

where P_l denotes the Legendre polynomial of degree l normalized according to the condition $P_l(1) = 1$. In this paper, we present systems of $(n+1)^2$ points on $\mathbb{S}^2$ that yield localized polynomial bases of the above form.

Key words. fundamental systems, localization, matrix condition, reproducing kernel.

AMS subject classifications. 41A05, 65D05, 15A12.

1. Introduction. Let $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$ denote the unit sphere embedded in the Euclidean space $\mathbb{R}^3$ and let $\Psi : [0,\pi] \times [0,2\pi) \longrightarrow \mathbb{R}^3$, $(\rho,\theta) \longmapsto (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ be its parameterization in spherical coordinates (ρ,θ). Corresponding to the surface element $dw(\xi)$, we have the $L^2(\mathbb{S}^2)$-inner product and norm

$$\begin{aligned}
\langle F, G\rangle &:= \int_{\mathbb{S}^2} F(\xi)\,\overline{G(\xi)}\,dw(\xi) \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\rho\; F(\Psi(\rho,\theta))\;\overline{G(\Psi(\rho,\theta))}\,d\rho\,d\theta, \\
\|F\|^2 &:= \langle F, F\rangle.
\end{aligned}$$

Furthermore, let $\mathrm{Harm}_l(\mathbb{R}^3)$ denote the space of harmonic homogeneous polynomials of degree l in three variables. Restricting these functions to $\mathbb{S}^2$, we obtain the so-called *spherical harmonics* of order l. Throughout this paper, we will focus on the space $V_n := \{P|_{\mathbb{S}^2} : P \in \Pi_n(\mathbb{R}^3)\}$ of spherical polynomials of degree at most n. It can be shown that

$$V_n = \bigoplus_{l=0}^{n} \mathrm{Harm}_l(\mathbb{S}^2), \tag{1.1}$$

where this direct sum decomposition has to be understood in the sense that any spherical polynomial of degree at most n is the restriction of a harmonic polynomial of degree less or equal to n to the sphere. Since $\dim \mathrm{Harm}_l(\mathbb{S}^2) = 2l+1$, it follows that $N := \dim V_n = \sum_{l=0}^{n}(2l+1) = (n+1)^2$. An $L^2(\mathbb{S}^2)$-orthonormal basis of V_n that is not localized on the sphere is given by

$$\left\{ Y_l^m(\Psi(\rho,\theta)) := \sqrt{\frac{2l+1}{4\pi}}\, P_l^{|m|}(\cos\rho)\, e^{im\theta},\; m = -l,\ldots,l,\; l = 0,\ldots,n \right\}, \tag{1.2}$$

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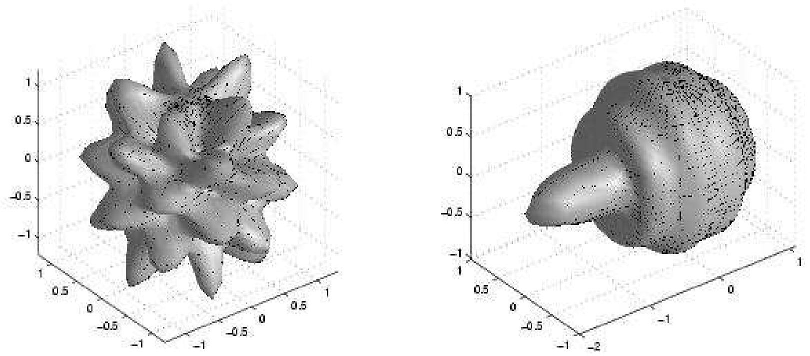

FIG. 1.1. *Spherical harmonic* $\mathcal{R}e\left(Y_{10}^{5}(\rho,\theta)\right)$ *and reproducing kernel* $K_{10}(\xi,\cdot)$ *with* $\xi = (-1,0,0)^T$.

where

$$P_l^m(t) := \left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}(1-t^2)^{m/2}\,\frac{d^m}{dt^m}P_l(t), \quad m = 0,\ldots,l,\ t\in[-1,1],$$

denote the associated Legendre functions and P_l stands for the Legendre polynomial of degree l normalized according to the condition $P_l(1) = 1$. From now on, this basis will be referred to as the *basis of spherical harmonics*.

A way of constructing better localized bases is by means of the *reproducing kernel* of the underlying space. Let $\{B_l^m \ : \ m = 1,\ldots,2l+1,\ l = 0,\ldots,n\}$ be an arbitrary $L^2(\mathbb{S}^2)$-orthonormal basis of V_n. It is straightforward to check that the reproducing kernel of $\mathrm{Harm}_l(\mathbb{S}^2)$ is given by

$$G_l(\xi,\eta) := \sum_{m=1}^{2l+1}\overline{B_l^m(\xi)}B_l^m(\eta), \qquad \xi,\eta\in\mathbb{S}^2.$$

Combining now the direct sum decomposition of V_n in (1.1) and the addition theorem for $\mathrm{Harm}_l(\mathbb{S}^2)$ (see Müller [4], page 10), one comes up with the following result.

