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A BDDC ALGORITHM FOR A MIXED FORMULATION OF FLOW IN POROUS MEDIA*

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Abstract. The BDDC (balancing domain decomposition by constraints) algorithms are similar to the balancing Neumann-Neumann methods, with a small number of continuity constraints enforced across the interface throughout the iterations. These constraints form a coarse, global component of the preconditioner. The BDDC methods are powerful for solving large sparse linear algebraic systems arising from discretizations of elliptic boundary value problems. In this paper, the BDDC algorithm is extended to saddle point problems generated from the mixed finite element methods used to approximate the scalar elliptic problems for flow in porous media. Edge/face average constraints are enforced and the same rate of convergence is obtained as for simple elliptic cases. The condition number bound is estimated and numerical experiments are discussed. In addition, a comparison of the BDDC method with an edge/face-based iterative substructuring method is provided.

Key words. BDDC, domain decomposition, saddle point problem, condition number, benign space, edge/face-based iterative substructuring method

AMS subject classifications. 65N30, 65N55, 65F10

1. Introduction. The BDDC algorithms, introduced by Dohrmann in [5], see also [13, 14], are nonoverlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In BDDC, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The relation between the BDDC and BNN is similar to that between the FETI-DP and one level FETI. Recently, the BDDC and FETI-DP algorithms for elliptic problems were rederived and a much shorter proof of the main result in [14] was given in [11].

Mixed formulations of elliptic problems, see [2], lead to large, sparse, symmetric, indefinite linear systems. Such methods have extensive applications, as in flow in porous media, where a good approximation to the velocity is required.

Overlapping domain decomposition methods for this kind of problem were developed in [6, 15, 16, 17]. These additive or multiplicative overlapping Schwartz alternating methods reduce the problem to a symmetric positive definite problem for a vector, divergence free in a finite element sense. Then two-level overlapping methods are applied to the reduced positive definite problem in the benign, divergence free subspace. The algorithms converge at a rate independent of the mesh parameters and the coefficients of the original equation.

In [9], two non-overlapping domain decomposition algorithms were proposed. They are unpreconditioned conjugate gradient methods for certain interface variables and are, to the best of our knowledge, the first iterative substructuring methods. The rate of convergence is independent of the coefficients of the original equations, but depends mildly on the mesh parameters. The consequence of the singular local Neumann problems that arise was addressed in [9]. Other non-overlapping domain decomposition methods were proposed in [8] and [4] with improved rates of convergence. A BNN version of the Method II of [9] was proposed in [3], see also [20]. The same rate of convergence is obtained as for simple elliptic cases.

Using mixed formulations of flow in porous media, we will obtain a saddle point problem which is closely related to that arising from the incompressible Stokes equations. We note

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that, in a recent paper [12], the BDDC algorithms have been applied to the incompressible Stokes equation, where the constraints enforced across the interface satisfy two assumptions. One assumption forces the iterates into the benign subspace in which the operator is positive definite and the other ensures a good bound for the condition number. In general, both these assumptions are required.

In this paper, we extend the BDDC algorithms to mixed formulations of flow in porous media. This work is directly related to [12], but our situation is also different. First of all, our problem is not originally formulated in the benign, divergence free subspace, and it will therefore be reduced to the benign subspace, as in [6, 15, 16, 17], at the beginning of the computation. In addition, only edge/face constraints are needed to force the iterates into the benign subspace and to ensure a good bound for the condition number, since Raviart-Thomas finite elements, see [2, Chapter III], are utilized. These elements have no degrees of freedom associated with vertices/edges in two/three dimensions. Also, the condition number estimate for the Stokes case can be simplified since the Stokes extension is equivalent to the harmonic extension, see [1]. However, this is not the case here, and different technical tools are required. We also note that our BDDC method is closely related to an edge/face-based substructuring iterative method. We will give a detailed description later.

An iterative substructuring method with Raviart-Thomas finite elements for vector field problems was proposed in [24, 21]. We will borrow some technical tools from these papers in our analysis of the BDDC algorithms.

The rest of the paper is organized as follows. The mixed formulation for the elliptic problems and its finite element discretization are described in Section 2. We reduce our system to an interface problem in Section 3. In Section 4, we introduce the BDDC methods for our mixed methods. We give some auxiliary results in Section 5. In Section 6, we provide an estimate of the form $C\left(1+\log\frac{H}{h}\right)^2$ of the condition number for the system with the BDDC preconditioner; these H and h are the diameters of the subdomains and elements, respectively. We also compare the BDDC methods with an edge/face-based algorithm in Section 7. Finally, some computational results are given in Section 8.

2. An elliptic problem discretized by mixed finite elements. We consider the following elliptic problem on a bounded polygonal domain Ω in two or three dimensions with a Neumann boundary condition:

$$\left\{\begin{array}{lll} -\nabla\cdot(a\nabla p)=f & \text{in} & \Omega \\ \mathbf{n}\cdot(a\nabla p)=g & \text{in} & \partial\Omega. \end{array}\right. \tag{2.1}$$

Here $\mathbf{n}$ is the outward normal to $\partial\Omega$ and a is a positive definite matrix function with entries in $L^\infty(\Omega)$ satisfying

$$\xi^T a(\mathbf{x})\xi \geq \alpha\|\xi\|^2, \quad \text{for a.e.} \quad \mathbf{x}\in\Omega, \tag{2.2}$$

for some positive constant α.

The functions $f\in L^2(\Omega)$ and $g\in H^{-1/2}(\partial\Omega)$ satisfy the compatibility condition

$$\int_\Omega f d\mathbf{x}+\int_{\partial\Omega} g d\mathbf{s}=0.$$

The equation (2.1) has a solution p which is unique up to a constant. Without loss of generality, we assume that $g=0$ and that f has mean value zero. We also require that the solution p has mean value zero over Ω; therefore we have a unique solution.

