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CHARACTERISTICS OF BESOV-NIKOL'SKIĬ CLASS OF FUNCTIONS *

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Abstract. In this paper we consider functions from a Besov-Nikol'skiĭ class. We give constructive characteristics of this class. We establish criteria for a function to be in this Besov-Nikol'skiĭ class by means of conditions on its Fourier coefficients. We also discuss embedding theorems between some classes of functions.

Key words. Moduli of smoothness, Besov-Nikol'skiĭ class, Best approximation, Fourier coefficients, Embedding theorems.

AMS subject classifications. 42A10, 42A16, 41A50, 42A32, 42A50.

1. Introduction. If $1 \le p < \infty$, let L_p be the space of 2π-periodic, measurable functions $f(x)$ such that $\|f\|_p = \left(\int\limits_0^{2\pi} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty$. Similarly, let L_∞ be the space of 2π-periodic, continuous functions $f(x)$ with $\|f\|_\infty = \max\limits_{x\in[0,2\pi]} |f(x)|$. The modulus of smoothness of order β $(\beta > 0)$ of a function $f \in L_p$, $1 \le p \le \infty$, is given by

$$\omega_\beta(f,t)_p = \sup_{|h|\le t} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} f(x+(\beta-\nu)h) \right\|_p .$$

If $F(\eta) \ge 0$ and $G(\eta) \ge 0$ for all η, then the notation $F(\eta) \ll G(\eta)$ will mean that there exists a positive constant C not depending on η and such that $F(\eta) \le C\,G(\eta)$ for all η. If $F(\eta) \ll G(\eta)$ and $G(\eta) \ll F(\eta)$ hold simultaneously, then we shall write $F(\eta) \asymp G(\eta)$.

Let $\alpha(t)$ be a function measurable on the interval $[0,1]$ and nonnegative. If there exists a real number σ such that $\int\limits_0^1 \alpha(t) t^\sigma\,dt < \infty$, then we shall say that $\alpha(t)$ satisfies the σ-condition.

Denote by $BH(\alpha,\beta,\psi,p,\theta,k)$ the *Besov-Nikol'skiĭ class* (see, e.g., [9],[10]), that is, the class of functions $f(x) \in L_p$, $1 \le p \le \infty$, such that

$$\left(\int\limits_0^\delta \alpha(t)\,\omega_{k+\beta}^\theta(f,t)_p\,dt + \delta^{\beta\theta} \int\limits_\delta^1 \alpha(t)\,t^{-\beta\theta}\,\omega_{k+\beta}^\theta(f,t)_p\,dt \right)^{\frac{1}{\theta}} \ll \psi(\delta),$$

where $\beta, k, \theta \in (0,\infty)$, and the function α satisfies the σ-condition with $\sigma = k\theta$ and $0 \le \delta \le 1$. Here and in the sequel, we shall assume that a function ψ is a majorant, i.e., ψ is a function, continuous and nonnegative on $[0,1]$, which possesses the following properties: $\psi(\delta_1) \le C_1\psi(\delta_2)$, $0 \le \delta_1 \le \delta_2 \le 1$; $\psi(2\delta) \le C_2\psi(\delta)$, $0 \le \delta \le \frac{1}{2}$, where the constants C_1 and æ$_2$ do not depend on δ_1, δ_2 and δ.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called quasi β-power-monotone increasing (decreasing) if there exist a natural number $N := N(\beta,\gamma)$ and a constant $K := K(\beta,\gamma) \ge 1$ such that

$$Kn^\beta\gamma_n \ge m^\beta\gamma_m \quad (n^\beta\gamma_n \le Km^\beta\gamma_m) \tag{1.1}$$

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holds for any $n \geq m \geq N$.

If (1.1) holds with $\beta = 0$, then we omit the attribute "β-power" in the inequality.

Furthermore, a sequence $\gamma := \{\gamma_n\}$ of positive terms will be called quasi-geometrically increasing (decreasing) if there exist natural numbers $\mu := \mu(\gamma)$, $N := N(\gamma)$ and a constant $K := K(\gamma) \geq 1$ such that

$$\gamma_{n+\mu} \geq 2\gamma_n \text{ and } \gamma_n \leq K\gamma_{n+1} \quad (\gamma_{n+\mu} \leq \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leq K\gamma_n)$$

hold for all $n \geq N$.

Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.2}$$

be the Fourier series of a function $f(x)$, and let $S_n(f)$ be the n-th partial sum of (1.2). Then $f(x)$ has *lacunary* Fourier coefficients if $a_n, b_n = 0$, $n \neq 2^k$, and $a_{2^k} = c_k, b_{2^k} = d_k; c_k, d_k \geq 0\,(k \in \mathbf{N} \cup \{0\}\,)$. If there exists $\gamma \geq 0$ such that $\frac{a_n}{n^\gamma} \downarrow 0,\ \frac{b_n}{n^\gamma} \downarrow 0$, then $f(x)$ has *quasimonotone* Fourier coefficients.

The best trigonometrical approximation to f is given by $E_n(f)_p = \inf\limits_{T \in \mathcal{T}_{n-1}} \|f(x) - T\|_p$, where $\mathcal{T}_{n-1}$ is the collection of trigonometrical polynomials of total degree smaller than n.