LEMMA 1.1. *The unique reproducing kernel of* V_n *is given by*

$$K_n(\xi,\eta) := \sum_{l=0}^{n} G_l(\xi,\eta) = \sum_{l=0}^{n}\frac{2l+1}{4\pi}P_l(\xi\cdot\eta) =: k_n(\xi\cdot\eta), \qquad \xi,\eta\in\mathbb{S}^2. \tag{1.3}$$

It should be observed that $K_n(\xi,\eta) = k_n(\xi\cdot\eta)$, as a zonal function, only depends on the Euclidean product of the vectors ξ and η. Therefore, it is invariant with respect to rotations, i.e., transformations of the group $SO(3)$.

1.1. Scaling Functions. In contrast to the spherical harmonics Y_l^m introduced in (1.2), the function $K_n(\xi,\cdot) : \mathbb{S}^2 \longrightarrow \mathbb{R}$ $(\xi\in\mathbb{S}^2)$ defined in (1.3) is the spherical polynomial with minimal $L^2(\mathbb{S}^2)$-norm among all spherical polynomials of degree at most n that attain the same value at the prescribed point ξ. The following lemma establishes this localization property.

LEMMA 1.2. *Let* $\xi\in\mathbb{S}^2$. *Then*

$$\left\|\frac{K_n(\xi,\cdot)}{K_n(\xi,\xi)}\right\| = \min\left\{\|P\| \ : P\in V_n,\ P(\xi) = 1\right\}. \tag{1.4}$$

The proof is a direct consequence of applying Cauchy-Schwarz's inequality and the addition theorem (see Laín Fernández [2]).

Our aim is to study the problem of characterizing sets of points $\{\xi_i\}_{i=1,\dots,N} \subset \mathbb{S}^2$ such that the functions $\{\varphi_i^n := K_n(\xi_i,\cdot)\}_{i=1,\dots,N}$ constitute a basis of the space V_n. The functions φ_i^n, $(i = 1,\dots,N)$ will be called *scaling functions*. The linear independence of the scaling functions is reflected in the regularity of an $N \times N$ matrix. Given $\{\xi_i\}_{i=1,\dots,N} \subset \mathbb{S}^2$, we can construct the interpolation matrix

$$\mathbf{A}_n := \begin{pmatrix} Y_0^0(\xi_1) & Y_0^0(\xi_2) & Y_0^0(\xi_3) & \dots & Y_0^0(\xi_N) \\ Y_1^{-1}(\xi_1) & Y_1^{-1}(\xi_2) & Y_1^{-1}(\xi_3) & \dots & Y_1^{-1}(\xi_N) \\ Y_1^0(\xi_1) & Y_1^0(\xi_2) & Y_1^0(\xi_3) & \dots & Y_1^0(\xi_N) \\ Y_1^1(\xi_1) & Y_1^1(\xi_2) & Y_1^1(\xi_3) & \dots & Y_1^1(\xi_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_n^{-n}(\xi_1) & Y_n^{-n}(\xi_2) & Y_n^{-n}(\xi_3) & \dots & Y_n^{-n}(\xi_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_n^n(\xi_1) & Y_n^n(\xi_2) & Y_n^n(\xi_3) & \dots & Y_n^n(\xi_N) \end{pmatrix} \in \mathbb{C}^{N\times N}. \tag{1.5}$$

By virtue of the addition theorem, one can directly see that the symmetric positive semidefinite matrix $\mathbf{\Phi}_n := \mathbf{A}_n^*\mathbf{A}_n$ has the entries

$$\mathbf{\Phi}_n(r,s) = \sum_{l=0}^{n}\sum_{m=-l}^{l} Y_l^{-m}(\eta_r)\, Y_l^m(\eta_s) = K_n(\eta_r,\eta_s) = \langle K_n(\eta_r,\cdot),\, K_n(\eta_s,\cdot)\rangle\,.$$

Therefore, the matrix $\mathbf{\Phi}_n$ is the Gram matrix of the scaling functions and will be positive definite, in particular regular, if and only if the scaling functions are linearly independent. As $\det \mathbf{\Phi}_n = |\det \mathbf{A}_n|^2$, we can study the regularity of either $\mathbf{A}_n$ or $\mathbf{\Phi}_n$ in order to determine whether the scaling functions constitute a basis of V_n or not. Since the regularity or singularity of the matrix $\mathbf{A}_n$ is independent of the basis of V_n on hand, we will assume from now on that the underlying basis is the basis (1.2) of spherical harmonics.

DEFINITION 1.3. *A set of points $\{\xi_i\}_{i=1,\dots,N} \subset \mathbb{S}^2$ for which the interpolation matrix $\mathbf{A}_n$ is nonsingular or equivalently the associated scaling functions $\{\varphi_i^n\}_{i=1,\dots,N}$ constitute a basis of V_n, is called a* fundamental system *for V_n.*

2. Construction of fundamental systems. With growing n, the analysis of the regularity of the interpolation matrices $\mathbf{A}_n$ becomes a very difficult task in general, so we have to restrict our analysis to specific choices of point constellations.

A possible way of constructing fundamental systems for V_n is due to v. Golitschek and Light [6] and Sünderman [5]. The key idea of their construction is to locate the N nodes on $n+1$ parallel circles, such that the kth $(k = 0,\dots,n)$ circle contains $2k+1$ points.

