Electronic Transactions on Numerical Analysis.
Volume 20, pp. 154-163, 2005.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

CONVERGENCE ANALYSIS OF THE ROTATED $\mathcal{Q}_1$ ELEMENT ON ANISOTROPIC RECTANGULAR MESHES*

SHIPENG MAO† AND SHAOCHUN CHEN‡

Abstract. The main aim of this paper is to study the convergence of the well-known nonconforming rotated $\mathcal{Q}_1$ element for the second order elliptic problems on anisotropic rectangular meshes, i.e., the meshes considered in our work do not satisfy the regular assumption. Lastly, a numerical test is carried out, which coincides with our theoretical analysis.

Key words. anisotropic, interpolation error, nonconforming, the rotated $\mathcal{Q}_1$ element

AMS subject classifications. 65N30, 65N15

1. Introduction. The regular assumption (cf. [8], [13]) of finite element meshes is a basic condition in the convergence analysis of finite element methods (FEMs), whereas some early papers have been written to prove error estimates under more general conditions (cf. [7], [19]). Recently, much attention is paid to FEMs on anisotropic meshes. In particular, for rectangular meshes we refer to Acosta [1], [2], Apel [3], [4], [5], [6], Chen [11], [12], Duran [16], [17], Shenk [24] and references therein. The studies mainly concentrated on some Lagrange type elements (conforming C^0 elements). But nonconforming methods are hardly treated, as far as we know, there are few papers on the nonconforming elements under anisotropic meshes. On the other hand, most of the former works are concentrated on the estimates of the interpolation error under anisotropic meshes, in particular, the readers are refer to [4] and [11] for some techniques of the anisotropic interpolation error estimates. However, some elements do not satisfy the anisotropic interpolation properties. A case in point is the famous rotated $\mathcal{Q}_1$ element of Rannacher and Turek [23]. In this paper, we will show that the interpolation error of the rotated $\mathcal{Q}_1$ element is not convergent, while it can be applied to anisotropic rectangular meshes.

The goal of this paper is to obtain the error estimates of the rotated $\mathcal{Q}_1$ nonconforming element on anisotropic rectangular meshes. Our estimates improve the previous known results in several aspects: Firstly, the anisotropic approximation error is obtained in a different way. The interpolation error of the rotated $\mathcal{Q}_1$ element is not convergent under anisotropic rectangular meshes. We overcome this difficulty by constructing another operator $T_h : H^2(\Omega) \longrightarrow V_h$ instead of the interpolation operator. Then we come to the interesting conclusion that the interpolation error may say nothing about the convergence of some FEMs. Secondly, our results improve the previous consistency error estimate of the rotated $\mathcal{Q}_1$ element. In section 3, we mainly study the consistency error of the rotated $\mathcal{Q}_1$ element under anisotropic rectangular meshes. Under moderate smoothness of the solution, accuracy with $O(h)$ and $O(h^2)$ order of the consistency error are both obtained by the trick of element cancellation. Lastly, the techniques developed in our analysis can give some hints to other nonconforming elements under anisotropic rectangular meshes.

Before the end of this section, we will recall some notations and terminology (or refer to [8], [13]). Let $(\cdot,\cdot)$ denote the usual L^2 -inner product and $\|u\|_{r,p,\Omega}$ (resp. $|u|_{r,p,\Omega}$) be the

* Received June 28, 2004. Accepted for publication June 19, 2005. Recommended by R. Lehoucq. The research is supported by NSF of China (No.10471133, 10590353) and the project of Creative Engineering of Henan Province of China.

† Department of Mathematics, Zhengzhou University, 450052, China. Current address: Institute of Computational Mathematics, Chinese Academy of Science, P.O. Box 2719, Beijing, 100080, China. (shipengmao@yahoo.com.cn).

‡Department of Mathematics, Zhengzhou University, 450052, China. (shchchen@zzu.edu.cn).

ETNA
Kent State University
etna@mcs.kent.edu

usual norm (resp. semi-norm) for the Sobolev space $W^{r,p}(\Omega)$. When $p=2$, denote $W^{r,2}(\Omega)$ by $H^r(\Omega)$.