We assume that we are interested in computing $-a\nabla p$ directly as is often required in flow in porous media. We then introduce the velocity $\mathbf{u}$:

$$\mathbf{u}=-a\nabla p,$$

and call p the pressure. We obtain the following system for the velocity $\mathbf{u}$ and the pressure p:

$$\left\{\begin{array}{lll} \mathbf{u} = -a\nabla p & \text{in} & \Omega \\ \nabla \cdot \mathbf{u} = f & \text{in} & \Omega \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{in} & \partial\Omega. \end{array}\right. \tag{2.3}$$

Let $c(\mathbf{x}) = a(\mathbf{x})^{-1}$ and define a Hilbert space by

$$H_0(div, \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \nabla \cdot \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

with the norm

$$\|\mathbf{v}\|^2_{H(div,\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + H_D^2 \|\nabla \cdot \mathbf{v}\|^2_{L^2(\Omega)},$$

where H_D is the diameter of Ω.

Given a vector $\mathbf{u} \in H(div, \Omega)$, it is possible to define its normal component $\mathbf{u} \cdot \mathbf{n}$ on $\partial\Omega$, as an element of $H^{-1/2}(\partial\Omega)$, and the following inequality holds

$$\|\mathbf{u} \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} \leq C \|\mathbf{u}\|^2_{H(div,\Omega)}, \tag{2.4}$$

with a constant C that is independent of H_D, the diameter of Ω, see [24, Section 2]. The trace operator that maps a vector in $H(div, \Omega)$ into its normal component in $H^{-1/2}(\partial\Omega)$ is thus continuous, and it can be shown to be surjective; see [7, Ch. I, Th. 2.5 and Cor. 2.8].

The weak form of (2.3) is as follows: find $\mathbf{u} \in H_0(div, \Omega)$ and $p \in L_0^2(\Omega) = \{q : q \in L^2(\Omega), \int_\Omega q d\mathbf{x} = 0\}$ such that,

$$\left\{\begin{array}{lcll} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & 0 & \forall \mathbf{v} \in H_0(div, \Omega) \\ b(\mathbf{u}, q) & = & -\int_\Omega f q d\mathbf{x} & \forall q \in L_0^2(\Omega), \end{array}\right.$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}^T c(\mathbf{x}) \mathbf{v} d\mathbf{x}$ and $b(\mathbf{u}, q) = -\int_\Omega (\nabla \cdot \mathbf{u}) q d\mathbf{x}$.

Let $\widehat{\mathbf{W}}$ be the lowest order Raviart-Thomas finite element space with a zero normal component on $\partial\Omega$, see [2, Chapter III, 3], and let Q be the space of piecewise constants with a zero mean value, which are finite dimensional subspaces of $H_0(div, \Omega)$ and $L_0^2(\Omega)$, respectively. The pair $\widehat{\mathbf{W}}, Q$ satisfy a uniform inf-sup condition, see [2, Chapter IV. 1.2]. The finite element discrete problem is: find $\mathbf{u}_h \in \widehat{\mathbf{W}}$ and $p_h \in Q$ such that,

$$\left\{\begin{array}{lcll} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & 0 & \forall \mathbf{v}_h \in \widehat{\mathbf{W}} \\ b(\mathbf{u}_h, q_h) & = & -\int_\Omega f q_h d\mathbf{x} & \forall q_h \in Q, \end{array}\right.$$

and the matrix form is:

$$\left[\begin{array}{cc} A & B^T \\ B & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u}_h \\ p_h \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ F_h \end{array}\right]. \tag{2.5}$$

The system (2.5) is symmetric indefinite with the matrix A symmetric, positive definite. For details on the range of negative and positive eigenvalues of (2.5), see [19].

3. Reduction to an interface problem. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and with $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse rectangles/hexahedra and that the number of such rectangles/hexahedra forming an individual subdomain is uniformly bounded. We note that the algorithm can be extended to different types of subdomains. In a more general case, we can still define faces, regarded as open sets that are shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains. We then introduce quasi-uniform triangulations of each subdomain. The global problem (2.5) is assembled from the subdomain problems

$$\begin{bmatrix} A^{(i)} & B^{(i)^T} \\ B^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h^{(i)} \\ p_h^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \end{bmatrix}. \tag{3.1}$$

The degrees of freedom of the Raviart-Thomas finite elements are the normal components on the boundary of each element only.

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i\neq j}\partial\Omega_{i,h} \cap \partial\Omega_{j,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$. We decompose the discrete velocity and pressure spaces $\widehat{\mathbf{W}}$ and Q into

$$\widehat{\mathbf{W}} = \mathbf{W}_I \bigoplus \widehat{\mathbf{W}}_\Gamma, \quad Q = Q_I \bigoplus Q_0. \tag{3.2}$$

$\widehat{\mathbf{W}}_\Gamma$ is the space of traces on Γ of functions of $\widehat{\mathbf{W}}$. $\mathbf{W}_I$ and Q_I are direct sums of subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$, and subdomain interior pressure spaces $Q_I^{(i)}$, i.e.,

$$\mathbf{W}_I = \bigoplus_{i=1}^N \mathbf{W}_I^{(i)}, \qquad Q_I = \bigoplus_{i=1}^N Q_I^{(i)}.$$

The elements of $\mathbf{W}_I^{(i)}$ are supported in the subdomain Ω_i and their normal components vanish on the subdomain interface $\Gamma_i = \Gamma \cap \partial\Omega_i$, while the elements of $Q_I^{(i)}$ are restrictions of elements in Q to Ω_i which satisfy $\int_{\Omega_i} q_I^{(i)} = 0$. Q_0 is the subspace of Q with constant values $q_0^{(i)}$ in the subdomain Ω_i that satisfy

$$\sum_{i=1}^N q_0^{(i)}\, m(\Omega_i) = 0, \tag{3.3}$$

where $m(\Omega_i)$ is the measure of the subdomain Ω_i. $R_0^{(i)}$ is the operator which maps functions in the space Q_0 to its constant component of the subdomain Ω_i.

We denote the subdomain velocity space by $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma$, the space of the interface velocity variables by $\mathbf{W}_\Gamma^{(i)}$, and the associate product space by $\mathbf{W}_\Gamma = \prod_{i=1}^N \mathbf{W}_\Gamma^{(i)}$.