Another description of specific sets of points, which admit unique polynomial interpolation, is given by Xu [7, 8]. In his construction, the fundamental systems arise from relating an interpolation problem on the unit disc to an interpolation problem on the unit sphere: one starts with $(n + 1)(n + 2)/2$ points lying on $[n/2]+1$ concentric circles inside the unit disc, such that one of the circles is the unit circle $\mathbb{S}^1$ itself and each circle contains $n + 1$ equidistantly distributed nodes. Finally, the points are projected onto the northern and southern hemispheres of $\mathbb{S}^2$, yielding thereby a symmetric distribution of N points on the sphere. Unfortunately, this construction only works in the case of even polynomial degree n.

Here and from now on, we will assume that n is an odd natural number and consider accordingly that the $(n+1)^2$ nodes are distributed on $n+1$ parallel circles, where each of them

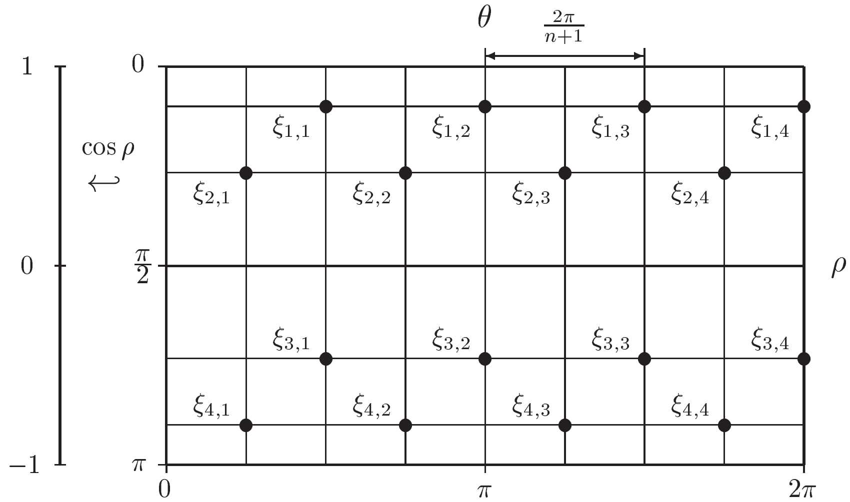

FIG. 2.1. *Case $n = 3$ and $\alpha = 1$: distribution of the points in a local-coordinates-grid: the points are distributed equidistantly on four symmetric latitudinal circles.*

bears $n+1$ equidistantly distributed points. Moreover, the latitudinal circles are symmetric with respect to the equator and the points on them exhibit a rotational symmetry with respect to the axis that joins the north and south poles. However, as the next lemma shows, a distribution with $(n+1)^2$ points lying simultaneously on $n+1$ meridians and on $n+1$ latitudes, does not yield a fundamental system for V_n.

LEMMA 2.1. *Let $n \in \mathbb{N}$ be an odd positive integer. Furthermore, denote with $\theta_k := 2\pi k/(n+1)$ $(k = 1,\ldots,n+1)$ and $\rho_j \in (0,\pi)$ $(j = 1,\ldots,n+1)$ pairwise different longitudinal and latitudinal angles, respectively. Then the system of points $\{\Psi(\rho_j,\theta_k)\}_{j,k=1,\ldots,n+1}$ does not constitute a fundamental system for V_n.*

The proof follows directly from constructing the associated interpolation matrix $\mathbf{A}_n$. It is straightforward to check that the rows corresponding to the functions $Y_l^{(n+1)/2}$ and $Y_l^{-(n+1)/2}$, $(l = (n+1)/2,\ldots,n)$ are linearly dependent.

In view of the preceding lemma we have to consider different longitudes on the chosen circles $\{(x_1,x_2,x_3) \in \mathbb{S}^2 : \ x_3 = \cos\rho_j\}$ $(j = 1,\ldots,n+1)$ in order to obtain fundamental systems. Based on this idea, the following theorem establishes a possible construction principle for fundamental systems when n is odd.

THEOREM 2.2. *Let n be an odd integer and let $0 < \rho_1 < \rho_2 < \ldots < \rho_{(n+1)/2} < \frac{\pi}{2}$ and $\rho_{n+2-j} := \pi - \rho_j$ $(j=1,\ldots,(n+1)/2)$ denote symmetric latitudinal angles. Then the set of points $M(\alpha) := \{\xi_{j,k} := \Psi(\rho_j,\theta_k^j),\ j,k = 1,\ldots,n+1\}$, where*

$$\theta_k^j = \begin{cases} \frac{2\pi k}{n+1}, & \text{if } j \text{ is odd,} \\ \frac{(2(k-1)+\alpha)\pi}{n+1}, & \text{if } j \text{ is even,} \end{cases}$$

with $\alpha \in (0,2)$, constitutes a fundamental system for V_n.

Proof. In order to establish the linear independence of the scaling functions corresponding to the point set $M(\alpha)$, we have to study the regularity of the interpolation matrix $\mathbf{A}_n$ in (1.5). The trick of the proof is to reduce our $(n+1)^2$-dimensional problem to $n+1$ problems of dimension $n+1$. Using the structure of the spherical harmonics as functions of separated variables, we can convert our original interpolation matrix $\mathbf{A}_n$ into an equivalent block diagonal one consisting of $n+1$ blocks $\mathbf{B}_m$ $(m = 0,\ldots,n)$ of dimension $(n+1)\times(n+1)$ by

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regular transformations.