An outline of this paper is as follows. In section 2, we consider some useful lemmas. In section 3, we give a constructive characteristic of the Besov-Nikol'skiĭ class. In section 4, we discuss criteria for a function to belong to a Besov-Nikol'skiĭ class through some conditions on its Fourier coefficients. In section 5, we consider Nikol'skiĭ and Besov classes and discuss embedding theorems between these classes and the Besov-Nikol'skiĭ class. We also obtain some results concerning $\widetilde{H}(k,p,\psi)$, $\widetilde{BH}(\alpha,\beta,\psi,p,\theta,k)$, $H(k,p,\psi)$ and $BH(\alpha,\beta,\psi,p,\theta,k)$ classes and their interrelations.

2. Auxiliary results. LEMMA 2.1 ([12]). *If $f(x) \in L_p$, $1 \leq p \leq \infty$, and if $n \in \mathbf{N}$, then*

$$E_n(f)_p \ll \omega_\beta(f, \frac{1}{n})_p \ll n^{-\beta} \sum_{\xi=1}^{n+1} \xi^{\beta-1} E_\xi(f)_p.$$

LEMMA 2.2 ([7]). *For any positive sequence $\gamma := \{\gamma_n\}$ the inequalities*

$$\sum_{k=n}^{\infty} \gamma_k \leq C\gamma_n \quad (n = 1, 2, \cdots; C \geq 1),$$

or

$$\sum_{k=1}^{n} \gamma_k \leq C\gamma_n \quad (n = 1, 2, \cdots; C \geq 1),$$

hold if and only if the sequence γ is quasi-geometrically decreasing (increasing), respectively.

LEMMA 2.3 ([8]). *Let a positive sequence $\gamma := \{\gamma_n\}$ be quasi-geometrically decreasing (increasing). Then there exists a positive (negative) ε such that the sequence $\{\gamma_n 2^{n\varepsilon}\}$ is quasi-geometrically decreasing (increasing).*

LEMMA 2.4 ([7]). *If $a_n \geq 0,\ \lambda_n > 0$, and if $p \geq 1$, then*

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(a) $\sum\limits_{n=1}^{\infty} \lambda_n \left(\sum\limits_{\nu=1}^{n} a_\nu \right)^p \le p^p \sum\limits_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum\limits_{\nu=n}^{\infty} \lambda_\nu \right)^p,$

(b) $\sum\limits_{n=1}^{\infty} \lambda_n \left(\sum\limits_{\nu=n}^{\infty} a_\nu \right)^p \le p^p \sum\limits_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum\limits_{\nu=1}^{n} \lambda_\nu \right)^p.$

LEMMA 2.5 ([4]). *If* $a_n \ge 0$, *and if* $0 < p \le 1$, *then* $\left(\sum\limits_{n=1}^{\infty} a_n \right)^p \le \sum\limits_{n=1}^{\infty} a_n^p$.

3. Constructive characteristics of the Besov-Nikol'skiĭ class. THEOREM 3.1. *Let* $\beta, k, \theta > 0$, *and* $\alpha(\cdot)$ *satisfies* σ*-condition with* $\sigma = k\theta$, *and*

$$2^{nk\theta} \int\limits_0^{1/2^n} \alpha(t)\, t^{k\theta} dt \ll \int\limits_{1/2^{n+1}}^{1/2^n} \alpha(t) dt \qquad \text{for} \quad n \in \mathbf{N} \cup \{0\}. \tag{3.1}$$

Suppose $f(x) \in L_p \, (1 \le p \le \infty)$, *then* $f(x) \in BH(\alpha, \beta, \psi, p, \theta, k)$ *if and only if for any* $n \in \mathbf{N} \cup \{0\}$

$$\left(2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \Lambda_\nu E_{2^\nu}^\theta(f)_p + \sum_{\nu=n+1}^{\infty} \Lambda_\nu E_{2^\nu}^\theta(f)_p \right)^{\frac{1}{\theta}} \ll \psi\left(\frac{1}{2^n} \right), \tag{3.2}$$

where $\Lambda_\nu = \int\limits_{1/2^{\nu+1}}^{1/2^\nu} \alpha(t)\, dt, \; \nu \in \mathbf{N} \cup \{0\}$.

Proof. Define $\mu_\nu = \int\limits_{1/(2\nu)}^{1/\nu} \alpha(t)\, dt, \nu \in \mathbf{N}$, i.e., $\mu_{2^\nu} = \Lambda_\nu = \int\limits_{1/2^{\nu+1}}^{1/2^\nu} \alpha(t)\, dt, \; \nu \in \mathbf{N} \cup \{0\}$.

Using (3.1) and Lemmas 2.2 and 2.3, we get that there exists an $\varepsilon > 0$ such that the sequence $\left\{ \Lambda_\nu 2^{-\nu(k\theta-\varepsilon)} \right\}$ is quasi-geometrically decreasing. Thus,

$$\Lambda_l 2^{-l(k\theta-\varepsilon)} \le C \Lambda_m 2^{-m(k\theta-\varepsilon)} \tag{3.3}$$

holds for any $m \le l$.