2. Nonconforming rotated $\mathcal{Q}_1$ element and its approximation error estimate on anisotropic rectangular meshes. For simplicity, assume that Ω is a polygon with boundaries parallel to the axes and let $\mathcal{J}_h$ be a partition of Ω by rectangular meshes which need not satisfy the regular assumption. $\forall K \in \mathcal{J}_h$, denote the barycenter of element K by (x_K, y_K), the length of edges parallel to x-axis and y-axis by $2h_{K1}, 2h_{K2}$ respectively, $h_K = \max\{h_{K1}, h_{K2}\}, h = \max\limits_{K\in\mathcal{J}_h} h_K$. Assume that $\widehat{K} = [-1,1]\times[-1,1]$ is the reference element, the four vertices are: $\widehat{a}_1 = (-1,-1), \widehat{a}_2 = (1,-1), \widehat{a}_3 = (1,1), \widehat{a}_4 = (-1,1)$, and its 4 sides are $\widehat{l}_1 = \widehat{a}_1\widehat{a}_2, \widehat{l}_2 = \widehat{a}_2\widehat{a}_3, \widehat{l}_3 = \widehat{a}_3\widehat{a}_4, \widehat{l}_4 = \widehat{a}_4\widehat{a}_1$. Then there exists an unique mapping $F_K : \widehat{K} \longrightarrow K$ defined as

$$\begin{cases} x = x_K + h_{K1}\xi, \\ y = y_K + h_{K2}\eta. \end{cases}$$

To begin with, we will introduce the finite element space of the rotated $\mathcal{Q}_1$ element.
Set

$$\hat{P} = span\{1, \xi, \eta, \xi^2 - \eta^2\},$$

and $\forall F \subset \partial K$, for any $v \in H^1(K)$, we define

$$M_F(v) = \frac{1}{|F|}\int_F v ds.$$

Then the finite element space is defined as

$$V_h = \{v \in L^2(\Omega) | v\circ F_K \in \hat{P}, v \text{ is continuous regarding } M_F(\cdot), M_F(v) = 0, \forall F \in \partial\Omega\}.$$

We will define two interpolations $\hat{I}$ and $\hat{\Pi}$ on $H^1(\hat{K}), H^2(\hat{K})$ respectively,

$$\begin{aligned} \hat{I}\hat{v} &= \frac{\widehat{v}_{12} + \widehat{v}_{23} + \widehat{v}_{34} + \widehat{v}_{41}}{4} + \frac{\widehat{v}_{23} - \widehat{v}_{41}}{2}\xi + \frac{\widehat{v}_{34} - \widehat{v}_{12}}{2}\eta \\ &+ \frac{3(-\widehat{v}_{12} + \widehat{v}_{23} - \widehat{v}_{34} + \widehat{v}_{41})}{8}(\xi^2 - \eta^2), \\ \hat{\Pi}\hat{v} &= \frac{\hat{v}_1 + \hat{v}_2 + \hat{v}_3 + \hat{v}_4}{4} + \frac{-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4}{4}\xi + \frac{-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4}{4}\eta \\ &+ \frac{\hat{v}_1 - \hat{v}_2 + \hat{v}_3 - \hat{v}_4}{4}\xi\eta, \end{aligned}$$

where $\hat{v}_{i(i+1)} = \frac{1}{|\hat{l}_i|}\int_{\widehat{l}_i} \hat{v} d\widehat{s}, \;\; \hat{v}_i = \hat{v}(\hat{a}_i), \;\; i = 1,2,3,4$. Let $I_h|_K = I_K$ and $\Pi_h|_K = \Pi_K$ be the interpolations of the rotated $\mathcal{Q}_1$ and bilinear element on K, respectively, where $I_K = \hat{I}\circ F_K^{-1}, \Pi_K = \hat{\Pi}\circ F_K^{-1}$.

For the convenience of simplicity, we consider the following Poisson problem:

$$\begin{cases} -\triangle u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then the weak form of (2.1) is:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega), & \text{such that} \\ a(u,v) = f(v), & \forall\, v \in H_0^1(\Omega), \end{cases} \tag{2.2}$$

where

$$a(u,v)=\int_\Omega \nabla u\nabla v dxdy,\quad f(v)=\int_\Omega fvdxdy.$$

The approximation of (2.2) reads as:

$$\left\{\begin{array}{ll}\text{Find } u_h\in V_h, & \text{such that}\\ a_h(u_h,v_h)=f(v_h), & \forall\, v_h\in V_h\end{array}\right. \tag{2.3}$$

with

$$a_h(u_h,v_h)=\sum_{K\in\mathcal{J}_h}\int_K \nabla u_h\nabla v_h dxdy.$$

We will define the broken norm as

$$\|\cdot\|_h=\left(\sum_{K\in\mathcal{J}_h}|\cdot|^2_{1,K}\right)^{\frac12}.$$

It is obvious that $\|\cdot\|_h$ is a norm on V_h.

The following theorem shows that the interpolation I_h is unstable or even divergent in the energy norm sense under anisotropic rectangular meshes.

