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THE GENERALIZATIONS OF NEWTON'S INTERPOLATION FORMULA DUE TO MÜHLBACH AND ANDOYER*

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Abstract. Newton's formula for constructing the interpolation polynomial is well–known. It makes use of divided differences. It was generalized around 1971–1973 by Mühlbach for interpolation by a linear family of functions forming a complete Chebyshev system. This generalization rests on a generalization of divided differences due to Popoviciu. In this paper, it is shown that Mühlbach's formula is related to the work of Andoyer which goes back to the beginning of the century.

Key words. interpolation, divided differences, biorthogonality.

AMS subject classifications. 65D05, 41A05.

1. Introduction. Newton's formula for interpolation by a polynomial was given by Isaac Newton (Woolsthorpe, 25.12.1642 – London, 20.3.1727) as Lemma 5 of Book III of his *Principia Mathematica* of 1687 [22] but it was known to him before since he mentioned it in a letter to the German scientist Henry Oldenburg (1618–1677) dated October 24, 1676.

According to this formula, the polynomial P_n such that

$$P_n(x_i) = f(x_i), \quad \text{for} \quad i = 0, \ldots, n,$$

is given by

$$\begin{aligned} P_n(x) &= [x_0] + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &\quad + \cdots + (x - x_0) \cdots (x - x_{n-1})[x_0, \ldots, x_n] \end{aligned}$$

where the quantities into square brackets are the usual divided differences defined recursively by

$$[x_{p_0}, \ldots, x_{p_n}] = \frac{[x_{p_0}, \ldots, x_{p_{n-1}}] - [x_{p_1}, \ldots, x_{p_n}]}{x_{p_0} - x_{p_n}}$$

with $[x_{p_i}] = f(x_{p_i})$. It is well known that these divided differences can also be written as a ratio of two determinants (see, for example, [9]). An account on the history of interpolation can be found in [6].

In this paper, we shall be interested in the generalization of Newton's formula for interpolation by a linear combination of the form

$$P_n(x) = a_0\varphi^0(x) + \cdots + a_n\varphi^n(x)$$

where the φ^i's are given functions which are assumed to satisfy

$$\begin{vmatrix} \varphi^0(z_0) & \cdots & \varphi^0(z_k) \\ \vdots & & \vdots \\ \varphi^k(z_0) & \cdots & \varphi^k(z_k) \end{vmatrix} \neq 0$$

for all choices of the distinct points $z_0, \ldots, z_k$ and for $k = 0, \ldots, n$. In that case, the functions $\varphi^0, \ldots, \varphi^n$ are said to form a complete Chebyshev system.

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2. Mühlbach's formula.

Interpolation by a linear family of functions forming a complete Chebyshev system was addressed by Mühlbach in his thesis [15] and in his subsequent papers [16, 17, 18, 19, 20]. He obtained a generalization of Newton's interpolation formula which is

$$P_n(x) = \sum_{i=0}^{n} \left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^i \\ x_0 & \cdots & x_i \end{array} \middle| f \right] g_i(x)$$

with $g_i(x) = \varphi^i(x) - R_{i-1}(x)$ where R_{i-1} is the linear combination of $\varphi^0, \ldots, \varphi^{i-1}$ which satisfies the interpolation conditions

$$R_{i-1}(x_j) = \varphi^i(x_j) \quad \text{for} \quad j = 0, \ldots, i-1.$$

Of course it holds

$$P_k(x) = P_{k-1}(x) + \left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^k \\ x_0 & \cdots & x_k \end{array} \middle| f \right] g_k(x) \tag{2.1}$$

for $k = 0, 1, \ldots$, with $P_{-1}(x) \equiv 0$.

