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RECURSIVE COMPUTATION OF CERTAIN INTEGRALS OF ELLIPTIC TYPE*

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Abstract. An algorithm for the numerical calculation of the integral function

$$N_n(x) = \int_0^{\pi/2} \frac{\cos^{2n}(\Phi)}{\sqrt{1-x\cdot\sin^2(\Phi)}} \cdot d\Phi \quad (0 \le x < 1;\ n = 0,1,2,\ldots),$$

distinguished solution of the second-order difference equation

$$(2n+1)\cdot x \cdot N_{n+1}(x) + 2n\cdot(1-2x)\cdot N_n(x) = (2n-1)\cdot(1-x)\cdot N_{n-1}(x) \quad (n = 1,2,\ldots),$$

that uses the recurrence relation and its related continued fraction expansion, is described and discussed. The numerical efficiency of the algorithm is analysed for various x values of the interval $(0 \le x < 1)$. A twelve digits tabulation of $N_n(x)$ for $n = 1(1)20$ and $x = 0(0.02)1$ is presented as example of the algorithm utilization.

Key words. recurrence relations, elliptic integrals, continued fractions

AMS subject classifications. 65Q05, 33E05, 11A55

1. Introduction. This paper presents an algorithm for the numerical computation of the integral function

$$N_n(x) = \int_0^{\pi/2} \frac{\cos^{2n}(\Phi)}{\sqrt{1-x\cdot\sin^2(\Phi)}} \cdot d\Phi \quad (0 \le x < 1;\ n = 0,1,2,\ldots), \tag{1.1}$$

that occurs in particular motion problems. Function (1.1), that can also be written in terms of hypergeometric function

$$N_n(x) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cdot F(\frac{1}{2}, \frac{1}{2}; n+1; x),$$

is a solution of the second-order difference equation

$$(2n+1)\cdot x \cdot y_{n+1}(x) + 2n\cdot(1-2x)\cdot y_n(x) = (2n-1)\cdot(1-x)\cdot y_{n-1}(x) \quad (n = 1,2,\ldots) \tag{1.2}$$

as it's readily verified writing

$$N_{n+1}(x) = N_n(x) - \int_0^{\frac{\pi}{2}} \frac{\sin^2(\Phi)\cdot\cos^{2n}(\Phi)}{\sqrt{1-x\cdot\sin^2(\Phi)}} \cdot d\Phi$$

and evaluating the integral by parts.
A second solution of (1.2), linearly independent of (1.1), is given by

$$\begin{aligned} G_n(x) = {} & \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cdot \ln(x) \cdot F(\frac{1}{2}, \frac{1}{2}; n+1; x) + \frac{\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}} \cdot \\ & \cdot \sum_{l=1}^{\infty} \left[\frac{(\Gamma(l+\frac{1}{2}))^2}{\Gamma(n+l+1)} \cdot \frac{x^l}{l!} \cdot D_l^n \right] - \frac{\sqrt{\pi^3}\cdot\Gamma(n+\frac{1}{2})}{2\cdot\Gamma(-n)\cdot\Gamma(n+1)} \cdot \\ & \cdot \sum_{l=1}^{n} \left[\frac{\Gamma(l-n)}{[\Gamma(l+\frac{1}{2})]^2} \cdot \frac{(l-1)!}{x^l} \right] \quad (|x| < 1) \end{aligned} \tag{1.3}$$

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where:

$$D_l^n = \psi(1) + \psi(n+1) - \psi(n+l+1) - \psi(l+1) + 2\cdot\psi(l+\frac{1}{2}) - 2\cdot\psi(\frac{1}{2}),$$

and $\psi(z)$ is the Euler's *digamma function*.
The functions (1.1) and (1.3) obey the Perron relations

$$\lim_{n\to\infty}\frac{N_{n+1}(x)}{N_n(x)} = t_1 = 1$$
$$\lim_{n\to\infty}\frac{G_{n+1}(x)}{G_n(x)} = t_2 = \frac{x-1}{x}$$

where t_1 and t_2 are roots of the characteristic polynomial of Poincaré

$$t^2\cdot x + (1-2x)\cdot t - (1-x) = 0$$

related to the recurrence relation (1.2), and also obey the relation

$$\lim_{n\to\infty}\frac{N_n(x)}{G_n(x)} = 0.$$

Therefore, according to the Pincherle theorem [1], $N_n(x)$ is the *distinguished solution* of the recurrence relation (1.2) and may be represented by the following continued fraction expansion, directly deriving from (1.2):

$$\frac{N_n(x)}{N_{n-1}(x)} = \frac{(2n-1)\cdot(1-x)}{2n\cdot(1-2x)+}\;\frac{(2n+1)^2\cdot x\cdot(1-x)}{2(n+1)\cdot(1-2x)+}\;\frac{(2n+3)^2\cdot x\cdot(1-x)}{2(n+2)\cdot(1-2x)+\dots}\quad(n=1,2,\dots). \tag{1.4}$$

This continued fraction is *periodic in the limit* and, according to the Pringsheim theorem [2], it's uniformly convergent in all the complex x-plane, with the exception of the points of the straight line $x = 1/2$.