THEOREM 2.1. *(cf.[22]) Let Ω be a rectangular and a uniform mesh division in each direction with the diameter h and h' respectively, and $1<\frac{h}{h'}<\sigma$. we have*

$$\|u-I_hu\|_h\le Ch(1+h/h')|u|_{2,\Omega}. \tag{2.4}$$

It's seen from Theorem 2.1 that the aspect ratio h/h' appears in the right hand side of (2.4). The following counterexample shows that this is the true case (cf. [5]).

Taking an element $K=[-h_x,h_x]\times[-h_y,h_y](h_x\gg h_y), u=x^2$, then by a direct calculation, we have

$$\hat{I}_{\hat{K}}\hat{u}=\frac23 h_x^2-\frac12 h_x^2(\xi^2-\eta^2),$$

$$\|\frac{\partial(u-I_Ku)}{\partial y}\|_{0,K}=h_y^{-1}(h_xh_y)^{\frac12}\|\frac{\partial(\hat{u}-\hat{I}_{\hat{K}}\hat{u})}{\partial\eta}\|_{0,\hat{K}}=\frac{2}{\sqrt3}h_x^{\frac52}h_y^{-\frac12},$$

$$|u|_{2,K}=(\int_K 2^2dxdy)^{\frac12}=4h_x^{\frac12}h_y^{\frac12}.$$

So

$$\frac{|u-I_Ku|_{1,K}}{|u|_{2,K}}\ge\frac{\|\frac{\partial(u-\Pi_Ku)}{\partial y}\|_{0,K}}{|u|_{2,K}}=\frac{\sqrt3}{6}\frac{h_x^2}{h_y}\longrightarrow\infty. \tag{2.5}$$

From the above discussion one may ask if the rotated $\mathcal{Q}_1$ element is convergent under anisotropic rectangular meshes, the answer is affirmative. In fact, from the numerical test and theoretical analysis in this paper, one can see that the rotated $\mathcal{Q}_1$ element preserves the

ETNA
Kent State University
etna@mcs.kent.edu

optimal order of convergence even if the interpolation error is divergent, which is a very interesting result.

In order to obtain the anisotropic approximation error estimate, we construct the operator $T_h : H^2(\Omega) \longrightarrow V_h$ with $T_K = I_K \Pi_K, T_h|_K = T_K, \forall K \in \mathcal{J}_h$.

LEMMA 2.2. *For the operator T_h, there holds:*

$$\|\hat{D}^\alpha(\hat{v} - \hat{T}\hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^\alpha \hat{v}|_{1,\hat{K}}, \forall\ \hat{v} \in H^2(\hat{K}),\ \ |\alpha| = 1. \tag{2.6}$$

Proof. A simple calculation gives

$$\hat{T}\hat{v} = \frac{\hat{v}_1 + \hat{v}_2 + \hat{v}_3 + \hat{v}_4}{4} + \frac{-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4}{4}\xi + \frac{-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4}{4}\eta.$$

For the case $\alpha = (1, 0)$,

$$\hat{D}^\alpha \hat{T}\hat{v} = \frac{1}{|\hat{K}|}\int_{-1}^{1} (\hat{D}^\alpha \hat{v}(\xi, -1)d\xi + \hat{D}^\alpha \hat{v}(\xi, 1))d\xi \triangleq F(\hat{D}^\alpha \hat{v}). \tag{2.7}$$

It can be verified that

$$\begin{cases} |F(\hat{w})| \leq C\|\hat{w}\|_{1,\hat{K}}, \forall \hat{w} \in H^1(\hat{K}), \\ F(\hat{w}) = 0,\ \forall\ \hat{w} \in P_0(\hat{K}). \end{cases}$$

Employing the Bramble-Hilbert lemma yields

$$\|\hat{D}^\alpha(\hat{v} - \hat{T}\hat{v})\|_{0,\hat{K}} = \|\hat{D}^\alpha \hat{v} - F(\hat{D}^\alpha \hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^\alpha \hat{v}|_{1,\hat{K}}.$$

Similarly, we can prove (2.6) for the case $\alpha = (0, 1)$. ∎

Assume u and u_h to be the unique solution of (2.1) and (2.3) respectively, $u \in H^2(\Omega)$. Then the second Strang's Lemma (cf. [8], [13]) gives

$$\|u - u_h\|_h \leq 2 \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{v_h \in V_h \setminus \{0\}} \frac{|a_h(u - u_h, v_h)|}{\|v_h\|_h}. \tag{2.8}$$

Now, we are in a position to bound the first term on the right hand of (2.8) firstly.