The quantities

$$\left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^i \\ x_0 & \cdots & x_i \end{array} \middle| f \right]$$

are the generalized divided differences introduced by Popoviciu [23] as a ratio of two determinants

$$\left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^i \\ x_m & \cdots & x_{m+i} \end{array} \middle| f \right] = \left| \begin{array}{ccc} \varphi^0(x_m) & \cdots & \varphi^0(x_{m+i}) \\ \vdots & & \vdots \\ \varphi^{i-1}(x_m) & \cdots & \varphi^{i-1}(x_{m+i}) \\ f(x_m) & \cdots & f(x_{m+i}) \end{array} \right| \Bigg/ \left| \begin{array}{ccc} \varphi^0(x_m) & \cdots & \varphi^0(x_{m+i}) \\ \vdots & & \vdots \\ \varphi^i(x_m) & \cdots & \varphi^i(x_{m+i}) \end{array} \right|.$$

As proved by Mühlbach [15, 16, 17], these generalized divided differences can be recursively computed by

$$\left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^i \\ x_m & \cdots & x_{m+i} \end{array} \middle| f \right] = \frac{\left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^{i-1} \\ x_{m+1} & \cdots & x_{m+i} \end{array} \middle| f \right] - \left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^{i-1} \\ x_m & \cdots & x_{m+i-1} \end{array} \middle| f \right]}{\left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^{i-1} \\ x_{m+1} & \cdots & x_{m+i} \end{array} \middle| \varphi^i \right] - \left[\begin{array}{ccc} \varphi^0 & \cdots & \varphi^{i-1} \\ x_m & \cdots & x_{m+i-1} \end{array} \middle| \varphi^i \right]}$$

with

$$\left[\begin{array}{c} \varphi^0 \\ x_m \end{array} \middle| f \right] = f(x_m)/\varphi^0(x_m) \quad \text{and} \quad \left[\begin{array}{c} \varphi^0 \\ x_m \end{array} \middle| \varphi^i \right] = \varphi^i(x_m)/\varphi^0(x_m).$$

The computation of P_n can also be performed via a generalization of the Neville–Aitken algorithm. This generalization was obtained by Mühlbach [18, 19] from the preceding recurrence relation for the generalized divided differences. It can also be derived by applying Sylvester's identity to the determinantal expression of P_n [3]. This algorithm was called the Mühlbach–Neville–Aitken algorithm. It gave rise to the E–algorithm which is the most general extrapolation algorithm known [2, 10, 12, 24]. See [4] for a survey including historical remarks about the synthesis of these algorithms, their generalizations and their applications. The corresponding subroutines, theoretical and numerical results can be found in [7]. These questions are very much related to biorthogonality as exposed in [9] and [5].

3. The work of Andoyer.

The *Encyklopädie der Mathematischen Wissenschaften* [13] is a series of books published from 1898 to 1904 under the supervision of Friedrich Wilhelm Franz Meyer (2.9.1856 – 11.4.1934), a Professor at the Universities of Clausthal and Königsberg. Each volume contains a series of articles mostly written by German mathematicians, and the whole project covers arithmetics and algebra, analysis, geometry, mechanics, physics, geodesics and geophysics, and astronomy. The idea of the project arose from a meeting in September 1894 between Meyer, Felix Klein (25.4.1849 – 22.6.1925) and Heinrich Weber (5.3.1842 – 17.5.1913). The article *Interpolation* appears in Tome II, vol.1, I D3, pp.799–820. It was written by Julius Bauschinger (1860–1934). This *Encyklopädie* was translated into French and published between 1904 and 1912 under the editorship of Jules Molk (1857–1914), a Professor at the University of Nancy [14]. This work was not only a translation. Each article was first revised by its German author and the French translator added new material (indicated between two *'s). The article on interpolation was translated and completed by the French astronomer and mathematician Henri Marie Andoyer (Paris, 1.10.1862 – Paris, 12.6.1929). It can be found in Tome I, vol. 4, fasc. 1, I–21. It is dated 20 March 1906. On pages 129 and 130 of the French edition, Andoyer gave a contribution of his (beginning page 127 and ending page 130) dealing with a general interpolation process. It seems that [14] is the only place where Andoyer published his results.

We shall now present the work of Andoyer using the same notation as in the preceding section. This notation is in fact Andoyer's except that $\varphi(x)$ is replaced by $P_n(x)$ and that Δ is replaced by δ to avoid possible confusion with the forward difference operator.