The subdomain saddle point problems (3.1) can be written as

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\ 0 & 0 & B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ p_{h,I}^{(i)} \\ \mathbf{u}_{h,\Gamma}^{(i)} \\ p_{h,0}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{h,I}^{(i)} \\ 0 \\ F_{h,\Gamma}^{(i)} \end{bmatrix}, \tag{3.4}$$

where $(\mathbf{u}_{h,I}^{(i)}, p_{h,I}^{(i)}, \mathbf{u}_{h,\Gamma}^{(i)}, p_{h,0}^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Gamma^{(i)}, Q_0^{(i)})$. We note that, by the divergence theorem, the lower left block of the matrix of (3.4) is zero since the bilinear form $b(\mathbf{v}_I^{(i)}, q_0^{(i)})$ always vanishes for any $\mathbf{v}_I^{(i)} \in \mathbf{W}_I^{(i)}$ and a constant $q_0^{(i)}$ in the subdomain Ω_i.

3.1. Obtaining a divergence free correction. First of all, we seek a discrete velocity $\mathbf{u}_h^* \in \widehat{\mathbf{W}}$ such that,

$$B\mathbf{u}_h^* = F_h. \tag{3.5}$$

Let $\widehat{\mathbf{W}}^H$ be the lowest order Raviart-Thomas finite element space on the coarse triangulation, associated with the subdomains, with zero normal components on $\partial\Omega$ and let Q^H be the space of piecewise constants with vanishing mean value. R_0^T is the natural interpolation operator from $\widehat{\mathbf{W}}^H \times Q^H$ to $\widehat{\mathbf{W}} \times Q$. We also use the same interpolation operator on the corresponding right hand side space. Let

$$\begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix} = R_0 \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} R_0^T, \tag{3.6}$$

and

$$\begin{bmatrix} \mathbf{u}_0^* \\ p_0^* \end{bmatrix} = R_0^T \begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix}^{-1} R_0 \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}.$$

We note that the coarse grid solution $\mathbf{u}_0^*$ does not necessarily satisfy (3.5), but that $B\mathbf{u}_0^* - F_h$ has mean value zero over each subdomain Ω_i, see [16, 6]. Then the local Neumann problems, with $\mathbf{u}_{h,\Gamma}^{(i)} = 0$ and the right hand sides $\begin{bmatrix} -A^{(i)}\mathbf{u}_0^{*,(i)} \\ F_h^{(i)} - B^{(i)}\mathbf{u}_0^{*,(i)} \end{bmatrix}$, $i = 1, \cdots, N$, are all well-posed. We can solve

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ p_{h,I}^{(i)} \end{bmatrix} = \begin{bmatrix} -(A^{(i)}\mathbf{u}_0^{*,(i)})_I \\ (F_h^{(i)} - B^{(i)}\mathbf{u}_0^{*,(i)})_I \end{bmatrix}, \quad i = 1, \cdots, N, \tag{3.7}$$

and set

$$\mathbf{u}_i^* = \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \cdots, N.$$

Let $\mathbf{u}_h^* = \mathbf{u}_0^* + \mathbf{u}_1^* + \cdots + \mathbf{u}_N^*$ which satisfies (3.5). We then write the solution of (2.5) as

$$\begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{u}_h^* \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix},$$

where the correction $(\mathbf{u}, p)^T$ satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} -A\mathbf{u}_h^* \\ 0 \end{bmatrix}. \tag{3.8}$$

This problem can be assembled from the subdomain problems:

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\ 0 & 0 & B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(i)} \\ 0 \\ \mathbf{f}_\Gamma^{(i)} \\ 0 \end{bmatrix}, \tag{3.9}$$

where $(\mathbf{u}_I^{(i)}, p_I^{(i)}, \mathbf{u}_\Gamma^{(i)}, p_0^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Gamma^{(i)}, Q_0^{(i)})$ and $\mathbf{f}_I^{(i)} = -\left(A^{(i)}\mathbf{u}^{*(i)}\right)_I$ and $\mathbf{f}_\Gamma^{(i)} = -\left(A^{(i)}\mathbf{u}^{*(i)}\right)_\Gamma$.

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3.2. A reduced interface problem. We now reduce the global problem (3.8) to an interface problem.

We define the subdomain Schur complements $S_\Gamma^{(i)}$ as: given $\mathbf{w}_\Gamma^{(i)} \in \mathbf{W}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)}\mathbf{w}_\Gamma^{(i)}$ such that,

$$
\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{w}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ S_\Gamma^{(i)}\mathbf{w}_\Gamma^{(i)} \end{bmatrix}. \tag{3.10}
$$

We know, from the definition in (3.10), that the action of $S_\Gamma^{(i)}$ can be evaluated by solving a Neumann problem on the subdomain Ω_i. We note that these Neumann problems are always well-posed, even without any constraints on the normal component of the velocity since we have removed the constant pressure component constraints. Furthermore, since the local matrices

$$
\begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)^T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}
$$

are symmetric, positive definite, we have, by an inertia argument,

LEMMA 3.1. *The subdomain Schur complements $S_\Gamma^{(i)}$ defined in (3.10) are symmetric, positive definite.*

Given the definition of $S_\Gamma^{(i)}$, the subdomain problems (3.9) are reduced to the subdomain interface problems

$$
\begin{bmatrix} S_\Gamma^{(i)} & B_{0\Gamma}^{(i)^T} \\ B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma^{(i)} \\ 0 \end{bmatrix}, \quad i = 1, 2, ..., N,
$$

where

$$
\mathbf{g}_\Gamma^{(i)} = \mathbf{f}_\Gamma^{(i)} - \left[A_{\Gamma I}^{(i)} \; B_{I\Gamma}^{(i)^T}\right] \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_I^{(i)} \\ 0 \end{bmatrix}.
$$

We denote the direct sum of $S_\Gamma^{(i)}$ by S_Γ. Let $R_\Gamma^{(i)}$ be the operator which maps functions in the continuous interface velocity space $\widehat{\mathbf{W}}_\Gamma$ to the subdomain components in the space $\mathbf{W}_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ. Then the global interface problem, assembled from the subdomain interface problems, can be written as: find $(\mathbf{u}_\Gamma, p_0) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$
\widehat{S} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}, \tag{3.11}
$$

where $\mathbf{g}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} \mathbf{g}_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} R_\Gamma^{(i)}$, and

$$
\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}. \tag{3.12}
$$

Thus, $\widehat{S}$ is an interface saddle point operator defined on the space $\widehat{\mathbf{W}}_\Gamma \times Q_0$. But by Lemma 3.1, this operator is symmetric positive definite on the benign subspace where $\widehat{B}_{0\Gamma}\mathbf{u}_\Gamma = 0$.