Multiplying from the left hand side by a proper permutation matrix $\mathbf{P}_1$, we can reorder the rows of $\mathbf{A}_n$ so that the matrix attains the form

$$\begin{pmatrix} \mathbf{Z}_0 \\ \hline \mathbf{Z}_1 \\ \hline \vdots \\ \hline \mathbf{Z}_n \end{pmatrix} := \begin{pmatrix} \mathbf{Y}_l^m & : & m \equiv 0 \bmod (n+1),\ l \geq |m| \\ \hline \mathbf{Y}_l^m & : & m \equiv 1 \bmod (n+1),\ l \geq |m| \\ \hline & \vdots & \\ \hline \mathbf{Y}_l^m & : & m \equiv n \bmod (n+1),\ l \geq |m| \end{pmatrix} \begin{matrix} \in \mathbb{C}^{(n+1)\times N} \\ \in \mathbb{C}^{(n+1)\times N} \\ \\ \in \mathbb{C}^{(n+1)\times N}. \end{matrix}$$

Here, $\mathbf{Y}_l^m$ denotes the row vector containing the evaluation of the spherical harmonic Y_l^m at the nodes $\{\xi_{j,k},\ j,k = 1,\dots,n+1\}$. Within the matrices $\mathbf{Z}_m \in \mathbb{C}^{(n+1)\times N}$ $(m = 0,\dots,n)$ the functions – each of them determines one row of $\mathbf{Z}_m$ – are ordered in the following way

$$Y_m^m,\ Y_{m+1}^m,\ \dots,\ Y_n^m,\ Y_{n+1-m}^{m-n-1},\ Y_{n+2-m}^{m-n-1},\ \dots,\ Y_n^{m-n-1}.$$

Furthermore, bearing in mind the distribution of the points on the parallel circles $\rho = \rho_j$ $(j = 1,\dots,n+1)$, we can split the $(n+1)\times N$-dimensional matrices $\mathbf{Z}_m$ $(m = 0,\dots,n)$ into $n+1$ square matrices of dimension $(n+1)\times(n+1)$

$$\mathbf{Z}_m = (\mathbf{Z}_m^1, \mathbf{Z}_m^2, \dots, \mathbf{Z}_m^{n+1}), \qquad m = 0,\dots,n,$$

where $\mathbf{Z}_m^j \in \mathbb{C}^{(n+1)\times(n+1)}$ contains the information relative to the points lying on the jth latitudinal circle.

Moreover, let $\omega := \exp(2\pi i/(n+1))$ and consider the $(n+1)\times(n+1)$-dimensional Fourier matrix $\mathbf{F}_{n+1}$ with entries

$$\mathbf{F}_{n+1}(j,k) := \frac{1}{\sqrt{n+1}}\left(\omega^{-(j-1)(k-1)}\right) = \frac{1}{\sqrt{n+1}}\left(e^{-\frac{2\pi i}{n+1}(j-1)(k-1)}\right). \tag{2.1}$$

Multiplication from the right hand side by the $N\times N$-dimensional block diagonal matrix $\mathbf{F} := \mathrm{diag}\,(\mathbf{F}_{n+1}, \mathbf{F}_{n+1}, \dots, \mathbf{F}_{n+1})$ yields

$$\mathbf{P}_1\,\mathbf{A}_n\,\mathbf{F} = \begin{pmatrix} \mathbf{Z}_0^1\,\mathbf{F}_{n+1} & \mathbf{Z}_0^2\,\mathbf{F}_{n+1} & \dots & \mathbf{Z}_0^{n+1}\,\mathbf{F}_{n+1} \\ \mathbf{Z}_1^1\,\mathbf{F}_{n+1} & \mathbf{Z}_1^2\,\mathbf{F}_{n+1} & \dots & \mathbf{Z}_1^{n+1}\,\mathbf{F}_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_n^1\,\mathbf{F}_{n+1} & \mathbf{Z}_n^2\,\mathbf{F}_{n+1} & \dots & \mathbf{Z}_n^{n+1}\,\mathbf{F}_{n+1} \end{pmatrix}. \tag{2.2}$$

Let us now focus our attention on the $(n+1)\times(n+1)$-dimensional matrices $\mathbf{Z}_m^j\,\mathbf{F}_{n+1}$ and compute their entries. Since the longitudinal angles of the points on hand vary from latitude to latitude, it is necessary to examine the cases of odd and even j separately. Also the structure of the functions involved in each of the matrices $\mathbf{Z}_m^j$ recommends the distinction of two cases. On the one hand, we study the entries of the $n-m+1$ first rows, i.e., the rows corresponding to the functions $Y_{m+\kappa}^m$ $(\kappa = 0,\dots,n-m)$, and, on the other hand, we analyze the remaining m rows relative to the functions $Y_{n+1-m+\kappa}^{m-n-1}$ $(\kappa = 0,\dots,m-1)$.