LEMMA 2.3. *Under the above assumptions, we have*

$$\inf_{v_h \in V_h} \|u - v_h\|_h \leq Ch|u|_{2,\Omega}. \tag{2.9}$$

Proof. By lemma 2.2, we have

$$\begin{aligned} \inf_{v_h \in V_h} \|u - v_h\|_h \leq \|u - T_h u\|_h &= \left(\sum_{K \in \mathcal{J}_h} |u - T_K u|^2_{1,K}\right)^{\frac{1}{2}} \\ &= \left(\sum_{K \in \mathcal{J}_h} \sum_{|\alpha|=1} \|D^\alpha(u - T_K u)\|^2_{0,K}\right)^{\frac{1}{2}} \\ &= \left(\sum_{K \in \mathcal{J}_h} \sum_{|\alpha|=1} h_K^{-2\alpha}(h_{K1}h_{K2})\|\hat{D}^\alpha(\hat{u} - \hat{T}\hat{u})\|^2_{0,\hat{K}}\right)^{\frac{1}{2}} \end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

$$
\begin{aligned}
&\le C\left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1} h_K^{-2\alpha}(h_{K1}h_{K2})|\hat{D}^\alpha\hat{u}|^2_{1,\hat{K}}\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1}\sum_{|\beta|=1} h_K^{2\beta}\|D^{\alpha+\beta}u\|^2_{0,K}\right)^{\frac{1}{2}}\\
&\le Ch|u|_{2,\Omega}.
\end{aligned}
$$

□

REMARK 2.1. *Recently, Cai, Douglas and Ye [9], [15] have proposed some new nonconforming midpoint-oriented four-node elements. Note that the associate interpolations have the same properties as the rotated $\mathcal{Q}_1$ element.*

REMARK 2.2. *In order to overcome the deficiency of the interpolation of the rotated $\mathcal{Q}_1$ element, reference [5] proposed a modified rotated $\mathcal{Q}_1$ element with the shape function space $\hat{P} = span\{1,\xi,\eta,\xi^2\}$ (cf. [6], [20]). However, the dissymmetry of the shape space will influence the element geometrical inflexibility (cf. [14]).*

3. Consistency error estimate on anisotropic rectangular meshes. In this section, we will turn to the second term on the right hand of (2.8), i.e., the consistency error. The standard technique of the consistency error estimate (cf. [8], [13]) is invalid under anisotropic meshes, then we develop a new one for the estimate of the consistency error.

For any $v \in H^1(K)$, we define

$$P_{0i}v = \frac{1}{|l_i|}\int_{l_i} v\,ds, \quad i = 1,2,3,4.$$

Then by Green's formula we have

$$
\begin{aligned}
a_h(u-u_h,v_h) = a_h(u,v_h) - (f,v_h) &= \sum_{K\in\mathcal{J}_h}\sum_{l_i\subset\partial K}\int_{l_i}\frac{\partial u}{\partial n}v_h ds\\
&= \sum_{K\in\mathcal{J}_h}[\int_{l_1} -(v_h - P_{01}v_h)(\frac{\partial u}{\partial y} - P_{01}\frac{\partial u}{\partial y})dx\\
&\quad + \int_{l_3}(v_h - P_{03}v_h)(\frac{\partial u}{\partial y} - P_{03}\frac{\partial u}{\partial y})dx\\
&\quad + \int_{l_2}(v_h - P_{02}v_h)(\frac{\partial u}{\partial x} - P_{02}\frac{\partial u}{\partial x})dy\\
&\quad - \int_{l_4}(v_h - P_{04}v_h)(\frac{\partial u}{\partial x} - P_{04}\frac{\partial u}{\partial x})dy\\
&= \sum_{K\in\mathcal{J}_h}[I_1 + I_3 + I_2 + I_4].
\end{aligned}
\tag{3.1}
$$

We will show that the conventional technique of nonconforming error estimate will become invalid under the consideration of anisotropic meshes. Take I_1 for example, in the conventional method, by the trace theorem and the Bramble-Hilbert lemma, we will estimate it as follows:

$$|I_1| = \left|\int_{l_1} -(v_h - P_{01}v_h)(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y})dx\right|$$

ETNA
Kent State University
etna@mcs.kent.edu

$$
\begin{aligned}
&\le \|v_h - P_{01}v_h\|_{0,l_1}\left\|\frac{\partial u}{\partial y} - P_{01}\frac{\partial u}{\partial y}\right\|_{0,l_1}\\
&= h_{K1}h_{K2}^{-\frac{1}{2}}\|\hat{v}_h - \hat{P}_{01}\hat{v}_h\|_{0,\hat{l}_1}\left\|\frac{\partial \hat{u}}{\partial \eta} - \hat{P}_{01}\frac{\partial \hat{u}}{\partial \eta}\right\|_{0,\hat{l}_1}\\
&\le Ch_{K1}h_{K2}^{-\frac{1}{2}}|\hat{v}_h|_{1,\hat{K}}\left|\frac{\partial \hat{u}}{\partial \eta}\right|_{1,\hat{K}}\\
&= C\frac{h_{K1}}{h_{K2}}\left(\sum_{|\alpha|=1} h_K^{2\alpha}\|D^\alpha v_h\|_{0,K}^2\right)^{\frac{1}{2}}\left|\frac{\partial u}{\partial y}\right|_{1,K}.
\end{aligned}
$$