The problem is the same as above. It consists in finding

$$P_n(x) = \sum_{i=0}^{n} a_i \varphi^i(x)$$

such that $P_n(x_j) = f(x_j)$ for $j = 0, \ldots, n$ where the quantities $f(x_j)$ are not all zero. For simplicity we shall sometimes make use of the notation

$$\varphi_j^i = \varphi^i(x_j) \quad \text{and} \quad f_j = f(x_j).$$

We have

$$\begin{array}{ccccccccc} P_n(x) & - & a_0\varphi^0(x) & - & \cdots & - & a_n\varphi^n(x) & = & 0, \\ f_0 & - & a_0\varphi_0^0 & - & \cdots & - & a_n\varphi_0^n & = & 0, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\ f_n & - & a_0\varphi_n^0 & - & \cdots & - & a_n\varphi_n^n & = & 0. \end{array}$$

Since the solution of this system is not identically zero, its determinant vanishes, and we obtain

$$\begin{vmatrix} P_n & \varphi^0 & \cdots & \varphi^n \\ f_0 & \varphi_0^0 & \cdots & \varphi_0^n \\ \vdots & \vdots & & \vdots \\ f_n & \varphi_n^0 & \cdots & \varphi_n^n \end{vmatrix} = 0, \tag{3.1}$$

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and it follows that

$$P_n(x) = -\begin{vmatrix} 0 & \varphi^0(x) & \cdots & \varphi^n(x) \\ f_0 & \varphi_0^0 & \cdots & \varphi_0^n \\ \vdots & \vdots & & \vdots \\ f_n & \varphi_n^0 & \cdots & \varphi_n^n \end{vmatrix} \Bigg/ \begin{vmatrix} \varphi_0^0 & \cdots & \varphi_0^n \\ \vdots & & \vdots \\ \varphi_n^0 & \cdots & \varphi_n^n \end{vmatrix}.$$

By combining the rows and the columns in (3.1) we shall now reduce by one the dimension of this determinant. For that purpose, Andoyer introduced the functions $\delta_k u(x)$ (he used the letter Δ instead of δ) defined recursively by

$$\delta_{k+1}u(x) = \delta_k u(x) - \frac{\delta_k \varphi^k(x)}{\delta_k \varphi^k(x_p)} \delta_k u(x_p)$$

where u is an arbitrary function (that will be later P_n or φ^i), p an arbitrary index (which can depend on k) greater or equal to k (Andoyer only considered the case $p = k$) and $\delta^0 u(x) = u(x)$. We have $\delta_{k+1}\varphi^k(x) \equiv 0$. Andoyer also defined the quantities $\delta^k u(x_i)$ by

$$\delta_{k+1}u(x_i) = \delta_k u(x_i) - \frac{\delta_k \varphi^k(x_i)}{\delta_k \varphi^k(x_p)} \delta_k u(x_p)$$

with $\delta^0 u(x_i) = u(x_i)$ where p is an arbitrary index (which can depend on k and i) strictly smaller than i and greater or equal to k, and $i = k+1, k+2, \ldots$. Obviously if $p = k$, which is always possible since p and k are both stricly smaller than i, then these two formulae reduce to a single one. Of course the quantities $\delta_k \varphi^k(x_p)$ are all assumed to be nonzero. It remains open how the necessary condition $\delta_k \varphi^k(x_p) \neq 0$ follows from the assumption that $\varphi^0, \ldots, \varphi^n$ form a complete Chebyshev system. For the choice $p = k$, made by Andoyer, this becomes clear from an induction proof which derives Mühlbach's recurrence formula for the generalized divided differences from Andoyer's calculations (I am indebted to the referee for this remark).

Let us multiply the row with the lower index p in the determinant (3.1) by $\delta_0\varphi^0(x)/\delta_0\varphi^0(x_p)$ for $p \geq 0$ and subtract it from the first row. Then, for the row with the lower index $i = n, n-1, \ldots, 1$, let us multiply the row with the lower index p, where $0 \leq p < i$, by $\delta_0\varphi^0(x_i)/\delta_0\varphi^0(x_p)$ and subtract it from that row. These linear combinations of the rows of the determinant (3.1) do not change its value. Making use of the definition of δ_1, we finally obtain

$$\begin{vmatrix} \delta_1 P_n(x) & 0 & \delta_1\varphi^1(x) & \cdots & \delta_1\varphi^n(x) \\ \delta_0 f(x_0) & \delta_0\varphi^0(x_0) & \delta_0\varphi^1(x_0) & \cdots & \delta_0\varphi^n(x_0) \\ \delta_1 f(x_1) & 0 & \delta_1\varphi^1(x_1) & \cdots & \delta_1\varphi^n(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_1 f(x_n) & 0 & \delta_1\varphi^1(x_n) & \cdots & \delta_1\varphi^n(x_n) \end{vmatrix} = 0.$$