Let n be an integer such that $\frac{1}{2^{n+1}} < \delta \le \frac{1}{2^n}$. Then using the properties of modulus of smoothness ($\omega_{k+\beta}(f,t)_p \asymp \omega_{k+\beta}(f, \frac{1}{2^n})_p$ for $\frac{1}{2^{n+1}} \le t \le \frac{1}{2^n}$, see [12]), we have

$$\int\limits_{2^{-(n+1)}}^{\delta} \alpha(t)\, \omega_{k+\beta}^\theta(f,t)_p dt + \delta^{\beta\theta} \int\limits_{\delta}^{2^{-n}} \alpha(t)\, t^{-\beta\theta}\, \omega_{k+\beta}^\theta(f,t)_p dt \asymp$$

$$\asymp \int\limits_{2^{-(n+1)}}^{2^{-n}} \alpha(t)\, dt\, \omega_{k+\beta}^\theta(f, \frac{1}{2^n})_p = \Lambda_n\, \omega_{k+\beta}^\theta(f, \frac{1}{2^n})_p$$

and hence,

$$I_1 + I_2 := \int\limits_0^{\delta} \alpha(t)\, \omega_{k+\beta}^\theta(f,t)_p\, dt + \delta^{\beta\theta} \int\limits_{\delta}^{1} \alpha(t)\, t^{-\beta\theta}\, \omega_{k+\beta}^\theta(f,t)_p\, dt \asymp \tag{3.4}$$

$$\asymp \sum_{\nu=n}^{\infty} \int\limits_{2^{-(\nu+1)}}^{2^{-\nu}} \alpha(t)\, \omega_{k+\beta}^\theta(f,t)_p\, dt + 2^{-n\beta\theta} \sum_{\nu=0}^{n-1} \int\limits_{2^{-(\nu+1)}}^{2^{-\nu}} \alpha(t)\, t^{-\beta\theta}\, \omega_{k+\beta}^\theta(f,t)_p\, dt \asymp$$

$$\asymp \sum_{\nu=n+1}^{\infty} \Lambda_\nu\, \omega_{k+\beta}^{\theta}(f,\frac{1}{2^\nu})_p + 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{\nu\beta\theta}\Lambda_\nu\, \omega_{k+\beta}^{\theta}(f,\frac{1}{2^\nu})_p.$$

Lemma 2.1 implies

$$I_1 + I_2 \ll \left(\sum_{\nu=n+1}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu + 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{-\nu k\theta}\Lambda_\nu\right)\left[\sum_{\xi=0}^{\nu} 2^{\xi(k+\beta)}E_{2^\xi}(f)_p\right]^{\theta}.$$

Let $\theta \le 1$. By Lemma 2.5 we obtain

$$I_1 + I_2 \ll \sum_{\nu=n+1}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu \left[\sum_{\xi=0}^{n} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p + \sum_{\xi=n+1}^{\nu} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p\right] + $$
$$+ 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{-\nu k\theta}\Lambda_\nu \sum_{\xi=0}^{\nu} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p.$$

Using (3.1), we get

$$\sum_{\nu=n}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu \le 2^{-n\beta\theta}\sum_{\nu=n}^{\infty} 2^{-\nu k\theta}\Lambda_\nu \ll 2^{-n(k+\beta)\theta}\Lambda_n. \tag{3.5}$$

Thus,

$$I_1 + I_2 \ll 2^{-n(k+\beta)\theta}\Lambda_n \sum_{\xi=0}^{n} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p + \sum_{\xi=n}^{\infty} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p \sum_{\nu=\xi}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu + $$
$$+ 2^{-n\beta\theta}\sum_{\xi=0}^{n} 2^{\xi(k+\beta)\theta}E_{2^\xi}^{\theta}(f)_p \sum_{\nu=\xi}^{n} 2^{-\nu k\theta}\Lambda_\nu \ll \sum_{\xi=n}^{\infty}\Lambda_\xi E_{2^\xi}^{\theta}(f)_p + 2^{-n\beta\theta}\sum_{\xi=0}^{n} 2^{\xi\beta\theta}\Lambda_\xi E_{2^\xi}^{\theta}(f)_p.$$

Now let $\theta > 1$. Define

$$I_{11} := \sum_{\nu=n+1}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu \left[\sum_{\xi=0}^{n} 2^{\xi(k+\beta)}E_{2^\xi}(f)_p\right]^{\theta},$$
$$I_{12} := \sum_{\nu=n+1}^{\infty} 2^{-\nu(k+\beta)\theta}\Lambda_\nu \left[\sum_{\xi=n+1}^{\nu} 2^{\xi(k+\beta)}E_{2^\xi}(f)_p\right]^{\theta},$$
$$I_{21} := 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{-\nu k\theta}\Lambda_\nu \left[\sum_{\xi=0}^{\nu} 2^{\xi(k+\beta)}E_{2^\xi}(f)_p\right]^{\theta}.$$

For I_{12} and I_{21}, Lemma 2.4(a) gives

$$I_{12} \ll \sum_{\nu=n+1}^{\infty} 2^{\nu(k+\beta)\theta}E_{2^\nu}^{\theta}(f)_p\left(2^{-\nu(k+\beta)\theta}\Lambda_\nu\right)^{1-\theta}\left[\sum_{\xi=\nu}^{\infty} 2^{-\xi(k+\beta)\theta}\Lambda_\xi\right]^{\theta} \ll \sum_{\nu=n+1}^{\infty}\Lambda_\nu E_{2^\nu}^{\theta}(f)_p.$$

$$I_{21} \ll 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{\nu(k+\beta)\theta}E_{2^\nu}^{\theta}(f)_p\left(2^{-\nu k\theta}\Lambda_\nu\right)^{1-\theta}\left[\sum_{\xi=\nu}^{n} 2^{-\xi k\theta}\Lambda_\xi\right]^{\theta} \ll 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{\nu\beta\theta}\Lambda_\nu E_{2^\nu}^{\theta}(f)_p.$$