When the regular assumption is satisfied, which yields $\frac{h_{K1}}{h_{K2}} \le C$, then we can get

$$
|I_1| \le Ch_K|u|_{2,K}|v_h|_{1,K}. \tag{3.2}
$$

However, we do not have the regular assumption under anisotropic rectangular meshes. On the contrary, we may have $\frac{h_{K1}}{h_{K2}} \longrightarrow \infty$ when $h_K \longrightarrow 0$, and the desire convergence result of (3.2) can not be obtained as usual. Thus it is more difficult for us to estimate nonconforming error on anisotropic meshes. Now, let us turn to (3.1) again.

From the construction of the element we know that $\frac{\partial v_h}{\partial x}$ is independent of y, hence

$$
\begin{aligned}
&(v_h - P_{01}v_h)(x, y_K - h_{K2})\\
&= v_h(x, y_K - h_{K2}) - \frac{1}{2h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}} v_h(t, y_K - h_{K2})dt\\
&= \frac{1}{2h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}} [v_h(x, y_K - h_{K2}) - v_h(t, y_K - h_{K2})]\,dt\\
&= \frac{1}{2h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \frac{\partial v_h}{\partial r}(r, y_K - h_{K2})dr\right)dt\\
&= \frac{1}{2h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \frac{\partial v_h}{\partial r}(r, y_K + h_{K2})dr\right)dt\\
&= (v_h - P_{03}v_h)(x, y_K + h_{K2})\\
&\triangleq w(x).
\end{aligned} \tag{3.3}
$$

It is easy to see that

$$
\begin{aligned}
|w(x)| \le& \frac{1}{4h_{K1}h_{K2}}\int_{x_K-h_{K1}}^{x_K+h_{K1}}\int_{x_K-h_{K1}}^{x_K+h_{K1}}\int_{y_K-h_{K2}}^{y_K+h_{K2}}\left|\frac{\partial v_h}{\partial x}(x,y)\right|dxdydt\\
\le& \frac{1}{2h_{K2}}\int_K\left|\frac{\partial v_h}{\partial x}(x,y)\right|dxdy \le \frac{1}{2h_{K2}}2\sqrt{h_{K1}h_{K2}}|v_h|_{1,K}.
\end{aligned} \tag{3.4}
$$

i) Suppose $u \in H^2(\Omega)$, we modified (3.1) slightly,

$$
\begin{aligned}
a_h(u - u_h, v_h) =& \sum_{K\in\mathcal{J}_h}\Big[\int_{l_1} -(v_h - P_{01}v_h)\frac{\partial u}{\partial y}dx + \int_{l_3}(v_h - P_{03}v_h)\frac{\partial u}{\partial y}dx\\
&+ \int_{l_2}(v_h - P_{02}v_h)\frac{\partial u}{\partial x}dy - \int_{l_4}(v_h - P_{04}v_h)\frac{\partial u}{\partial x}dy\\
=& \sum_{K\in\mathcal{J}_h}[J_1 + J_3 + J_2 + J_4].
\end{aligned} \tag{3.5}
$$

ETNA
Kent State University
etna@mcs.kent.edu

Substituting (3.3) into $J_1 + J_3$, we get

$$
\begin{aligned}
J_1 + J_3 &= \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)\left[\frac{\partial u}{\partial y}(x, y_K + h_{K2}) - \frac{\partial u}{\partial y}(x, y_K - h_{K2})\right] dx \\
&= \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)\left[\int_{y_K-h_{K2}}^{y_K+h_{K2}} \frac{\partial^2 u}{\partial y^2} dy\right] dx \\
&\triangleq \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)Q(x)dx.
\end{aligned}
\tag{3.6}
$$