(I) Let j be odd.

a) For $1 \leq r \leq n-m+1$, we obtain

$$\left(\mathbf{Z}_m^j\,\mathbf{F}_{n+1}\right)(r,s) = \frac{1}{\sqrt{n+1}}\sum_{k=1}^{n+1} Y_{m-1+r}^m\left(\Psi\left(\rho_j, \frac{2\pi k}{n+1}\right)\right)\,\omega^{(1-k)(s-1)}$$

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$$= K_r^m\, P_{m-1+r}^m(\cos\rho_j) \sum_{k=1}^{n+1} \omega^{mk+s-1-sk+k}$$

$$= K_r^m\, P_{m-1+r}^m(\cos\rho_j)\, \omega^{s-1} \sum_{k=1}^{n+1} \omega^{(m-s+1)\,k}, \tag{2.3}$$

where K_r^m denotes the constant

$$K_r^m := \sqrt{\frac{2\,(m-1+r)+1}{4\pi\,(n+1)}}\,, \qquad r = 1,\dots,n-m+1. \tag{2.4}$$

Note that the sum in (2.3) is nonzero if and only if $(m-s+1) \equiv 0 \bmod (n+1)$. However, since $m \in \{0,\dots,n\}$ and $s \in \{1,\dots,n+1\}$, this implies that $s = m+1$. Accordingly, we obtain

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{n+1}\right)(r,s) = \begin{cases} (n+1)\, K_r^m\, P_{m-1+r}^m(\cos\rho_j)\, \omega^m, & \text{if } s = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

b) Similarly, if $n-m+2 \le r \le n+1$, then

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{n+1}\right)(r,s) = \begin{cases} (n+1)\, K_r^{2r}\, P_{r-1}^{n+1-m}(\cos\rho_j)\, \omega^m, & \text{if } s = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

(II) Now let j be even.

a) For $1 \le r \le n-m+1$, it holds

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{n+1}\right)(r,s) = \frac{1}{\sqrt{n+1}} \sum_{k=1}^{n+1} Y_{m-1+r}^m\left(\Psi\left(\rho_j, \frac{(2(k-1)+\alpha)\pi}{n+1}\right)\right) \omega^{(1-k)(s-1)}$$

$$= K_r^m\, P_{m-1+r}^m(\cos\rho_j) \sum_{k=1}^{n+1} \omega^{\alpha m/2-m+km-sk+s-1+k}$$

$$= K_r^m\, P_{m-1+r}^m(\cos\rho_j)\, \omega^{\alpha m/2-m+s-1} \sum_{k=1}^{n+1} \omega^{(m-s+1)\,k}, \tag{2.5}$$

where K_r^m was already introduced in (2.4). Again, the sum in (2.5) is nonzero if and only if $s = m+1$. Accordingly, it follows that

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{n+1}\right)(r,s) = \begin{cases} (n+1)\, K_r^m\, P_{m-1+r}^m(\cos\rho_j)\, \omega^{\frac{\alpha m}{2}}, & \text{if } s = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

b) In a similar way, if $n-m+2 \le r \le n+1$, then

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{n+1}\right)(r,s) = \begin{cases} (n+1)\, K_r^{2r}\, P_{r-1}^{n+1-m}(\cos\rho_j)\, \omega^{\frac{\alpha m}{2}}\, e^{-\alpha\pi i}, & \text{if } s = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

In view of these calculations, we realize that each of the matrices $\mathbf{Z}_m^j\, \mathbf{F}_{n+1}$ possesses only one nonzero column, namely the $(m+1)$st one. Moreover, bearing in mind the structure of the matrix in (2.2), we observe that we can obtain a block diagonal matrix if we multiply from the right hand side by the following permutation matrix

$$\mathbf{P}_2 := \big[\mathbf{e}_1,\, \mathbf{e}_{n+2},\, \mathbf{e}_{2(n+1)+1}, \dots,\, \mathbf{e}_{n(n+1)+1},\, \mathbf{e}_2, \mathbf{e}_{n+3},\, \mathbf{e}_{2(n+1)+2} \\ \dots, \mathbf{e}_{n(n+1)+2}, \dots,\, \mathbf{e}_{n+1},\, \mathbf{e}_{2(n+1)},\, \mathbf{e}_{3(n+1)}, \dots,\, \mathbf{e}_{(n+1)^2}\,\big]\,.$$

Thereby, the product $\mathbf{P}_1\, \mathbf{A}_n\, \mathbf{F}\, \mathbf{P}_2$ attains the form $\operatorname{diag}(\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_n)$, where the matrix $\mathbf{B}_m \in \mathbb{C}^{(n+1)\times(n+1)}$ $(m = 0,\dots,n)$ has the following entries:

- If $1 \le r \le n-m+1$, then

$$\mathbf{B}_m(r,j) = \begin{cases} (n+1)\, K_r^m\, P_{m-1+r}^m(\cos\rho_j)\,\omega^m, & \text{if } j \text{ odd},\\ (n+1)\, K_r^m\, P_{m-1+r}^m(\cos\rho_j)\,\omega^{\frac{\alpha m}{2}}, & \text{if } j \text{ even}. \end{cases}$$

- If $n-m+2 \le r \le n+1$, then

$$\mathbf{B}_m(r,j) = \begin{cases} (n+1)\, K_r^{2r}\, P_{r-1}^{n+1-m}(\cos\rho_j)\,\omega^m, & \text{if } j \text{ odd},\\ (n+1)\, K_r^{2r}\, P_{r-1}^{n+m-1}(\cos\rho_j)\,\omega^{\frac{\alpha m}{2}}\, e^{-\alpha\pi i}, & \text{if } j \text{ even}. \end{cases}$$

Summarizing, we have obtained a simpler representation of the $N\times N$-dimensional matrix $\mathbf{A}_n$ as the block diagonal matrix $\operatorname{diag}(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n)$ via multiplication by regular matrices. Hence, we have reduced the study of the regularity of $\mathbf{A}_n$ to the analysis of the regularity of each of the lower dimensional matrices $\mathbf{B}_m$ $(m=0,\ldots,n)$.