From (3.8), we know that the correction $(\mathbf{u}_\Gamma, p)^T$ lies in this benign subspace. We will propose a preconditioner for (3.11) which keeps all the iterates in this benign subspace. Therefore, the iterates remain in the benign subspace in which the preconditioned operator is positive definite and a preconditioned conjugate gradient method can be applied.

4. The BDDC methods. We follow [12, Section 4] closely in this section. We introduce a partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ by

$$\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widehat{\mathbf{W}}_\Pi \bigoplus \left(\prod_{i=1}^N \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{W}}_\Pi$ is the coarse level, primal interface velocity space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see [11] and [10]. The space $\mathbf{W}_\Delta$ is the direct sum of the $\mathbf{W}_\Delta^{(i)}$, which is spanned by the remaining interface velocity degrees of freedom with a zero average over each edge/face. In the space $\widetilde{W}_\Gamma$, we have relaxed most continuity constraints on the velocity across the interface but retained all primal continuity constraints, which has the important advantage that all the linear systems are nonsingular in the computation. This is the main difference from an edge/face-based iterative substructuring domain decomposition method, where we will encounter singular local problems; see Section 7.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to the components $\mathbf{W}_\Gamma^{(i)}$ related to the subdomain Ω_i. $R_\Delta^{(i)}$ maps functions from $\widehat{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Delta^{(i)}$, its dual subdomain component. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\mathbf{W}}_\Gamma$ to its subspaces $\widehat{\mathbf{W}}_\Pi$ and $R_\Pi^{(i)}$ is the operator which maps vectors in $\widehat{\mathbf{W}}_\Pi$ into their components in $W_\Pi^{(i)}$. $\overline{R}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{W}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{c_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} c_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We assume that $c_i(x)$ is a constant in each subdomain. We then note that $\delta_i^\dagger(x)$ is constant on each edge/face since the nodes on each edge/face are shared by the same pair of subdomains. Multiplying each row of $R_\Delta^{(i)}$ with the scaling factor $\delta_i^\dagger(x)$ gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. We also use the notation

$$\widetilde{R} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \quad \text{and} \quad \widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}.$$

We also denote by $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$, the right hand side spaces corresponding to $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$, respectively. We will use the same restriction, extension, and scaled restriction operators for the spaces $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$ as for $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$.

We define the partially assembled interface velocity Schur complement $\widetilde{S}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{F}}_\Gamma$ by

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{4.1}$$

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$\widetilde{S}_\Gamma$ can also be defined by: for any given $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$, $\widetilde{S}_\Gamma \mathbf{w}_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ satisfies

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)^T} & A_{\Delta I}^{(1)^T} & & \widetilde{A}_{\Pi I}^{(1)^T} \\ B_{II}^{(1)} & 0 & B_{I\Delta}^{(1)} & & \widetilde{B}_{I\Pi}^{(1)} \\ A_{\Delta I}^{(1)} & B_{I\Delta}^{(1)^T} & A_{\Delta\Delta}^{(1)} & & \widetilde{A}_{\Pi\Delta}^{(1)^T} \\ & & & \ddots & \vdots \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{I\Pi}^{(1)^T} & \widetilde{A}_{\Pi\Delta}^{(1)} & \dots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(1)} \\ p_I^{(1)} \\ \mathbf{w}_\Delta^{(1)} \\ \vdots \\ \mathbf{w}_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Delta^{(1)} \\ \vdots \\ (\widetilde{S}_\Gamma \mathbf{w}_\Gamma)_\Pi \end{bmatrix}. \tag{4.2}$$

Here,

$$\widetilde{A}_{\Pi I}^{(i)} = R_\Pi^{(i)^T} A_{\Pi I}^{(i)},\ \widetilde{A}_{\Pi\Delta}^{(i)} = R_\Pi^{(i)^T} A_{\Pi\Delta}^{(i)},\ \widetilde{A}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)},\ \widetilde{B}_{I\Pi}^{(i)} = B_{I\Pi}^{(i)} R_\Pi^{(i)}.$$

Given the definition $\widetilde{S}_\Gamma$ on the partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$, we can also obtain $\widehat{S}_\Gamma$ from $\widetilde{S}_\Gamma$ by assembling the dual interface velocity part on the subdomain interface, i.e.,

$$\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma. \tag{4.3}$$

We can also define the operator $\widetilde{B}_{0\Gamma}$, partially assembled from the subdomain operators $B_{0\Gamma}^{(i)}$, which maps the partially assembled interface velocity to the subdomain constant pressures. Then $\widehat{B}_{0\Gamma}$ can also be obtained from $\widetilde{B}_{0\Gamma}$ by assembling the dual interface velocity part on the subdomain interface, i.e., $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma}\widetilde{R}_\Gamma$.

Therefore, we can write the global interface saddle point problem operator $\widehat{S}$, introduced in Equation (3.11), as

$$\widehat{S} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma & \widetilde{R}_\Gamma^T \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma}\widetilde{R}_\Gamma & 0 \end{bmatrix}. \tag{4.4}$$

The BDDC preconditioner for solving the global interface saddle point problem (3.11) is then

$$M^{-1} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}. \tag{4.5}$$

We use the notation

$$\widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix},$$

then the preconditioned BDDC algorithm is of the form: find $(\mathbf{u}_\Gamma, \mathbf{p}_0) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \widehat{S} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}. \tag{4.6}$$

We define two subspaces $\widehat{W}_{\Gamma,B}$ and $\widetilde{W}_{\Gamma,B}$ of $\widehat{W}_\Gamma$ and $\widetilde{W}_\Gamma$, respectively, as in [12, Definition 1]:

$$\widehat{\mathbf{W}}_{\Gamma,B} = \{\mathbf{w}_\Gamma \in \widehat{\mathbf{W}}_\Gamma \mid \widehat{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\},$$
$$\widetilde{\mathbf{W}}_{\Gamma,B} = \{\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \mid \widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\}.$$

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We call $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ the *benign subspaces* of $\widehat{\mathbf{W}}_{\Gamma} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma} \times Q_0$, respectively. With Lemma 3.1, it is easy to check that both operators $\widehat{S}_{\Gamma}$ and $\widetilde{S}_{\Gamma}$, given in (3.12) and (4.1), are symmetric, positive definite when restricted to the benign subspaces $\widehat{\mathbf{W}}_{\Gamma} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma} \times Q_0$, respectively and we also have

LEMMA 4.1. *For any* $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, $\widetilde{R}_D^T \mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$.