A combination of (3.4) and (3.6) yields

$$
J_1 + J_3 \leq Ch_{K1}|u|_{2,K}|v_h|_{1,K}. \tag{3.7}
$$

By the same argument, we can prove

$$
J_2 + J_4 \leq Ch_{K2}|u|_{2,K}|v_h|_{1,K}. \tag{3.8}
$$

Then we have

$$
a_h(u - u_h, v_h) \leq Ch|u|_{2,\Omega}\|v_h\|_h. \tag{3.9}
$$

ii) Suppose $u \in H^3(\Omega)$, we consider (3.1) again. Set $v = \frac{\partial u}{\partial y}$, there hold

$$
v(x, y_K - h_{K2}) - P_{01}v = \frac{1}{2h_{K1}} \int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \frac{\partial v}{\partial r}(r, y_K - h_{K2})dr\right) dt, \tag{3.10}
$$

and

$$
v(x, y_K + h_{K2}) - P_{03}v = \frac{1}{2h_{K1}} \int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \frac{\partial v}{\partial r}(r, y_K + h_{K2})dr\right) dt. \tag{3.11}
$$

Substituting the above two formulas and (3.3) into $I_1 + I_3$, we get

$$
\begin{aligned}
I_1 + I_3 &= \frac{1}{2h_{K1}} \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)\left[\int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \left(\frac{\partial v}{\partial r}(r, y_K + h_{K2})\right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{\partial v}{\partial r}(r, y_K - h_{K2})\right) dr\right) dt\right] dx \\
&= \frac{1}{2h_{K1}} \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)\left(\int_{x_K-h_{K1}}^{x_K+h_{K1}} \left(\int_t^x \int_{y_K-h_{K2}}^{y_K+h_{K2}} \frac{\partial^2 v}{\partial r \partial y}(r, y)drdy\right) dt\right) dx \\
&\triangleq \frac{1}{2h_{K1}} \int_{x_K-h_{K1}}^{x_K+h_{K1}} w(x)Q(x)dx.
\end{aligned}
\tag{3.12}
$$

Obviously,

$$
\begin{aligned}
|Q(x)| &\leq 2h_{K1} \int_K \left|\frac{\partial^2 v}{\partial x \partial y}(x, y)\right| dxdy \leq 2h_{K1}.2\sqrt{h_{K1}h_{K2}}|v|_{2,K} \\
&\leq 4h_{K1}\sqrt{h_{K1}h_{K2}}|u|_{3,K}.
\end{aligned}
\tag{3.13}
$$

Substituting (3.4) and (3.13) into (3.12) gives

$$
J_1 + J_3 \leq Ch_{K1}^2|u|_{3,K}|v_h|_{1,K}. \tag{3.14}
$$

ETNA
Kent State University
etna@mcs.kent.edu

Similarly,

$$J_2 + J_4 \leq Ch_{K2}^2 |u|_{3,K} |v_h|_{1,K}. \tag{3.15}$$

Thus we have obtained

$$a_h(u - u_h, v_h) \leq Ch^2 |u|_{3,\Omega} \|v_h\|_h. \tag{3.16}$$

Then we can get the following anisotropic error estimate of the nonconforming rotated $\mathcal{Q}_1$ element.

THEOREM 3.1. *Suppose $u \in H^2(\Omega)$, under anisotropic rectangular meshes, we have the following error estimates*

$$\|u - u_h\|_h \leq Ch|u|_{2,\Omega}, \qquad \|u - u_h\|_{0,\Omega} \leq Ch^2 |u|_{2,\Omega} \tag{3.17}$$

If further assume $u \in H^3(\Omega)$, the consistency error is of one higher order, i.e.,

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{|a_h(u - u_h, v_h)|}{\|v_h\|_h} \leq Ch^2 |u|_{3,\Omega}. \tag{3.18}$$

Proof. (3.18) is a direct consequence of (3.16). Substituting (2.9) and (3.9) into (2.8), we can obtain the first identity of (3.17). By the usual dual argument as standard finite element theory (cf. [8], [13]), we will get the second identity of (3.17). Then the proof is completed. □

REMARK 3.1. *The results of (3.17) can be applied to the nonconforming elements discussed in [5], [6], [9], [15], [18], [21], [23]. However, the result of (3.18) does not stand for the midpoint-oriented elements proposed by [23].*

4. Numerical experiment. In order to investigate the numerical behavior of the rotated $\mathcal{Q}_1$ element under anisotropic rectangular meshes, we consider the second order problem (2.1) with $f(x, y) = -\pi(\cos(\pi x)\sin(\pi y) + \cos(\pi y)\sin(\pi x)) \in L^2(\Omega)$, and $\Omega = [0, 1] \times [0, 1]$. It can be verified that the exact solution of problem (2.1) is $u(x, y) = \sin(\pi x)\sin(\pi y)$. The meshes on Ω can be obtained in the following way, the edges of Ω parallel to $x - axis$ ($y - axis$ resp.) are divided into n segments with $n + 1$ points $(1 - cos(\frac{i\pi}{n}))/2, i = 0, 1, ..., \frac{n}{2}, (1 + sin(\frac{i\pi}{n} - \frac{\pi}{2}))/2, i = \frac{n}{2} + 1, ..., n$(m resp.). The mesh obtained in this way for 16×16 is illustrated in Fig 4.1. Note that the large aspect ratio meshes can be obtained by adjusting the ratio value of $\frac{n}{m}$ (refer to Table 4.1 for the aspect ratio $\max\limits_{K \in \mathcal{J}_h} \frac{h_K}{\rho_K}$).