2. Computation procedures. A simple procedure allows us to transform the second-order difference equation (1.2) in two first-order difference equations. Let

$$R_n(x) = \frac{(2n+1)^2\cdot x\cdot(1-x)}{2(n+1)\cdot(1-2x)+}\;\frac{(2n+3)^2\cdot x\cdot(1-x)}{2(n+2)\cdot(1-2x)+}\;\frac{(2n+5)^2\cdot x\cdot(1-x)}{2(n+3)\cdot(1-2x)+\dots}\quad(n=1,2,3,\dots) \tag{2.1}$$

we have from (1.4) and (2.1):

$$R_n(x) = 2n\cdot\left[\frac{(2n-1)^2\cdot x\cdot(1-x)}{2n\cdot R_{n-1}(x)} - (1-2x)\right]\quad(n=1,2,3,\dots) \tag{2.2}$$

$$N_n(x) = N_{n-1}(x)\cdot\frac{R_{n-1}(x)}{(2n-1)\cdot x}\quad(n=1,2.3,\dots) \tag{2.3}$$

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The recursive relations (2.2) and (2.3) allows us to calculate the function $N_n(x)$ for various values of x and n, starting from the initial conditions:

$$\begin{aligned} R_0(x) &= \frac{[E(x) - (1-x) \cdot K(x)]}{K(x)} \\ N_0(x) &= K(x) \end{aligned} \tag{2.4}$$

where $K(x)$ and $E(x)$ are the well-known complete elliptic integrals of the first and the second kind:

$$K(x) = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1 - x \cdot \sin^2(\Phi)}}; \qquad E(x) = \int_0^{\pi/2} \sqrt{1 - x \cdot \sin^2(\Phi)} \cdot d\Phi.$$

The recursive relations (2.2) and (2.3) are applicable for $0.5 \leq x < 1$, interval in which the numerical results of $R_n(x)$, for any n, are only affected by round-off errors. For $0 \leq x \leq 0.5$ the recursive relation (2.2) is affected by numerical instability as n increases. To this purpose the expression

$$N_c \cong -0.43429 \cdot n \cdot \ln \left[1 - \frac{1-2x}{x \cdot (1-x)} \right] \quad (x > 0),$$

obtained from (2.2) can provide a good approximation of the digits lost number in function of the cycles number n and the x values.
In the interval $0 \leq x \leq 0.5$ we can apply (2.2) backward:

$$R_{n-1}(x) = \frac{(2n-1)^2 \cdot x \cdot (1-x)}{2n \cdot (1-2x) + R_n(x)} \quad (n = M, M-1, \ldots) \tag{2.5}$$

Being

$$\begin{aligned} \frac{N_n(x)}{N_{n-1}(x)} &= \frac{(2n-1) \cdot (1-x)}{2n \cdot (1-2x) + R_n(x)} \quad (n = 1, 2, \ldots, M) \\ N_0(x) &= K(x), \end{aligned} \tag{2.6}$$

the integral function $N_n(x)$ can be written as follows:

$$\begin{aligned} N_n(x) = K(x) \cdot \frac{(2n-1) \cdot (1-x)}{2n(1-2x) + R_n(x)} \cdot \frac{(2n-3) \cdot (1-x)}{2(n-1) \cdot (1-2x) + R_{n-1}(x)} \cdots \\ \cdots \frac{3 \cdot (1-x)}{4 \cdot (1-2x) + R_2(x)} \cdot \frac{1-x}{2 \cdot (1-2x) + R_1(x)} \quad (n \leq M). \end{aligned} \tag{2.7}$$

This procedure requires the knowledge of only initial condition that is $N_0(x) = K(x)$, and the computation of only one continued fraction that is (2.1), for $n = M$. With the backward calculation (2.5) the reason of the accuracy loss due to the difference in (2.2) is removed. The values of $N_n(x)$ for $x = 1$ are directly obtainable from (1.1):

$$N_0(1) = \infty; \quad N_1(1) = 1; \quad N_n(1) = \frac{2n-2}{2n-1} \cdot N_{n-1}(1) \; (n \geq 2)$$

.

3. Efficiency analysis of the continued fraction $R_n(x)$. The function (2.1) can also be written in the following equivalent form

$$R_n(x) = 2n \cdot (1-2x) \cdot F[n,x] \tag{3.1}$$

where

$$F[n,x] = \cfrac{(2n+1)^2/[4n\cdot(n+1)]\cdot[x(1-x)/(1-2x)^2]}{1+\cfrac{(2n+3)^2/[4(n+1)\cdot(n+2)]\cdot[x(1-x)/(1-2x)^2]}{1+\cfrac{(2n+5)^2/[4(n+2)\cdot(n+3)]\cdot[x(1-x)/(1-2x)^2]}{1+\cdots}}} \quad (n=1,2,\ldots) \tag{3.2}$$

is a continued fraction, better fitting the requirements for automatic computation. We have investigated the numerical efficiency of the continued fraction $F[n,x]$ using the classical methodology proposed by Wynn [3]. In detail, we have built tables containing, for various values of x, the least order number m of the convergent C_m of the continued fraction $F[n,x]$ necessary to achieve a given accuracy, ϵ, according to the following inequality

$$\left|\frac{C_m - C_\infty}{C_\infty}\right| \le \epsilon = \frac{1}{2}\cdot 10^{-h} \quad (h=1,2,3,\ldots).$$