Proof. We need to show that for any $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, $\widetilde{R}_D^T \mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$. Given $\mathbf{w} = (\mathbf{w}_{\Gamma}, p_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, we have $\widetilde{B}_{0\Gamma} \mathbf{w}_{\Gamma} = 0$ and

$$\widetilde{R}_D^T \mathbf{w} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_{\Gamma} \\ p_0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_{\Gamma} \\ p_0 \end{bmatrix} \in \widehat{\mathbf{W}}_{\Gamma} \times Q_0. \tag{4.7}$$

We only need to show that $\widehat{B}_{0\Gamma} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_{\Gamma} = 0$ and we find that

$$\widehat{B}_{0\Gamma} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_{\Gamma} = \widetilde{B}_{0\Gamma} \widetilde{R}_{\Gamma} \widetilde{R}_{D,\Gamma}^T \mathbf{w}_{\Gamma} = \widetilde{B}_{0\Pi} \mathbf{w}_{\Pi} = 0.$$

Here we use the definitions of $\widehat{B}_{0\Gamma}$ and $\widetilde{B}_{0\Gamma}$ for the first equality. For the second, we use the fact that the Raviart-Thomas finite element functions only have degrees of freedom on edges/faces. In our BDDC algorithm, we choose the continuous primal interface velocity space $\mathbf{W}_{\Pi}$ and the subdomain dual interface velocity spaces $\mathbf{W}_{\Delta}^{(i)}$ such that, if $\mathbf{u}_{\Delta}^{(i)} \in \mathbf{W}_{\Delta}^{(i)}$, then $\mathbf{u}_{\Delta}^{(i)}$ has a zero edge/face average for each edge/face. In fact, $\widetilde{R}_{\Gamma} \widetilde{R}_{D,\Gamma}^T$ computes the average of the dual interface velocities $\mathbf{w}_{\Delta}$, then distributes them back to each subdomain and leaves $\mathbf{w}_{\Pi}$ the same. We recall that the weights at these nodes are the same for each edge/face since these nodes are shared by the same pair of subdomains. The averaged dual interface velocity still has a zero edge/face average for each edge/face. For the third equality, we use that $\widetilde{B}_{0\Gamma} \mathbf{w}_{\Gamma} = \widetilde{B}_{0\Pi} \mathbf{w}_{\Pi} = 0$, since $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$. □

Therefore, we can conclude that the preconditioned BDDC operator, defined in (4.6), is positive definite in the benign subspace $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.

5. Some auxiliary results. We first list some results for Raviart-Thomas finite element function spaces needed in our analysis. These results were originally given in [24, 21, 23].

We consider the interpolation operator Π_{RT}^H from $\widehat{\mathbf{W}}$ onto $\widehat{\mathbf{W}}^H$. Recall that $\widehat{\mathbf{W}}^H$ is the Raviart-Thomas finite element space on the coarse mesh with mesh size H, which is defined in terms of the degrees of freedom $\lambda_{\mathcal{F}}$, by

$$\lambda_{\mathcal{F}}(\Pi_{RT}^H \mathbf{u}) := \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} \mathbf{u} \cdot \mathbf{n} ds, \quad \mathcal{F} \subset \mathcal{F}_H.$$

We consider the stability of the interpolant Π_{RT}^H in the next lemma.

LEMMA 5.1. *There exists a constant* C*, which depends only on the aspect ratios of* $K \in \mathcal{T}_H$ *and of the elements of* $\mathcal{T}_h$*, such that, for all* $\mathbf{u} \in \widehat{\mathbf{W}}$*,*

$$\|div(\Pi_{RT}^H \mathbf{u})\|_{L^2(K)}^2 \leq \|div \mathbf{u}\|_{L^2(K)}^2,$$

$$\|\Pi_{RT}^H \mathbf{u}\|_{L^2(K)}^2 \leq C \left(1 + \log \frac{H}{h}\right) \left(\|\mathbf{u}\|_{L^2(K)}^2 + H_K^2 \|div \mathbf{u}\|_{L^2(K)}^2\right).$$

Proof. See [24, Lemma 4.1]. □

We define $N(\partial\Omega_i)$ as the the space of functions that are constant on each element of the edges/faces of the boundary of Ω_i and its subspace $N_0(\partial\Omega_i)$, of functions that have mean

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value zero on $\partial\Omega_i$. Let N^H be the space of functions μ defined on Γ, such that, for each subdomain Ω_i and each edge/face $\mathcal{F}$ of Ω_i, μ is constant on $\mathcal{F}$. We note that N^H is the space of normal components on Γ of vectors in $\widehat{\mathbf{W}}^H$.

The stable extension operator, defined in the next lemma, provides a divergence-free extension of boundary data given on $\partial\Omega_i$.

LEMMA 5.2. *There exists an extension operator $\tilde{\mathcal{H}}_i : N_0(\partial\Omega_i) \longrightarrow \mathbf{W}^{(i)}$, such that, for any $\mu \in N_0(\partial\Omega_i)$,*

$$div\tilde{\mathcal{H}}_i\mu = 0, \quad \textit{for } x \in \Omega_i,$$

and

$$\|\tilde{\mathcal{H}}_i\mu\|_{L^2(\Omega_i)} \le C\|\mu\|_{H^{-1/2}(\partial\Omega_i)}. \tag{5.1}$$

Here C is independent of h, H, and μ.

Proof. See [24, Lemma 4.3]. □

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/faces. Let $\zeta_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$, i.e., the function that is identically one on $\mathcal{F}$ and zero on $\partial\Omega_i\backslash\mathcal{F}$. We clearly have

$$\sum_{\mathcal{F}\subset\partial\Omega_i} \zeta_{\mathcal{F}}(x) = 1, \qquad \text{almost everywhere on } \ \partial\Omega_i\backslash\partial\Omega.$$

Given a function $\mu \in N(\partial\Omega_i)$ and a face $\mathcal{F} \subset \partial\Omega_i$, let

$$\mu_{\mathcal{F}} := \zeta_{\mathcal{F}}\mu \in N(\partial\Omega_i).$$

We have the following estimates for the edge/face components of the particular functions in $N(\partial\Omega_i)$ with a vanishing average on the subdomain edges/faces.