In Table 4.1, we list the numerical results of the rotated $\mathcal{Q}_1$ nonconforming element, here $ei = \frac{\|u - I_h u\|_h}{h}$ and $eh = \frac{\|u - u_h\|_h}{h}$, where $I_h u$ denotes the interpolation of the exact solution. From the results of this table, we can see that the optimal energy norm error between u and u_h is obtained under large aspect ratio meshes (strongly anisotropic meshes), while the interpolation error is divergent in such case. These results show that the constant C at the right hand side of the interpolation error is dependent on the aspect ratio, while that of the error of u and u_h is independent of the aspect ratio, which coincides with our theoretical analysis.

For the sake of a comparison with the rotated $\mathcal{Q}_1$ nonconforming element, we also have computed the five-node element proposed by Han in [18]. The shape space is $span\{1, \xi, \eta, \xi^2, \eta^2\}$, and the degrees of freedom are $\{\hat{v}_{12}, \hat{v}_{23}, \hat{v}_{34}, \hat{v}_{14}, \hat{v}_5\}$ on $\hat{K}$, where $\hat{v}_5 = \frac{\int_{\hat{K}} \hat{v}\, d\xi d\eta}{|\hat{K}|}$. The results are listed in Table 4.2, from which we can see that both the interpolation error and finite element error are of optimal order on anisotropic rectangular meshes.

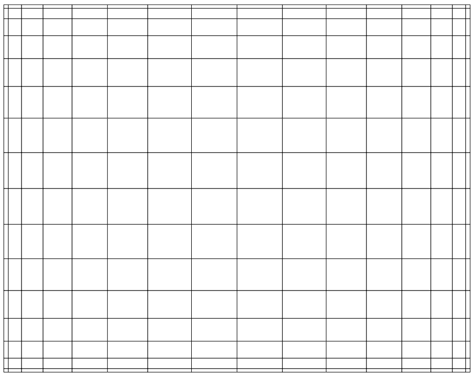

FIG. 4.1. *The anisotropic rectangular mesh for the case $n = m = 16$*

TABLE 4.1
Numerical results of the rotated $\mathcal{Q}_1$ element

$m \times n$	2×2	2×8	2×32	2×128	2×512	2×1024
eh	1.92	1.98	2.04	2.04	2.04	2.04
ei	1.84	3.49	15.24	68.80	303.67	633.91
$\max\limits_{K\in\mathcal{J}_h} \frac{h_K}{\rho_K}$	1.41	18.58	293.69	4695.56	75125.35	300500.71

5. Conclusions. This paper has been studied the convergence properties of the famous rotated $\mathcal{Q}_1$ nonconforming element under anisotropic rectangular meshes. The interpolation error in the discrete H^1-norm is not convergent on anisotropic rectangular meshes. But this does not influence the convergence of the rotated $\mathcal{Q}_1$ element. From this paper we can see that the interpolation error does not indicate the convergence of some elements. What's more, the techniques developed in this paper can be applied to other nonconforming elements.

The higher order of the consistency error estimate obtained in this paper is of interest in the superconvergence analysis of nonconforming elements (cf. [10], [25]). In fact, combined the result of this paper with $a_h(u - I_hu, v_h) \le Ch^2|u|_{3,\Omega}\|v_h\|_h$ (If we can prove this!) yields a superclose result $\|I_hu - u_h\|_h \le Ch^2|u|_{3,\Omega}$. Some superconvergence results of the five-node element proposed in [18] can be obtained in this way (will be addressed elsewhere). However, the superconvergence of the rotated $\mathcal{Q}_1$ element is still an open problem, which will be a future work of us.

Acknowledgments. The authors would like to express their sincere thanks to the anonymous referees for many helpful suggestions and corrections of the English and typesetting mistakes.

REFERENCES

[1] G. ACOSTA AND R. G. DURAN, *The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations*, SIAM J. Numer. Anal., 37 (1999), pp. 18-36.
[2] G. ACOSTA, *Langrange and average interpolation over 3D anisotropic meshes*, J. Comput. Appl. Math., 135 (2001), pp. 91-109.
[3] T. APEL AND M. DOBROWOLSKI, *Anisotropic interpolation with applications to the finite element method*, Computing, 47 (1992), pp. 277-293.
[4] T. APEL, *Anisotropic finite element: local estimates and applications*,Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1999.