In order to face this problem, we will make use of some properties of the calculus of determinants. Note also that it is enough to restrict our analysis to the cases $m=0,\ldots,(n+1)/2$, since $\mathbf{B}_{n+1-m}$ is equivalent to $\mathbf{B}_m$ for $m=0,\ldots,(n+1)/2$. Indeed, the latter one is a permuted and scaled version of $\mathbf{B}_m$.

Extracting the constants appearing in the entries of each of the matrices $\mathbf{B}_m$ $(m = 0,\ldots,(n+1)/2)$, first rowwise, then columnwise, and introducing the notation $x_j := \cos\rho_j$ $(j=1,\ldots,n+1)$, we obtain

$$\det \mathbf{B}_m = (n+1)^{n+1} \left(\prod_{r=1}^{n-m+1} K_r^m\right) \left(\prod_{s=n-m+2}^{n+1} K_s^{2s}\right) e^{m\pi i}\, e^{\frac{\alpha m \pi i}{2}} \times$$

$$\times \begin{vmatrix} P_m^m(x_1) & P_m^m(x_2) & \ldots & P_m^m(x_{n+1}) \\ P_{m+1}^m(x_1) & P_{m+1}^m(x_2) & \ldots & P_{m+1}^m(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ P_n^m(x_1) & P_n^m(x_2) & \ldots & P_n^m(x_{n+1}) \\ P_{n+1-m}^{n+1-m}(x_1) & e^{-\alpha\pi i} P_{n+1-m}^{n+1-m}(x_2) & \ldots & e^{-\alpha\pi i} P_{n+1-m}^{n+1-m}(x_{n+1}) \\ P_{n+2-m}^{n+1-m}(x_1) & e^{-\alpha\pi i} P_{n+2-m}^{n+1-m}(x_2) & \ldots & e^{-\alpha\pi i} P_{n+2-m}^{n+1-m}(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ P_n^{n+1-m}(x_1) & e^{-\alpha\pi i} P_n^{n+1-m}(x_2) & \ldots & e^{-\alpha\pi i} P_n^{n+1-m}(x_{n+1}) \end{vmatrix}. \tag{2.6}$$

Note that in the lower half of the above matrix, the factor $e^{-\alpha\pi i}$ only appears in the even columns. For the sake of symmetry, let us multiply the last m rows by $e^{\alpha\pi i/2}$. Thereby, it remains to show that the determinant

$$\begin{vmatrix} P_m^m(x_1) & P_m^m(x_2) & \ldots & P_m^m(x_{n+1}) \\ P_{m+1}^m(x_1) & P_{m+1}^m(x_2) & \ldots & P_{m+1}^m(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ P_n^m(x_1) & P_n^m(x_2) & \ldots & P_n^m(x_{n+1}) \\ e^{\frac{\alpha\pi i}{2}} P_{n+1-m}^{n+1-m}(x_1) & e^{-\frac{\alpha\pi i}{2}} P_{n+1-m}^{n+1-m}(x_2) & \ldots & e^{-\frac{\alpha\pi i}{2}} P_{n+1-m}^{n+1-m}(x_{n+1}) \\ e^{\frac{\alpha\pi i}{2}} P_{n+2-m}^{n+1-m}(x_1) & e^{-\frac{\alpha\pi i}{2}} P_{n+2-m}^{n+1-m}(x_2) & \ldots & e^{-\frac{\alpha\pi i}{2}} P_{n+2-m}^{n+1-m}(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{\alpha\pi i}{2}} P_n^{n+1-m}(x_1) & e^{-\frac{\alpha\pi i}{2}} P_n^{n+1-m}(x_2) & \ldots & e^{-\frac{\alpha\pi i}{2}} P_n^{n+1-m}(x_{n+1}) \end{vmatrix} \tag{2.7}$$

is nonzero. Exploiting the parity of the associated Legendre functions, we can transform the above matrix by elementary row operations into $\mathbf{C}_m\,\mathbf{D}$, where $\mathbf{D}$ is the $(n+1)\times(n+1)$-dimensional diagonal matrix

$$\mathbf{D} = \mathrm{diag}\ \left((1-x_1^2)^m, (1-x_2^2)^m, \ldots, (1-x_{n+1}^2)^m\right) \in \mathbb{R}^{(n+1)\times(n+1)},$$

and $\mathbf{C}_m$ is given by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{n+1} \\ x_1^2 & x_2^2 & \cdots & x_{n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-m} & x_2^{n-m} & \cdots & x_{n+1}^{n-m} \\ e^{\frac{\alpha\pi i}{2}}(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}(1-x_2^2)^\beta & \cdots & e^{\frac{-\alpha\pi i}{2}}(1-x_{n+1}^2)^\beta \\ e^{\frac{\alpha\pi i}{2}}x_1(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}x_2(1-x_2^2)^\beta & \cdots & e^{\frac{-\alpha\pi i}{2}}x_{n+1}(1-x_{n+1}^2)^\beta \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{\alpha\pi i}{2}}x_1^{m-1}(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}x_2^{m-1}(1-x_2^2)^\beta & \cdots & e^{\frac{-\alpha\pi i}{2}}x_{n+1}^{m-1}(1-x_{n+1}^2)^\beta \end{pmatrix}$$

with $\beta = (n+1)/2 - m$. Consequently, in order to conclude the proof of Theorem 2.2, it remains to establish the regularity of the $(n+1)\times(n+1)$-dimensional matrices $\mathbf{C}_m$ $(m = 0, \ldots, (n+1)/2)$.