LEMMA 5.3. *Let $\mu \in N(\partial\Omega_i)$ with $\int_{\partial\Omega_i}\mu ds = 0$, and for any $\mathcal{F} \subset \partial\Omega_i$, $\int_{\mathcal{F}}\mu ds = \int_{\mathcal{F}}\mu_{\mathcal{F}} ds = 0$. There then exists a constant C, independent of h and μ_H, such that, for any $\mu_H \in N^H$,*

$$\begin{aligned}&\|\mu_{\mathcal{F}}\|^2_{\mathcal{H}^{-1/2}(\partial\Omega_i)}\\ &\le C\left(1+\log\frac{H}{h}\right)\left((1+\log\frac{H}{h})\|\mu+\mu_H\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu\|^2_{H^{-1/2}(\partial\Omega_i)}\right).\end{aligned} \tag{5.2}$$

Proof. See [24, Lemma 4.4]. □

The following lemma compares norms of traces on the subdomain boundaries that share an edge/face.

LEMMA 5.4. *Let Ω_i and Ω_j be two subdomains with a common edge/face $\mathcal{F}$. Let $\mu_{\mathcal{F}}$ be a function in $H^{-1/2}(\partial\Omega_i)$, that vanishes outside $\mathcal{F}$. Then, there is a constant C that depends only on the aspect ratios of Ω_i and Ω_j, such that*

$$\|\mu_F\|_{H^{-1/2}(\partial\Omega_i)} \le C\|\mu_F\|_{H^{-1/2}(\partial\Omega_j)}.$$

Proof. See [21, Lemma 5.5.2]. □

We next list some results for the benign subspace $\widetilde{W}_{\Gamma,B} \times Q_0$.

Let $\|\mathbf{w}\|^2_{\widetilde{S}} = \mathbf{w}^T\widetilde{S}\mathbf{w}$ and $\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = \mathbf{w}_\Gamma^T\widetilde{S}_\Gamma\mathbf{w}_\Gamma$. We then have

LEMMA 5.5. *Given any $\mathbf{w} \in \widetilde{W}_{\Gamma,B} \times Q_0$, we have*

$$\|\mathbf{w}\|^2_{\widetilde{S}} = \|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma}.$$

Proof.

$$\|\mathbf{w}\|_{\widetilde{S}}^2 = \mathbf{w}^T\widetilde{S}\mathbf{w} = \begin{bmatrix}\mathbf{w}_\Gamma^T & q_0^T\end{bmatrix}\begin{bmatrix}\widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0\end{bmatrix}\begin{bmatrix}\mathbf{w}_\Gamma \\ q_0\end{bmatrix} = \mathbf{w}_\Gamma^T\widetilde{S}_\Gamma\mathbf{w}_\Gamma = \|\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}^2.$$

□

We define the *average operator* by $E_D = \widetilde{R}\widetilde{R}_D^T$. We see that, for any vector $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_\Gamma \times Q_0$,

$$E_D\begin{bmatrix}\mathbf{w}_\Gamma \\ q_0\end{bmatrix} = \begin{bmatrix}\widetilde{R}_\Gamma & \\ & I\end{bmatrix}\begin{bmatrix}\widetilde{R}_{D,\Gamma}^T & \\ & I\end{bmatrix}\begin{bmatrix}\mathbf{w}_\Gamma \\ q_0\end{bmatrix} = \begin{bmatrix}E_{D,\Gamma}\mathbf{w}_\Gamma \\ q_0\end{bmatrix}, \tag{5.3}$$

where $E_{D,\Gamma} = \widetilde{R}_\Gamma\widetilde{R}_{D,\Gamma}^T$, which computes the average of the interface velocities across the subdomain interface. Lemma 4.1 shows that after averaging a benign vector across a subdomain interface the result is still benign.

An estimate of the norm of the E_D operator restricted to the benign subspace $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ is given in the next lemma.

LEMMA 5.6. *There exists a positive constant C, which is independent of H and h, and the number of subdomains, such that,*

$$\|E_D\mathbf{w}\|_{\widetilde{S}}^2 \le C\left(1 + \log\frac{H}{h}\right)^2 \|\mathbf{w}\|_{\widetilde{S}}^2, \quad \forall \mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0.$$

Proof. Given any $\mathbf{w} = (\mathbf{w}_\Gamma, q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, we know, from Lemma 4.1, that $\widetilde{R}_D^T\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$. Therefore, $E_D\mathbf{w} = \widetilde{R}_D\widetilde{R}_D^T\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$. We have, by Lemma 5.5, that

$$\begin{aligned}
&\|E_D\mathbf{w}\|_{\widetilde{S}}^2 \\
&\le 2\left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \|\mathbf{w} - E_D\mathbf{w}\|_{\widetilde{S}}^2\right) \\
&\le 2\left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \|\mathbf{w}_\Gamma - E_{D,\Gamma}\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}^2\right) \\
&= 2\left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \|\overline{R}_\Gamma\left(\mathbf{w}_\Gamma - E_{D,\Gamma}\mathbf{w}_\Gamma\right)\|_{S_\Gamma}^2\right) \\
&= 2\left(\|\mathbf{w}\|_{\widetilde{S}}^2 + \sum_{i=1}^N \|\overline{R}_\Gamma^{(i)}\left(\mathbf{w}_\Gamma - E_{D,\Gamma}\mathbf{w}_\Gamma\right)\|_{S_\Gamma^{(i)}}^2\right).
\end{aligned} \tag{5.4}$$

Let $\mathbf{w}_i = \overline{R}^{(i)}\mathbf{w}_\Gamma$ and set

$$\mathbf{v}_i(x) := \overline{R}_\Gamma^{(i)}(\mathbf{w}_\Gamma - E_{D,\Gamma}\mathbf{w}_\Gamma)(x) = \sum_{j\in\mathcal{N}_x} \delta_j^\dagger(\mathbf{w}_i(x) - \mathbf{w}_j(x)), \quad x \in \partial\Omega^i \cap \Gamma. \tag{5.5}$$

Here $\mathcal{N}_x$ is the set of indices of the subdomains that have x on their boundaries. Since a fine edge/face only belongs to exactly two subdomains, for an edge/face $\mathcal{F}^{ij} \subset \partial\Omega_i$ that is also shared by Ω_j, we have

$$\mathbf{v}_i = \delta_j^\dagger\mathbf{w}_i - \delta_j^\dagger\mathbf{w}_j, \text{ on } \mathcal{F}^{ij}. \tag{5.6}$$

We note that the simple inequality

$$c_i{\delta_j^\dagger}^2 \le \min(c_i, c_j), \tag{5.7}$$

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holds for $\gamma \in [1/2, \infty)$.