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We now estimate I_{11}. Let $\theta' = \frac{\theta}{\theta-1}$ and $\tau = \frac{\varepsilon}{\theta}$ with ε defined by (3.3). An elementary calculation, using the Hölder inequality, gives that

$$\sum_{\xi=0}^{n} 2^{\xi(k+\beta)} E_{2^\xi}(f)_p \ll \left[\sum_{\xi=0}^{n} 2^{\xi(k+\beta-\tau)\theta} E_{2^\xi}^\theta(f)_p\right]^{\frac{1}{\theta}} \left[\sum_{\xi=0}^{n} 2^{\xi\tau\theta'}\right]^{\frac{1}{\theta'}} \ll 2^{n\tau}\left[\sum_{\xi=0}^{n} 2^{\xi(k+\beta-\tau)\theta} E_{2^\xi}^\theta(f)_p\right]^{\frac{1}{\theta}}. \tag{3.6}$$

Using (3.5) and then (3.6) and (3.3), we obtain

$$I_{11} \ll 2^{-n(k+\beta)\theta}\Lambda_n \left[\sum_{\xi=0}^{n} 2^{\xi(k+\beta)} E_{2^\xi}(f)_p\right]^{\theta} \ll 2^{-n(k+\beta-\tau)\theta}\Lambda_n \sum_{\xi=0}^{n} 2^{\xi(k+\beta-\tau)\theta} E_{2^\xi}^\theta(f)_p \ll$$
$$\ll 2^{-n\beta\theta}\Lambda_n 2^{-n(k\theta-\varepsilon)} \sum_{\xi=0}^{n} 2^{\xi\beta\theta} 2^{\xi(k\theta-\varepsilon)} E_{2^\xi}^\theta(f)_p \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} 2^{\xi\beta\theta}\Lambda_\xi E_{2^\xi}^\theta(f)_p.$$

Hence, we claim that (3.2) $\Longrightarrow f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$.

On the contrary, if $f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$, then Lemma 2.1 and (3.4) imply (3.2).
□

4. Fourier coefficients of functions from Besov-Nikol'skiĭ class. THEOREM 4.1. *Let $\beta, k, \theta > 0$, and $\alpha(\cdot)$ satisfies the σ-condition with $\sigma = k\theta$, and*

$$\int_{1/2^n}^{1} \alpha(t)dt + 2^{nk\theta} \int_{0}^{1/2^n} \alpha(t)\, t^{k\theta} dt \ll \int_{1/2^{n+1}}^{1/2^n} \alpha(t)dt \qquad \text{for} \quad n \in \mathbf{N}\cup\{0\}. \tag{4.1}$$

If $f(x) \in L_p(1 \le p \le \infty)$ has lacunary Fourier coefficients, then $f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$ if and only if, for any $n \in \mathbf{N}\cup\{0\}$, there holds

$$\left(2^{-n\beta\theta}\sum_{\nu=0}^{n}(c_\nu+d_\nu)^\theta\Lambda_\nu 2^{\nu\beta\theta} + \sum_{\nu=n+1}^{\infty}(c_\nu+d_\nu)^\theta\Lambda_\nu\right)^{\frac{1}{\theta}} \ll \psi\left(\frac{1}{2^n}\right), \tag{4.2}$$

where $\Lambda_\nu = \int\limits_{1/2^{\nu+1}}^{1/2^\nu} \alpha(t)\,dt,\ \nu \in \mathbf{N}\cup\{0\}$.

Proof. Let n be an integer such that $\frac{1}{2^{n+1}} < \delta \le \frac{1}{2^n}$. We note that $\alpha(t)$ satisfies (3.1) as well, which follows from (4.1). Thus, it is sufficient to prove

$$(4.2) \iff J_1+J_2 := 2^{-n\beta\theta}\sum_{\nu=0}^{n} 2^{\nu\beta\theta}\Lambda_\nu E_{2^\nu}^\theta(f)_p + \sum_{\nu=n+1}^{\infty}\Lambda_\nu E_{2^\nu}^\theta(f)_p \ll \psi^\theta\left(\frac{1}{2^n}\right).$$

It is clear that condition (4.1) implies

$$\sum_{\nu=0}^{n}\Lambda_\nu \ll \Lambda_n. \tag{4.3}$$

Using Lemma 2.2 and 2.3, we have that there exist $\varepsilon > 0$ and $C > 1$ such that the inequality

$$\Lambda_l \, 2^{-l\varepsilon} \leq C \, \Lambda_m \, 2^{-m\varepsilon} \tag{4.4}$$

holds for any $l \leq m$.

For the case $1 \leq p < \infty$ we shall need a well-known statement for functions with lacunary Fourier coefficients ([14]): $E_{2^n}(f)_p \asymp \|f - S_{2^n-1}(f)\|_p \asymp \left[\sum\limits_{k=n}^{\infty} \left(c_k^2 + d_k^2\right)\right]^{\frac{1}{2}}$. For brevity, we shall suppose that the function f is even ($d_k = 0$). Put

$$J_1 \asymp 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \Lambda_\nu \left[\sum_{\xi=\nu}^{\infty} c_\xi^2\right]^{\frac{\theta}{2}} \asymp 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \Lambda_\nu \left[\sum_{\xi=\nu}^{n} c_\xi^2\right]^{\frac{\theta}{2}} +$$

$$+ 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \Lambda_\nu \left[\sum_{\xi=n+1}^{\infty} c_\xi^2\right]^{\frac{\theta}{2}} =: J_{11} + J_{12}, \qquad J_2 \asymp \sum_{\nu=n+1}^{\infty} \Lambda_\nu \left[\sum_{\xi=\nu}^{\infty} c_\xi^2\right]^{\frac{\theta}{2}} =: J_{21}.$$