TABLE 4.2
Numerical results of the five-node element

$m \times n$	2×2	2×8	2×32	2×128	2×512	2×1024
eh	1.49	1.46	1.48	1.48	1.48	1.48
ei	1.39	1.37	1.39	1.40	1.40	1.40
$\max_{K \in \mathcal{J}_h} \frac{h_K}{\rho_K}$	1.41	18.58	293.69	4695.56	75125.35	300500.71

[5] T. APEL, S. NICAISE, AND J. SCHÖBERL,*Crouzeix-Raviart type finite elements on anisotropic meshes*, Numer. Math., 89 (2001), pp. 193-223.

[6] T. APEL, S. NICAISE, AND J. SCHÖBERL,*A nonconforming finite elements method with anisotropic mesh grading for the Stokes problem in domains with edges*, IMA J. Numer. Anal., 21 (2001), pp. 843-856.

[7] I. BABUSKA AND AK. AZIZ,, *On the angle condition in the finite element method*, SIAM J. Numer. Anal., 13 (1976), pp. 214-226.

[8] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, New York, Springer-Verlag, 1994.

[9] Z. Q. CAI, J. DOUGLAS JR., AND X. YE, *A stable nonconforming quadrilateral finite element method for the stationary Stokes and Navier-Stokes equations*, Calcolo, 36 (1999), pp. 215-232.

[10] C. M. CHEN AND Y. Q. HUANG, *High accuracy theory of finite element methods*, Hunan Science Press, P. R. China, 1995.

[11] S. C. CHEN, D. Y. SHI, AND Y. C. ZHAO, *Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes*, IMA J. Numer. Anal., 24 (2004), pp. 77-95.

[12] S. C. CHEN, D. Y. SHI, AND Y. C. ZHAO, *Anisotropic interpolations with application to nonconforming elements*, Appl. Numer. Math., 49 (2004), pp. 135-152.

[13] P. G. CIARLET, *The Finite Element method for Elliptic Problems*, Amsterdam, North-Holland, 1978.

[14] W. J. CHEN AND X. L. YANG, *Variational basement and geometric inviarants of generalized conforming element*, Comput. Struct. Mech. Appl., 13 (1992), pp. 245-252.

[15] J. DOUGLAS JR., J. E. SANTOS, D. SHEEN AND X. YE, *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems*, RAIRO Modél. Math. Anal. Num ér., 33 (1999), pp. 747-770.

[16] R. G. DURAN, *Error estimates for narrow 3D finite elements*, Math. Comp., 68 (1999), pp. 187-199.

[17] R. G. DURAN AND A. L. LOMBARDI, *Error estimates on anisotropic Q_1 elements for functions in weighted sobolev spaces*, Math. Comp., 73 (2005), to appear.

[18] H. D. HAN, *Nonconforming elements in the mixed finite element method*, J. Comput. Math., 2 (1984), pp. 223-233.

[19] P. JAMET, *Estimations d'erreur pour des $\acute{e}l\acute{e}ments$ finis droits presque $d\acute{e}g\acute{e}n\acute{e}r\acute{e}s$*, RAIRO Modél. Math. Anal. Num ér., 10 (1976), pp. 46-61.

[20] J. LAZAAR AND S. NICAISE, *A nonconforming finite element method with anisotropic mesh grading for the incompressible Navier-Stokes equations in domains with edges*, Calcolo, 39 (2002), pp. 123-168.

[21] B. LI AND M. LUSKIN, *Nonconforming finite element approximation of crystalline microstructures*, Math. Comp., 67 (1998), pp. 917-946.

[22] P. B. MING AND Z. C. SHI, *Convergence analysis for quadrilateral rotated Q_1 element*, in Advances in Computation: Theory and Practice, Scientific Computing and Applications, eds:Peter Minev and Yanping Lin, Nova Science Publications, Inc, 7 (2001), pp. 115-124.

[23] R. RANNACHER AND ST. TUREK, *Simple nonconforming quadrilateral stokes element*, Numer. Methods Partial Differential Equations, 8 (1992), pp. 97-111.

[24] N. AL. SHENK, *Uniform error estimates for certain narrow Lagrange finite elements*, Math. Comp., 63 (1994), pp. 105-119.

[25] L. B. WAHLBIN, *Superconvergence in Galerkin Finite Element Methods*, Lecture Notes in Mathematicas, Vol. 1605, Springer, Berlin, 1995.