As it is shown in Theorem 2.7 of Laín Fernández [1], all these matrices $\mathbf{C}_m$ $(m = 0, \ldots, (n+1)/2)$ are regular, which completes the proof. □

2.1. Conditions of the matrices $\mathbf{A}_n$. According to Lemma 2.1, the point distribution described in Theorem 2.2 does not yield a fundamental system for V_n when $\alpha = 0$ or $\alpha = 2$. In order to understand better the influence of the parameter α on the stability of the interpolation problem, let us now study the asymptotic behavior of the condition number cond $\mathbf{A}_1(\alpha)$ when α approximates the extremal values 0 or 2. On account of the symmetric distribution of the points in Theorem 2.2, we have that cond $\mathbf{A}_n(\alpha) =$ cond $\mathbf{A}_n(2-\alpha)$. Hence, we will assume without loss of generality that $\alpha \in (0, 1]$.

LEMMA 2.3. *Let $n = 1$ and let $\{\xi_1, \xi_2, \xi_3, \xi_4\} \subset \mathbb{S}^2$ be a fundamental system of the form described in Theorem 2.2. Then*

$$\arg\min\{\mathrm{cond}\ \mathbf{A}_1(\alpha) :\ \alpha \in (0, 1]\} = 1,$$

and cond $\mathbf{A}_1(1) = 1$ *if and only if the underlying latitude* $\cos\rho_1 = 1/\sqrt{3}$ *is the positive zero of the Legendre polynomial P_2. Moreover,*

$$\mathrm{cond}\ \mathbf{A}_1(\alpha) = O\left(\alpha^{-1}\right) \qquad \textit{as} \qquad \alpha \to 0.$$

Proof. In order to estimate the condition number of the matrix $\mathbf{A}_1(\alpha)$ in dependence on the shift α, let us compute the eigenvalues of the Gram matrix $\mathbf{\Phi}_1(\alpha)$. For the sake of simplicity, let $a := \cos(\alpha\pi/2)$ and $b := \cos^2\rho_1$. It is straightforward to check that the matrix $\mathbf{\Phi}_1(\alpha)$ attains the form

$$\frac{1}{4\pi}\begin{pmatrix} 4 & 6b-2 & 1+3(a(b-1)-b) & 1+3(a(1-b)-b) \\ 6b-2 & 4 & 1+3(a(1-b)-b) & 1+3(a(b-1)-b) \\ 1+3(a(b-1)-b) & 1+3(a(1-b)-b) & 4 & 6b-2 \\ 1+3(a(1-b)-b) & 1+3(a(b-1)-b) & 6b-2 & 4 \end{pmatrix}.$$

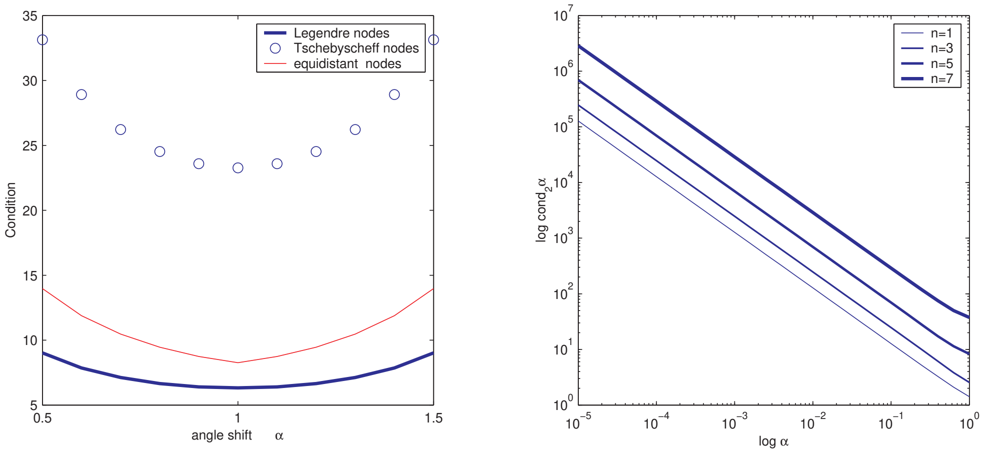

FIG. 2.2. *On the left hand side: Condition number of* $\mathbf{A}_3$ *in dependence of the shift* α *for different latitudinal heights. Note that because of the symmetry of the sphere, cond* $\mathbf{A}_3(\alpha) = $ *cond* $\mathbf{A}_3(2-\alpha)$*. The minimal condition number is achieved for* $\alpha = 1$*. On the right hand side: log-log plot of cond* $\mathbf{A}_n(\alpha)$*, when* α *tends to* 0.

Since this matrix is block circulant with circulant blocks, one can compute its eigenvalues via diagonalization by Fourier matrices

$$\lambda_1(a) = 4,\ \lambda_2(a) = 12b,\ \lambda_3(a) = 6(a-1)(b-1),\ \lambda_4(a) = -6(1+a)(b-1).$$

Considering now the cases $b \leq \frac{1}{3}$ and $b > \frac{1}{3}$ separately, it can be seen that the quotient $\lambda_{\max}(\mathbf{\Phi}_1(\alpha))/\lambda_{\min}(\mathbf{\Phi}_1(\alpha))$ attains its minimum at the intersection point of the lines $\lambda_3(a)$ and $\lambda_4(a)$, i.e. at the point $a = 0$ or equivalently $\alpha = 1$.