To estimate J_{11}, J_{21} we now write, for $\theta \leq 2$, using Lemma 2.5 and (4.3), that

$$J_{11} \ll 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \Lambda_\nu \sum_{\xi=\nu}^{n} c_\xi^\theta \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} c_\xi^\theta \sum_{\nu=1}^{\xi} 2^{\nu\beta\theta} \Lambda_\nu \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} c_\xi^\theta 2^{\xi\beta\theta} \Lambda_\xi;$$

$$J_{21} \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \sum_{\nu=n+1}^{\xi} \Lambda_\nu \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \Lambda_\xi.$$

We note that (4.4) gives the inequality $\Lambda_l \leq C\Lambda_m$ $(l \leq m)$. Thus, for $\theta \leq 2$ we have

$$J_{12} \ll \Lambda_n \sum_{\xi=n+1}^{\infty} c_\xi^\theta \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \Lambda_\xi.$$

Let $\theta > 2$. Using Lemma 2.4(b) and (4.3), we get

$$J_{11} \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} c_\xi^\theta \left(2^{\xi\beta\theta} \Lambda_\xi\right)^{1-\frac{\theta}{2}} \left[\sum_{\nu=0}^{\xi} 2^{\nu\beta\theta} \Lambda_\nu\right]^{\frac{\theta}{2}} \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} c_\xi^\theta 2^{\xi\beta\theta} \Lambda_\xi$$

and

$$J_{21} \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \left(\Lambda_\xi\right)^{1-\frac{\theta}{2}} \left[\sum_{\nu=n+1}^{\xi} \Lambda_\nu\right]^{\frac{\theta}{2}} \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \Lambda_\xi.$$

Let $\theta' = \frac{\theta}{\theta-2}$ and $\tau = \frac{\varepsilon}{\theta}$ with ε defined by (4.4). Using Hölder inequality, we have

$$\sum_{\xi=n+1}^{\infty} c_\xi^2 \ll \left[\sum_{\xi=n+1}^{\infty} c_\xi^\theta 2^{\xi\tau\theta}\right]^{\frac{2}{\theta}} \left[\sum_{\xi=n+1}^{\infty} 2^{-2\xi\tau\theta'}\right]^{\frac{1}{\theta'}} \ll 2^{-2n\tau} \left[\sum_{\xi=n+1}^{\infty} c_\xi^\theta 2^{\xi\tau\theta}\right]^{\frac{2}{\theta}}. \tag{4.5}$$

Combining (4.3), (4.5) and (4.4), we obtain

$$J_{12} \ll \Lambda_n \left[\sum_{\xi=n+1}^{\infty} c_\xi^2\right]^{\frac{\theta}{2}} \ll 2^{-n\tau\theta} \Lambda_n \sum_{\xi=n+1}^{\infty} c_\xi^\theta 2^{\xi\tau\theta} \ll 2^{-n\varepsilon} \Lambda_n \sum_{\xi=n+1}^{\infty} c_\xi^\theta 2^{\xi\varepsilon} \ll \sum_{\xi=n+1}^{\infty} c_\xi^\theta \Lambda_\xi.$$

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Finally, we have $J_1+J_2 \ll \sum\limits_{\xi=n+1}^{\infty} c_\xi^\theta 2^{\xi\theta}\Lambda_\xi + 2^{-n\beta\theta}\sum\limits_{\xi=0}^{n} c_\xi^\theta 2^{\xi\beta\theta}\Lambda_\xi$ and the sufficiency of (4.2) has been proved. The necessity of (4.2) follows from evident estimates: $J_1 \geq 2^{-n\beta\theta}\sum\limits_{\xi=0}^{n} c_\xi^\theta 2^{\xi\beta\theta}\Lambda_\xi, \quad J_2 \geq \sum\limits_{\xi=n+1}^{\infty} c_\xi^\theta \Lambda_\xi.$

In the case of $p=\infty$ the proof can be developed in the same manner using the following inequality ([11]): $E_{2^n}(f)_\infty \asymp \sum\limits_{k=n}^{\infty}(c_k+d_k)$. □

THEOREM 4.2. *Let $\beta, k, \theta > 0$, and $\alpha(\cdot)$ satisfies a σ-condition with $\sigma = k\theta$, and*

$$\int_{1/n}^{1}\alpha(t)\,dt \asymp n^{k\theta}\int_{0}^{1/n}\alpha(t)\,t^{k\theta}\,dt \asymp n\int_{1/(n+1)}^{1/n}\alpha(t)\,dt \qquad \text{for} \quad n=2,3,\cdots \tag{4.6}$$

Suppose $f(x)\in L_p\,(1<p<\infty)$ has quasimonotone Fourier coefficients, then $f(x)\in BH(\alpha,\beta,\psi,p,\theta,k)$ if and only if, for any $n\in\mathbf{N}$,

$$\left(n^{-\beta\theta}\sum_{\nu=1}^{n}(a_\nu+b_\nu)^\theta\lambda_\nu\nu^{\beta\theta+\theta-\frac{\theta}{p}}+\sum_{\nu=n+1}^{\infty}(a_\nu+b_\nu)^\theta\lambda_\nu\nu^{\theta-\frac{\theta}{p}}\right)^{\frac{1}{\theta}} \ll \psi\left(\frac{1}{n}\right),$$

where $\lambda_\nu = \int\limits_{1/(\nu+1)}^{1/\nu}\alpha(t)\,dt,\ \nu\in\mathbf{N}.$

Theorem 4.2 can be proved in the same way as Theorem 4.1. Another way of proof is presented in [13].