Since $\mathbf{v}_i \cdot \mathbf{n}$ has a vanishing mean value on each face of Ω_i, we can define, by Lemma 5.2, $\mathbf{v}_i^E = \tilde{\mathcal{H}}_i(\mathbf{v}_i \cdot \mathbf{n})$. Then

$$div \mathbf{v}_i^E = 0, \quad \text{for } x \in \Omega_i, \tag{5.8}$$

and,

$$\|\mathbf{v}_i^E\|^2_{L^2(\Omega_i)} \le C \|\mathbf{v}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)}. \tag{5.9}$$

We then obtain

$$\begin{aligned} \|\mathbf{v}_i\|^2_{S_\Gamma^{(i)}} &= c_i \|\mathbf{v}_i^E\|^2_{L^2(\Omega_i)} \le C c_i \|\mathbf{v}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} \\ &\le C c_i \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} \|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i \cdot \mathbf{n})\|^2_{H^{-1/2}(\partial\Omega_i)}. \end{aligned} \tag{5.10}$$

Using (5.6), we have, with $\overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}}$ the average over $\mathcal{F}^{ij}$,

$$\begin{aligned} & c_i \|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i \cdot \mathbf{n})\|^2_{H^{-1/2}(\partial\Omega_i)} \\ = {} & c_i \|\zeta_{\mathcal{F}^{ij}} \delta_j^\dagger (\mathbf{w}_i - \mathbf{w}_j) \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} \\ \le {} & 2 c_i \delta_j^{\dagger^2} \left(\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)} \right. \\ + {} & \left. \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_j \cdot \mathbf{n} - \overline{(\mathbf{w}_j \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)} \right) \\ \le {} & 2 c_i \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)} \\ + {} & 2 c_j \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_j \cdot \mathbf{n} - \overline{(\mathbf{w}_j \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_j)}. \end{aligned}$$

Here we use Lemma 5.4 and (5.7) for the last inequality.

We only need to estimate the first term since the second term can be estimated similarly.

Since $\mathbf{w}$ is in the benign space, $\mathbf{w}_i \cdot \mathbf{n}$ has vanishing mean value on $\partial\Omega_i$. By Lemma 5.2, we can construct

$$\mathbf{w}_i^E = \tilde{\mathcal{H}}_i(\mathbf{w}_i \cdot \mathbf{n}),$$

such that,

$$div \mathbf{w}_i^E = 0, \quad \text{for } x \in \Omega_i.$$

Let $\mathbf{w}_0^E \in \widehat{\mathbf{W}}$ be defined by

$$\mathbf{w}_0^E = \begin{cases} \mathbf{w}_i^E & \text{in } \Omega_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{u}_H = \Pi_{RT}^H \mathbf{w}_0$ and $\mu_H = \mathbf{u}_H \cdot \mathbf{n}$. By the definition of Π_{RT}^H, we know that $\zeta_{\mathcal{F}^{ij}} \mu_H = \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}}$, and for any $\mathcal{F} \subset \partial\Omega_i$, $\int_{\mathcal{F}} (\mathbf{w}_i \cdot \mathbf{n} - \mu_H) ds = 0$. Using Lemma 5.3, we have

(5.11)

$$\begin{aligned} & \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \overline{(\mathbf{w}_i \cdot \mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)} \\ = {} & \|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i \cdot \mathbf{n} - \mu_H)\|^2_{H^{-1/2}(\partial\Omega_i)} \\ \le {} & C \left(1 + \log \frac{H}{h}\right) \left((1 + \log \frac{H}{h}) \|\mathbf{w}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mathbf{w}_i \cdot \mathbf{n} - \mu_H\|^2_{H^{-1/2}(\partial\Omega_i)} \right) \\ \le {} & C \left(1 + \log \frac{H}{h}\right) \left((1 + \log \frac{H}{h}) \|\mathbf{w}_i \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu_H\|^2_{H^{-1/2}(\partial\Omega_i)} \right), \end{aligned}$$

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where we use the triangle inequality for the last inequality.

By Lemma 5.1, we know that

$$\|div\mathbf{u}_H\|^2_{L^2(\Omega_i)} \le \|div\mathbf{w}_0^E\|^2_{L^2(\Omega_i)} = \|div\mathbf{w}_i^E\|^2_{L^2(\Omega_i)} = 0. \tag{5.12}$$

and

$$\begin{aligned}\|\mathbf{u}_H\|^2_{L^2(\Omega_i)} &\le C\left(1+\log\frac{H}{h}\right)\left(\|\mathbf{w}_0^E\|^2_{L^2(\Omega_i)} + H^2\|div\mathbf{w}_0^E\|^2_{L^2(\Omega_i)}\right)\\ &= C\left(1+\log\frac{H}{h}\right)\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}.\end{aligned} \tag{5.13}$$

Using (5.11), (2.4), (5.12), and (5.13), we obtain:

$$\begin{aligned}&\|\zeta_{\mathcal{F}^{ij}}(\mathbf{w}_i\cdot\mathbf{n} - \overline{(\mathbf{w}_i\cdot\mathbf{n})}_{\mathcal{F}^{ij}})\|^2_{H^{-1/2}(\partial\Omega_i)}\\ &\le C\left(1+\log\frac{H}{h}\right)\left((1+\log\frac{H}{h})\|\mathbf{w}_i\cdot\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega_i)} + \|\mu_H\|^2_{H^{-1/2}(\partial\Omega_i)}\right)\\ &\le C\left(1+\log\frac{H}{h}\right)\left((1+\log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{H(div,\Omega_i)} + \|\mathbf{u}_H\|^2_{H(div,\Omega_i)}\right)\\ &\le C\left(1+\log\frac{H}{h}\right)\left((1+\log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)} + (1+\log\frac{H}{h})\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}\right)\\ &\le C\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_i^E\|^2_{L^2(\Omega_i)}\\ &\le \frac{C}{c_i}\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_i\|^2_{S_\Gamma^{(i)}}.\end{aligned}$$

Here we use that $div\mathbf{w}_i^E = 0$ for the third inequality.