Accordingly, for $\alpha = 1$, the Gram matrix $\mathbf{\Phi}_1(1)$ possesses the eigenvalues

$$\lambda_1 = 4, \quad \lambda_2 = 6(1 + \cos 2\rho_1), \quad \lambda_{3,4} = -3(-1 + \cos 2\rho_1).$$

It is now straightforward to prove that the matrix has equal eigenvalues if and only if $\cos\rho_1 = 1/\sqrt{3}$. For this value of ρ_1, the condition number of the matrix is equal to one and the scaling functions form an orthogonal basis of V_1.

Having computed the eigenvalues of $\mathbf{\Phi}_1(\alpha)$, we can immediately conclude that cond $A_1(\alpha) = O\left(\alpha^{-1}\right),\ \alpha \to 0$ by analyzing the cases $\cos^2\rho_1 \leq 1/3$ and $\cos^2\rho_1 > 1/3$ separately. ☐

At this point it should be mentioned that for the special value of $\rho_1 = \arccos\left(1/\sqrt{3}\right)$, the points constituting the set $M(1)$, i.e.

$$\xi_{1,1} = \left(-\sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}}\right), \quad \xi_{1,2} = \left(\sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}}\right),$$
$$\xi_{2,1} = \left(0, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right), \quad \xi_{2,2} = \left(0, -\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right),$$

are the vertices of a regular tetrahedron inscribed in the sphere.

Motivated by the results obtained for the case $n = 1$, it is natural to ask whether it is possible to make similar assertions for higher values of n. Unfortunately, so far it was only possible to conduct numerical experiments, which indeed support the conjecture that for

$n \geq 2$ the condition number cond $\mathbf{A}_n(\alpha)$ behaves like $O(\alpha^{-1})$, resp. $O((2-\alpha)^{-1})$, when the shift α approximates the singular values 0 or 2.

On the right hand side of Figure 2.2, we display the behavior of the condition number cond $\mathbf{A}_n(\alpha)$ for small values of α and for the odd values of n between 1 and 7. In order to bring out the asymptotic behavior of cond $\mathbf{A}_n(\alpha)$, we have used logarithmic scales in the plot. According to our expectation, we obtain a straight line with slope equal to -1. It can also be observed that with increasing n the intersection of the graph with the y-axis moves upwards. In the numerical tests, we have chosen equidistantly distributed latitudinal angles for our calculations. Remember that the considered fundamental systems in Theorem 2.2 allowed arbitrary heights of the latitudinal circles.

On the left hand side of Figure 2.2, we illustrate the behavior of cond $\mathbf{A}_3(\alpha)$ for the values of $\alpha \in [1/2, 3/2]$. Note that for $n = 1$ the minimal condition number is attained at $\alpha = 1$, which corresponds to Lemma 2.3.

On account of Lemma 2.3, for $n = 1$ orthogonality of the scaling functions could be achieved by considering the zeros of the Legendre polynomial of the next higher degree as latitudes. Motivated by this fact, we now choose as heights for our parallel circles the zeros of orthogonal polynomials of degree four. In particular we consider the zeros of Legendre and Tschebyscheff polynomials of the first kind. This choice fits into the approach pursued by Fischer and Prestin [3], where polynomial wavelets on the interval were studied and orthogonality of the scaling functions was achieved by employing the zeros of the underlying orthogonal polynomials as nodes. As we might have expected, the condition number of the matrix $\mathbf{A}_3(\alpha)$ improves when we choose the zeros of the Legendre polynomials as heights of the latitudinal circles. The choice of equidistantly distributed heights $z_k = \cos\left((2k-1)\,\pi/(2\,(n+1))\right)$ yields similar condition numbers to the case of the Legendre nodes.

REFERENCES

[1] N. LAÍN FERNÁNDEZ, *Polynomial Bases on the Sphere*, PhD thesis, in preparation.
[2] N. LAÍN FERNÁNDEZ, *Polynomial Bases on the Sphere*, in Proceedings IDoMAT, 2001, Int. Dortmund Meeting in Approximation Theory, Witten-Bommerholz, Birkhäuser, 2001.
[3] B. FISCHER AND J. PRESTIN, *Wavelets based on orthogonal polynomials*, *Math. Comp*, 66 (1997), pp. 1593–1618.
[4] C. MÜLLER, *Spherical Harmonics*, Springer, Berlin, 1966.
[5] B. SÜNDERMANN, *Projektionen auf Polynomräumen in mehreren Veränderlichen*, PhD thesis, Diss. Dortmund, 1983.
[6] M. V. GOLITSCHEK AND W. LIGHT, *Interpolation by Polynomials and Radial Basis Functions on Spheres*, Constr. Approx., 17 (2001), pp. 1–18.
[7] Y. XU, *Polynomial interpolation on the unit sphere*, SIAM J. Numer. Anal., to appear.
[8] Y. XU, *Polynomial interpolation on the unit sphere and on the unit ball*, Adv. in Com. Math., to appear.