5. Embedding theorems. Recall that the sequences $\{\Lambda\},\{\mu\},\{\lambda\}$ are defined by $\Lambda_\nu = \int\limits_{1/2^{\nu+1}}^{1/2^\nu}\alpha(t)\,dt,\ \nu\in\mathbf{N}\cup\{0\},\ \mu_\nu = \int\limits_{1/(2\nu)}^{1/\nu}\alpha(t)\,dt,\ \lambda_\nu = \int\limits_{1/(\nu+1)}^{1/\nu}\alpha(t)\,dt,\ \nu\in\mathbf{N}.$ By $H(k,p,\psi)$ and $B(\alpha,\beta,p,\theta)$ we denote Nikol'skiĭ and Besov classes, i.e., $H(\beta,p,\psi) = \{f\in L_p : \omega_\beta(f,\delta)_p \ll \psi(\delta)\}$, and $B(\alpha,\beta,p,\theta) = \left\{f\in L_p : \int\limits_0^1 \alpha(t)\,\omega_\beta^\theta(f,t)_p\,dt<\infty\right\}$. The class $\widetilde{H}(\beta,p,\psi)$ consists of all functions $f\in L_p$ that satisfy $\omega_\beta(\tilde{f},\delta)_p \ll \psi(\delta)$, where $\tilde{f}$ is the conjugate function of f ([14]). In the same way we define the class $\widetilde{BH}(\alpha,\beta,\psi,p,\theta,k)$. Now we can formulate the following embedding theorems.

THEOREM 5.1. *Suppose $\beta,k,\theta>0,\ 1\leq p\leq\infty$, and $\alpha(\cdot)$ satisfies the σ-condition with $\sigma=k\theta$. If the majorant $\psi(t)$ satisfies* ([1])*:*

$(B)\qquad \sum\limits_{k=n}^{\infty}\psi\left(\frac{1}{2^k}\right)=O\left[\psi\left(\frac{1}{2^n}\right)\right],$

$(B_\beta)\qquad \sum\limits_{k=0}^{n}2^{k\beta}\psi\left(\frac{1}{2^k}\right)=O\left[2^{n\beta}\psi\left(\frac{1}{2^n}\right)\right],$

then the classes $BH(\alpha,\beta,\psi,p,\theta,k)$ and $H\left(k+\beta,p,\left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta)\right)$ coincide.

Proof. Let n be an integer such that $\frac{1}{2^{n+1}}<\delta\leq\frac{1}{2^n}$. Note ([1]) that if $\psi(\delta)\in B$ and $\psi(\delta)\in B_\beta$, then for all $\theta>0$ we have

$$\sum_{k=n}^{\infty}\psi^\theta\left(\frac{1}{2^k}\right)\ll\psi^\theta\left(\frac{1}{2^n}\right)\ll\psi^\theta(\delta),\ \sum_{k=0}^{n}2^{k\beta\theta}\psi^\theta\left(\frac{1}{2^k}\right)\ll 2^{n\beta\theta}\psi^\theta\left(\frac{1}{2^n}\right)\ll\delta^{-\beta\theta}\psi^\theta(\delta).$$

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Therefore,

$$\int_0^{\delta} \alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt \ll \sum_{\nu=n}^{\infty} \int_{2^{-(\nu+1)}}^{2^{-\nu}} \alpha(t)\,(\mu_{[1/t]})^{-1}\psi^{\theta}(t)dt \ll \sum_{\nu=n}^{\infty} \psi^{\theta}\left(\frac{1}{2^{\nu}}\right) \ll \psi^{\theta}\left(\delta\right)$$

and

$$\delta^{\beta\theta}\int_{\delta}^{1} t^{-\beta\theta}\alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt \ll \delta^{\beta\theta}\sum_{k=0}^{n} 2^{k\beta\theta}\psi^{\theta}\left(\frac{1}{2^{k}}\right) \ll \psi^{\theta}\left(\delta\right).$$

In other words, if $f(x) \in H(k+\beta, p, \left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$, then $f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$. Let $f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$. Then

$$\psi^{\theta}\left(\delta\right) \gg \int_0^{\delta} \alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt + \delta^{\beta\theta}\int_{\delta}^{1} t^{-\beta\theta}\alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt \gg$$

$$\gg \left(\int_{2^{-(n+1)}}^{\delta} \alpha(t)\,dt + \delta^{\beta\theta}\int_{\delta}^{2^{-n}} t^{-\beta\theta}\alpha(t)\,dt\right)\omega_{k+\beta}^{\theta}(f,\delta)_p \gg \mu_{[1/\delta]}\,\omega_{k+\beta}^{\theta}(f,\delta)_p,$$

i.e., $f(x) \in H(k+\beta, p, \left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$. ☐

Using Theorems 4.2 and 5.1, one can prove

COROLLARY 5.2. *Under the conditions of Theorem 5.1, if $f(x) \in L_p\,(1<p<\infty)$ has quasimonotone Fourier coefficients and $\alpha(\cdot)$ satisfies (4.6), then*

$$f(x) \in H(k+\beta, p, \left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta)) \Longleftrightarrow a_n, b_n \ll \mu_n^{-\frac{1}{\theta}}\psi\left(\frac{1}{n}\right)n^{-(1-\frac{1}{p})}.$$