Thus, assuming that $\delta_0\varphi^0(x_0) \neq 0$, we have

$$\begin{vmatrix} \delta_1 P_n(x) & \delta_1\varphi^1(x) & \cdots & \delta_1\varphi^n(x) \\ \delta_1 f(x_1) & \delta_1\varphi^1(x_1) & \cdots & \delta_1\varphi^n(x_1) \\ \vdots & \vdots & & \vdots \\ \delta_1 f(x_n) & \delta_1\varphi^1(x_n) & \cdots & \delta_1\varphi^n(x_n) \end{vmatrix} = 0.$$

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It follows that

$$\delta_1 P_n(x) = -\begin{vmatrix} 0 & \delta_1\varphi^1(x) & \cdots & \delta_1\varphi^n(x) \\ \delta_1 f(x_1) & \delta_1\varphi^1(x_1) & \cdots & \delta_1\varphi^n(x_1) \\ \vdots & \vdots & & \vdots \\ \delta_1 f(x_n) & \delta_1\varphi^1(x_n) & \cdots & \delta_1\varphi^n(x_n) \end{vmatrix} \Bigg/ \begin{vmatrix} \delta_1\varphi^1(x_1) & \cdots & \delta_1\varphi^n(x_1) \\ \vdots & & \vdots \\ \delta_1\varphi^1(x_n) & \cdots & \delta_1\varphi^n(x_n) \end{vmatrix}.$$

The same process can now be repeated on this new determinant and, after the k–th step, we obtain for $k = 1, \cdots, n$

$$\begin{vmatrix} \delta_k P_n(x) & \delta_k\varphi^k(x) & \cdots & \delta_k\varphi^n(x) \\ \delta_k f(x_k) & \delta_k\varphi^k(x_k) & \cdots & \delta_k\varphi^n(x_k) \\ \vdots & \vdots & & \vdots \\ \delta_k f(x_n) & \delta_k\varphi^k(x_n) & \cdots & \delta_k\varphi^n(x_n) \end{vmatrix} = 0.$$

Let us multiply the row with the lower index p in this determinant by $\delta_k\varphi^k(x)/\delta_k\varphi^k(x_p)$ for $p \geq k$ and subtract it from the first row. Then, for the row with the lower index $i = n, n-1, \ldots, k+1$, let us multiply the row with the lower index p, where $k \leq p < i$, by $\delta_k\varphi^k(x_i)/\delta_k\varphi^k(x_p)$ and subtract it from that row. These linear combinations do not change the value of the determinant. Making use of the definition of δ_{k+1}, we obtain

$$\begin{vmatrix} \delta_{k+1} P_n(x) & 0 & \delta_{k+1}\varphi^{k+1}(x) & \cdots & \delta_{k+1}\varphi^n(x) \\ \delta_k f(x_k) & \delta_k\varphi^k(x_k) & \delta_k\varphi^{k+1}(x_k) & \cdots & \delta_k\varphi^n(x_k) \\ \delta_{k+1} f(x_{k+1}) & 0 & \delta_{k+1}\varphi^{k+1}(x_{k+1}) & \cdots & \delta_{k+1}\varphi^n(x_{k+1}) \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_{k+1} f(x_n) & 0 & \delta_{k+1}\varphi^{k+1}(x_n) & \cdots & \delta_{k+1}\varphi^n(x_n) \end{vmatrix} = 0.$$