Finally, we obtain

$$c_i\|\zeta_{\mathcal{F}^{ij}}(\mathbf{v}_i\cdot\mathbf{n})\|^2_{H^{-1/2}(\partial\Omega_i)} \le C\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_i\|^2_{S_\Gamma^{(i)}}. \tag{5.14}$$

Since $\mathbf{w}$ is benign, we have, from Lemma 5.5, that $\|\mathbf{w}\|_{\widetilde{S}} = \|\mathbf{w}_\Gamma\|_{\widetilde{S}_\Gamma}$; then by Equations (5.4), (5.5), (5.10), and (5.14), we have

$$\|E_D\mathbf{w}\|^2_{\widetilde{S}} \le C\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = C\left(1+\log\frac{H}{h}\right)^2\|\mathbf{w}\|^2_{\widetilde{S}}.$$

□

6. Condition number estimate for the BDDC preconditioner. We are now ready to formulate and prove our main result; this follows directly from the proof of [12, Theorem 1] by using Lemma 4.1 and Lemma 5.6.

THEOREM 6.1. *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the benign space $\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$ and*

$$\langle\mathbf{u},\mathbf{u}\rangle_{\widehat{S}} \le \left\langle M^{-1}\widehat{S}\mathbf{u},\mathbf{u}\right\rangle_{\widehat{S}} \le C\left(1+\log\frac{H}{h}\right)^2\langle\mathbf{u},\mathbf{u}\rangle_{\widehat{S}},\ \forall\mathbf{u}\in\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0. \tag{6.1}$$

Here, C is a constant which is independent of h and H.

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TABLE 8.1
Condition number bounds and iteration counts, for a pair of the BDDC and the FBA algorithms, with a change of the number of subdomains. $H/h = 8$ *and* $c \equiv 1$.

Num. of sub.	BDDC		FBA	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	5	1.66	5	2.43
8×8	8	2.95	8	2.90
12×12	9	3.08	7	2.75
16×16	9	3.13	7	2.72
20×20	8	3.15	7	2.71

TABLE 8.2
Condition number bounds and iteration counts, for a pair of the BDDC and the FBA algorithms, with a change of the size of subdomain problems. 8×8 *subdomains and* $c \equiv 1$.

	BDDC		FBA	
$\frac{H}{h}$	Iter.	Cond. Num.	Iter.	Cond. Num.
4	8	2.17	7	2.12
8	8	2.95	8	2.90
12	9	3.47	9	3.45
16	9	3.88	9	3.83
20	9	4.20	9	4.15

7. Comparison with an edge/face-based iterative substructuring domain decomposition method. We define an edge/face-based iterative substructuring domain decomposition method as a hybrid method (see [22, Section 2.5.2]). Similar to the BNN method, as defined in [18, Section 4], the coarse problems and the local problems are treated multiplicatively and additively, respectively, in this preconditioner. We use a different coarse component, i.e., a different choice of the matrix L_0 for the coarse problem, but the same local problems as in [18, Section 4]. Here, each column of L_0 corresponds to an edge/face on the interface of Ω and is given by the positive scaling factor $\delta_i^\dagger(x)$. It is clear and we can prove that the condition number with this preconditioner is also bounded by $C\left(1+\log\frac{H}{h}\right)^2$. We will call this method the FBA.

The size and sparsity of the coarse problems of the BDDC and the FBA are the same. However, the two algorithms are different. The FBA is a hybrid algorithm and a coarse problem has to be solved before the rest of the iterations. In contrast, only the variables have to be changed at the beginning of computation with the BDDC, to accommodate the edge/face constraints. In addition, the FBA requires two Dirichlet local problems and one singular local Neumann problem in each iteration, whereas the BDDC requires one local Dirichlet problem and two nonsingular local Neumann problem. In the latter algorithm, singular problems are avoided. Numerical experiments show that FBA is somewhat slower than BDDC.

8. Numerical experiments. We have applied our BDDC and FBA algorithms to the model problem (2.1), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (2.1) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure, respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and con-

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TABLE 8.3
Condition number bounds and iteration counts, for a pair of the BDDC and the FBA algorithms, with a change of the number of subdomains. $H/h = 8$ and c is in a checkerboard pattern.

Num. of sub.	BDDC		FBA	
$n_x \times n_y$	Iter.	Cond. Num.	Iter.	Cond. Num.
4×4	3	1.03	5	2.20
8×8	3	1.06	7	2.44
12×12	3	1.07	7	2.49
16×16	3	1.08	7	2.51
20×20	3	1.08	7	2.53

TABLE 8.4
Condition number bounds and iteration counts, for a pair of the BDDC and the FBA algorithms, with a change of the size of subdomain problems. 8×8 subdomains and c is in a checkerboard pattern.

$\frac{H}{h}$	BDDC		FBA	
	Iter.	Cond. Num.	Iter.	Cond. Num.
4	3	1.04	7	2.00
8	3	1.06	7	2.44
12	4	1.10	8	2.69
16	4	1.11	8	2.88
20	4	1.12	8	3.02

dition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we take the coefficient $c \equiv 1$. Table 8.1 gives the iteration counts and the estimate of the condition numbers, with a change of the number of subdomains. We find that the condition number is independent of the number of subdomains for both algorithms. Table 8.2 gives the results with a change of the size of the subdomain problems.

In the second set of experiments, we take the coefficient $c = 1$ in half the subdomains and $c = 100$ in the neighboring subdomains, in a checkerboard pattern. Table 8.3 gives the iteration counts, and condition number estimates with a change of the number of subdomains. We find that the condition numbers are independent of the number of subdomains for both algorithms. Table 8.4 gives the results with a change of the size of the subdomain problems.

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