THEOREM 5.3. *Suppose $\beta, k, \theta > 0$, $1 \le p \le \infty$, and $\alpha(\cdot)$ satisfies the σ-condition with $\sigma = k\theta$. If the majorant $\psi(t)$ satisfies the following condition: $\psi(t) \ge C$ for $0 \le t \le 1$, then the classes $BH(\alpha,\beta,\psi,p,\theta,k)$ and $B(\alpha,k+\beta,p,\theta)$ coincide.*

Proof. Let $f(x) \in B(\alpha,k+\beta,p,\theta)$. We see that

$$\left(\int_0^{\delta} \alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt + \delta^{\beta\theta}\int_{\delta}^{1} t^{-\beta\theta}\alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt\right)^{\frac{1}{\theta}} \ll$$

$$\ll \left(\int_0^{1} \alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt\right)^{\frac{1}{\theta}} \ll C \le \psi\left(\delta\right).$$

Let $f(x) \in BH(\alpha,\beta,\psi,p,\theta,k)$. Then $\int_0^1 \alpha(t)\,\omega_{k+\beta}^{\theta}(f,t)_p dt \ll \psi^{\theta}\left(1\right) < \infty$. ☐

Using Theorems 4.2 and 5.3, one can prove

COROLLARY 5.4. *Under the conditions of Theorem 5.3, if $f(x) \in L_p\,(1<p<\infty)$ has quasimonotone Fourier coefficients and $\alpha(\cdot)$ satisfies (4.6), then*

$$f(x) \in B(\alpha,k+\beta,p,\theta) \Longleftrightarrow \sum_{\nu=1}^{\infty}(a_{\nu}+b_{\nu})^{\theta}\lambda_{\nu}\nu^{\theta-\frac{\theta}{p}} < \infty.$$

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THEOREM 5.5. *Let* $\beta, k > 0$, $0 < \theta \le 1$, $p = 1, \infty$, *and* $\alpha(t) = t^{-1}$. *Then*

$$BH(\alpha, \beta, \psi, p, \theta, k) \subset \widetilde{H}(\beta, p, \psi) \subset \widetilde{H}(k+\beta, p, \psi).$$

Proof. Let $f(x) \in BH(\alpha, \beta, \psi, p, \theta, k)$ and $\alpha(t) = t^{-1}$. It is easy to prove that $\Lambda_\nu \asymp 1$ $(\nu \in \mathbf{N} \cup \{0\})$. We need here the Stechkin inequality ([1]): $E_{2^n}(\tilde{f})_p \ll \sum_{\nu=n}^{\infty} E_{2^\nu}(f)_p$. If we combine this with Lemma 2.1, 2.5 and Theorem 3.1, we get

$$\omega_{k+\beta}(\tilde{f}, \frac{1}{2^n})_p \ll \omega_\beta^\theta(\tilde{f}, \frac{1}{2^n})_p \ll \left(2^{-n\beta} \sum_{\nu=0}^{n} 2^{\nu\beta} E_{2^\nu}(\tilde{f})_p\right)^\theta \ll$$

$$\ll 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \left(\sum_{\xi=\nu}^{\infty} E_{2^\xi}(f)_p\right)^\theta \ll 2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \left(\sum_{\xi=\nu}^{n} E_{2^\xi}(f)_p\right)^\theta +$$

$$+2^{-n\beta\theta} \sum_{\nu=0}^{n} 2^{\nu\beta\theta} \left(\sum_{\xi=n+1}^{\infty} E_{2^\xi}(f)_p\right)^\theta \ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} E_{2^\xi}^\theta(f)_p \sum_{\nu=0}^{\xi} 2^{\nu\beta\theta} + \sum_{\xi=n+1}^{\infty} E_{2^\xi}^\theta(f)_p \ll$$

$$\ll 2^{-n\beta\theta} \sum_{\xi=0}^{n} 2^{\xi\beta\theta} E_{2^\xi}^\theta(f)_p + \sum_{\xi=n+1}^{\infty} E_{2^\xi}^\theta(f)_p \ll \psi^\theta\left(\frac{1}{2^n}\right). \tag{5.1}$$

□

REMARK 5.6. *Let* $\beta_1, \beta_2 > 0$, $0 < \theta \le 1$. *If* $f(x) \in L_p$, $p = 1, \infty$, *then*

$$\omega_{\beta_1}(\tilde{f}, \delta)_p \ll \left(\int_0^\delta t^{-1}\, \omega_{\beta_2}^\theta(f,t)_p dt + \delta^{\beta_1\theta} \int_\delta^1 t^{-\beta_1\theta-1}\, \omega_{\beta_2}^\theta(f,t)_p dt\right)^{\frac{1}{\theta}}.$$

Proof. Applying Lemma 2.1 to (5.1), we obtain

$$\omega_{\beta_1}^\theta(\tilde{f}, \frac{1}{2^n})_p \ll 2^{-n\beta_1\theta} \sum_{\xi=0}^{n} 2^{\xi\beta_1\theta} \omega_{\beta_2}^\theta(f, 2^{-\xi})_p + \sum_{\xi=n+1}^{\infty} \omega_{\beta_2}^\theta(f, 2^{-\xi})_p \ll$$

$$\ll 2^{-n\beta_1\theta} \sum_{\xi=0}^{n-1} \omega_{\beta_2}^\theta(f, 2^{-\xi})_p \int_{1/2^{\xi+1}}^{1/2^\xi} t^{-\beta_1\theta-1} dt + \omega_{\beta_2}^\theta(f, 2^{-n})_p +$$