Thus, assuming that $\delta_k\varphi^k(x_k) \neq 0$, we have

$$\begin{vmatrix} \delta_{k+1} P_n(x) & \delta_{k+1}\varphi^{k+1}(x) & \cdots & \delta_{k+1}\varphi^n(x) \\ \delta_{k+1} f(x_1) & \delta_{k+1}\varphi^{k+1}(x_{k+1}) & \cdots & \delta_{k+1}\varphi^n(x_{k+1}) \\ \vdots & \vdots & & \vdots \\ \delta_{k+1} f(x_n) & \delta_{k+1}\varphi^{k+1}(x_n) & \cdots & \delta_{k+1}\varphi^n(x_n) \end{vmatrix} = 0.$$

It follows that

$$\delta_k P_n(x) = -\begin{vmatrix} 0 & \delta_k\varphi^k(x) & \cdots & \delta_k\varphi^n(x) \\ \delta_k f(x_k) & \delta_k\varphi^k(x_k) & \cdots & \delta_k\varphi^n(x_k) \\ \vdots & \vdots & & \vdots \\ \delta_k f(x_n) & \delta_k\varphi^k(x_n) & \cdots & \delta_k\varphi^n(x_n) \end{vmatrix} \Bigg/ \begin{vmatrix} \delta_k\varphi^k(x_k) & \cdots & \delta_k\varphi^n(x_k) \\ \vdots & & \vdots \\ \delta_k\varphi^k(x_n) & \cdots & \delta_k\varphi^n(x_n) \end{vmatrix}. \tag{3.2}$$

When $k = n+1$, we have

$$\delta_{n+1} P_n(x) \equiv 0,$$

and thus

$$\begin{aligned} \delta_{n+1} P_n(x) &= \delta_n P_n(x) - \frac{\delta_n f(x_n)}{\delta_n\varphi^n(x_n)}\delta_n\varphi^n(x) = 0, \\ \delta_n P_n(x) &= \delta_{n-1} P_n(x) - \frac{\delta_{n-1} f(x_{n-1})}{\delta_{n-1}\varphi^{n-1}(x_{n-1})}\delta_{n-1}\varphi^{n-1}(x), \\ \cdots\cdots \;&\cdots\; \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \delta_1 P_n(x) &= \delta_0 P_n(x) - \frac{\delta_0 f(x_0)}{\delta_0\varphi^0(x_0)}\delta_0\varphi^0(x). \end{aligned}$$

By summing up these equations we obtain, since $\delta_0 P_n(x) = P_n(x)$,

$$P_n(x) = \sum_{i=0}^{n} \frac{\delta_i f(x_i)}{\delta_i \varphi^i(x_i)} \delta_i \varphi^i(x). \tag{3.3}$$

This is the generalization of Newton's interpolation formula obtained by Andoyer.

Of course it holds that

$$P_k(x) = P_{k-1}(x) + \frac{\delta_k f(x_k)}{\delta_k \varphi^k(x_k)} \delta_k \varphi^k(x) \tag{3.4}$$

for $k = 0, 1, \ldots$, with $P_{-1}(x) \equiv 0$.

Writing the preceding relations for an arbitrary function u and for the indexes $1, \ldots, k$, and summing them up, leads to

$$u(x) = \delta_k u(x) + Q_{k-1}(x) \tag{3.5}$$

with

$$Q_{k-1}(x) = \sum_{i=0}^{k-1} \frac{\delta_i u(x_i}{\delta_i \varphi^i(x_i)} \delta_i \varphi^i(x).$$

By formula (3.4), Q_{k-1} is the unique element of $\mathrm{span}\{\varphi^0, \ldots, \varphi^{k-1}\}$ which satisfies the interpolation conditions $Q_{k-1}(x_i) = u(x_i)$ for $i = 0, \ldots, k-1$.

Let us also remark that writing (3.5) for $u \equiv P_n$ and subtracting from (3.3) gives

$$\delta_k P_n(x) = \sum_{i=k}^{n} \frac{\delta_i f(x_i)}{\delta_i \varphi^i(x_i)} \delta_i \varphi^i(x)$$

which corresponds to (3.2).

Another by–product of the theory is a formula for the error. Replacing k by $n+1$ in (3.5) and writing it for the function f, leads to

$$f(x) - P_n(x) = \delta_{n+1} f(x).$$

This formula should be compared to the error formula given by Mühlbach [17].

The technique of Andoyer for reducing the order of the determinants is essentially the same as the method proposed by Felice Chiò (Crescentino, 29.4.1813 – Torino, 28.5.1871) in 1853 [8]. It is called *compression* or *condensation*, see [1, p.46], and it was virtually used by Carl Friedrich Gauss (Braunschweig, 23.04.1777 – Göttingen, 22.02.1855) more than forty years earlier when evaluating symmetric determinants. The transformation of a determinant into one of the next lower order can also be viewed as a generalization of the procedure given by Charles Hermite (Dieuze, 24.12.1822 – Paris, 14.01.1901) in 1849 [11] for the order 4. On these questions, consult [21, pp.79–81].

4. The connection.

We shall now prove that the formula (2.1) of Mühlbach and the formula (3.4) of Andoyer, when p is always chosen to be equal to k, are the same. For that purpose, we have to show that

$$\frac{\delta_k f(x_k)}{\delta_k \varphi^k(x_k)} = \left[\begin{array}{ccc|c} \varphi^0 & \cdots & \varphi^k & f \\ x_0 & \cdots & x_k & \end{array} \right]$$

and that $g_k(x) = \delta_k \varphi^k(x)$.

Let us begin by proving the last result. Since (3.4) holds for any function f, we have for $f \equiv \varphi^k$

$$R_k(x) = R_{k-1}(x) + \delta_k \varphi^k(x).$$

But, since $R_k, \varphi^k \in \operatorname{span}\left\{\varphi^0, \ldots, \varphi^k\right\}$, we have $R_k \equiv \varphi^k$, and it follows that

$$\varphi^k(x) - R_{k-1}(x) = g_k(x) = \delta_k \varphi^k(x).$$

This result can also be obtained directly from (3.5) by taking $u \equiv \varphi^k$ since, in that case, $Q_{k-1} \equiv R_{k-1}$.

The first result will now be proved by induction. It is true for $k = 0$ and for any function f. Let us assume that the result is true for the index k. We have, from the recursive definition of δ_{k+1}, and for the choice $p = k$ that

$$\frac{\delta_{k+1} f(x_{k+1})}{\delta_{k+1} \varphi^{k+1}(x_{k+1})} = \frac{\delta_k \varphi^k(x_k) \delta_k f(x_{k+1}) - \delta_k \varphi^k(x_{k+1}) \delta_k f(x_k)}{\delta_k \varphi^k(x_k) \delta_k \varphi^{k+1}(x_{k+1}) - \delta_k \varphi^k(x_{k+1}) \delta_k \varphi^{k+1}(x_k)}.$$

Dividing the numerator and the denominator in the right hand side by $\delta_k \varphi^k(x_k) \delta_k \varphi^k(x_{k+1})$, this ratio is equal to

$$\frac{\dfrac{\delta_k f(x_{k+1})}{\delta_k \varphi^k(x_{k+1})} - \dfrac{\delta_k f(x_k)}{\delta_k \varphi^k(x_k)}}{\dfrac{\delta_k \varphi^{k+1}(x_{k+1})}{\delta_k \varphi^k(x_{k+1})} - \dfrac{\delta_k \varphi^{k+1}(x_k)}{\delta_k \varphi^k(x_k)}}$$

which is the recurrence formula of Mühlbach for its generalized divided differences since

$$\frac{\delta_k f(x_{k+1})}{\delta_k \varphi^k(x_{k+1})} = \left[\begin{array}{ccc|c} \varphi^0 & \cdots & \varphi^k & \multirow{2}{*}{f} \\ x_1 & \cdots & x_{k+1} & \end{array} \right]$$

and since the result holds for any function f.

Thus, we proved that the formulae of Mühlbach and Andoyer (when $p = k$) are the same.

Let us also remark that

$$\frac{\delta_k f(x_k)}{\delta_k \varphi^k(x_k)} = \frac{f(x_k) - P_{k-1}(x_k)}{\delta_k \varphi^k(x_k)}$$

and that the procedure of Andoyer and the preceding results can be extended to the general interpolation problem as described in [9] and developed in [5].

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