$$+ \sum_{\xi=n+1}^{\infty} \omega_{\beta_2}^\theta(\ f\ , 2^{-\xi})_p \int_{1/2^{\xi+1}}^{1/2^\xi} t^{-1} dt \ll 2^{-n\beta_1\theta} \sum_{\xi=0}^{n-1} \int_{1/2^{\xi+1}}^{1/2^\xi} t^{-\beta_1\theta-1}\, \omega_{\beta_2}^\theta(f,t)_p\, dt +$$

$$+\ 2^{-n\beta_1\theta} \int_{1/2^{n+1}}^{1/2^n} t^{-\beta_1\theta-1}\, \omega_{\beta_2}^\theta(f,t)_p dt + \sum_{\xi=n+1}^{\infty} \int_{1/2^{\xi+1}}^{1/2^\xi} t^{-1}\, \omega_{\beta_2}^\theta(f,t)_p\, dt.$$

Let n be an integer such that $\frac{1}{2^{n+1}} < \delta \le \frac{1}{2^n}$. Hence,

$$\omega^{\theta}_{\beta_1}(\tilde{f},\delta)_p \ll 2^{-n\beta_1\theta}\sum_{\xi=0}^{n-1}\int\limits_{1/2^{\xi+1}}^{1/2^{\xi}} t^{-\beta_1\theta-1}\,\omega^{\theta}_{\beta_2}(f,t)_p\,dt + \delta^{\beta_1\theta}\int\limits_{1/2^{n+1}}^{1/2^n} t^{-\beta_1\theta-1}\,\omega^{\theta}_{\beta_2}(f,t)_p dt +$$

$$+\sum_{\xi=n+1}^{\infty}\int\limits_{1/2^{\xi+1}}^{1/2^{\xi}} t^{-1}\,\omega^{\theta}_{\beta_2}(f,t)_p\,dt \ll \int\limits_0^{\delta} t^{-1}\,\omega^{\theta}_{\beta_2}(f,t)_p dt + \delta^{\beta_1\theta}\int\limits_{\delta}^{1} t^{-\beta_1\theta-1}\,\omega^{\theta}_{\beta_2}(f,t)_p dt.$$

□

COROLLARY 5.7. *Suppose the sequence $\{\mu_n\}$ is quasi-monotone increasing and quasi ε-power-monotone decreasing with $\varepsilon = -k\theta$. Under the conditions of Theorem 5.1, if $p = 1,\infty$, then the classes $BH(\alpha,\beta,\psi,p,\theta,k)$, $\widetilde{BH}(\alpha,\beta,\psi,p,\theta,k)$, $H(k+\beta,p,\left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$ and $\widetilde{H}(k+\beta,p,\left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$ coincide.*

Proof. By Theorem 5.1, it is sufficient to prove that the classes $H(k+\beta,p,\left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$ and $\widetilde{H}(k+\beta,p,\left(\mu_{[1/\delta]}\right)^{-\frac{1}{\theta}}\psi(\delta))$ coincide. This easily follows from the conditions on $\{\mu_n\}$ and the inequality of Remark 5.6 ($\theta = 1, \beta_1 = \beta_2 = k+\beta$). □

THEOREM 5.8. *Suppose $\beta,k,\theta > 0$, $p = 1,\infty$, and $\alpha(\cdot)$ satisfies the σ-condition with $\sigma = k\theta$, and*

$$\int\limits_{1/2^n}^{1}\alpha(t)dt \ll \int\limits_{1/2^{n+1}}^{1/2^n}\alpha(t)dt \qquad \textit{for} \quad n\in\mathbf{N}.$$

Then the classes $BH(\alpha,\beta,\psi,p,\theta,k)$ and $\widetilde{BH}(\alpha,\beta,\psi,p,\theta,k)$ coincide.

Proof. According to Theorem 3.1,

$$f(x)\in BH(\alpha,\beta,\psi,p,\theta,k) \Longleftrightarrow 2^{-n\beta\theta}\sum_{\nu=0}^{n}2^{\nu\beta\theta}\Lambda_\nu E^{\theta}_{\nu}(f)_p + \sum_{\nu=n+1}^{\infty}\Lambda_\nu E^{\theta}_{\nu}(f)_p \ll \psi^{\theta}\left(\frac{1}{2^n}\right).$$

Using the inequality ([1]) $E_{2^n}(\tilde{f})_p \ll \sum\limits_{\nu=n}^{\infty} E_{2^\nu}(f)_p$ and following the line of proof of Theorem 3.1 and 3.2 it easy to show that $2^{-n\beta\theta}\sum\limits_{\nu=0}^{n}2^{\nu\beta\theta}\Lambda_\nu E^{\theta}_{\nu}(\tilde{f})_p + \sum\limits_{\nu=n+1}^{\infty}\Lambda_\nu E^{\theta}_{\nu}(\tilde{f})_p \ll \psi^{\theta}\left(\frac{1}{2^n}\right)$, i.e., $f(x)\in\widetilde{BH}(\alpha,\beta,\psi,p,\theta,k)$. □

Note. Let $k,\beta,\theta > 0$, $k+\beta\in\mathbf{N}$ and $\alpha(t) = t^{-r\theta-1}$, $0 < r < k$. Then Theorems 3.1, 5.1, 5.3 are proved in [6]. If additionally $f(x)$ has monotonic Fourier coefficients, then Theorem 4.2 is established in [3], in Corollary 5.2 in [5], and Corollary 5.4 in [2].

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