Numerical efficiency data of the continued fraction $F[n,x]$ for $0 \le x < 1$ and $n=1$ are shown in Table 3.1. These data are valid also for $F[n,x]$ with $n > 1$, as verified from numerical experiments. The data of Table 3.1 can be used as starting points for the ascending

TABLE 3.1
Numerical efficiency of the continued fraction $F[n,x]$ for $n=1$ and $0 \le x \le 1$.

h		4	6	8	10	12	14
x				m			
0.00	1.00	1	1	1	1	1	1
0.05	0.95	4	5	7	9	10	12
0.10	0.90	5	7	9	11	13	15
0.15	0.85	6	9	12	14	17	19
0.20	0.80	8	11	14	18	21	24
0.25	0.75	10	14	18	22	27	30
0.30	0.70	13	18	24	29	34	39
0.35	0.65	17	25	32	40	47	53
0.40	0.60	26	38	49	60	72	81
0.45	0.55	53	76	99	-	-	-
0.50	0.50	-	-	-	-	-	-

calculation of the continued fraction $F[n,x]$, for any n , described by the following procedure:

$$\begin{aligned} F[n,x] &= F_1 \\ F_j &= \left[\frac{(2n+2j-1)^2\cdot x\cdot(1-x)}{4\,(n+j-1)\cdot(n+j)\cdot(1-2x)^2}\right] / \left[1+F_{j+1}\right] \quad (j=m,m-1,\ldots). \\ F_{m+1} &= 0 \end{aligned} \tag{3.3}$$

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The numerical efficiency of the continued fraction $F[n,x]$ can be improved introducing the function

$$\frac{1}{2}\cdot\left[-1+\sqrt{1+\frac{4x\cdot(1-x)}{(1-2x)^2}}\right]=\cfrac{x\cdot(1-x)/(1-2x)^2}{1+\cfrac{x\cdot(1-x)/(1-2x)^2}{1+\cfrac{x\cdot(1-x)/(1-2x)^2}{1+\cdots}}}$$

that can be a good approximation, for large values of m, of the continued fraction

$$\cfrac{(2n+2m+1)^2/[4(n+m)\cdot(n+m+1)]\cdot[x\cdot(1-x)/(1-2x)^2]}{1+\cfrac{(2n+2m+3)^2/[4(n+m+1)\cdot(n+m+2)]\cdot[x\cdot(1-x)/(1-2x)^2]}{1+\cfrac{(2n+2m+5)^2/[4(n+m+2)\cdot(n+m+3)]\cdot[x\cdot(1-x)/(1-2x)^2]}{1+\cdots}}}.$$

We obtain in this way

$$(3.4)\ F[n,x]=\cfrac{(2n+1)^2/[4n\cdot(n+1)]\cdot[x\cdot(1-x)/(1-2x)^2]}{1+\cfrac{(2n+3)^2/[4(n+1)\cdot(n+2)]\cdot[x\cdot(1-x)/(1-2x)^2]}{1+\cdots\cfrac{(2n+2m-1)^2/[4(n+m-1)\cdot(n+m)]\cdot[x\cdot(1-x)/(1-2x)^2]}{[1+\sqrt{1+[4x\cdot(1-x)/(1-2x)^2]}]/2}}}.$$

The numerical efficiency data of the continued fraction $F[n,x]$ so modified, collected for $n=1$ in the Table 3.2, show a significant efficiency improvement in comparison to the original $F[n,x]$. In this case the recursive procedure for the ascending calculation of continued

TABLE 3.2
Numerical efficiency of the continued fraction $F[n,x]$ for $n=1$ and $0\le x\le 1$·, with introduction of the analytical approximation for large values of m (see the recursive procedure (3.5))

h		4	6	8	10	12	14
x		m					
0.00	1.00	1	1	1	1	1	1
0.05	0.95	3	4	5	7	8	9
0.10	0.90	3	5	7	9	11	12
0.15	0.85	4	6	8	11	13	15
0.20	0.80	4	7	10	13	16	18
0.25	0.75	5	8	12	16	19	23
0.30	0.70	6	10	15	19	24	29
0.35	0.65	7	13	19	25	32	38
0.40	0.60	9	17	27	37	47	56
0.45	0.55	14	29	47	67	87	-
0.50	0.50	-	-	-	-	-	-

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fraction $F[n,x]$ is modified as follows:

$$\begin{aligned} &F[n,x] = F_1 \\ &F_j = \left[\frac{(2n+2j-1)^2 \cdot x \cdot (1-x)}{4\,(n+j-1)\cdot(n+j)\cdot(1-2x)^2}\right] / \left[1+F_{j+1}\right] \quad (j = m, m-1, \ldots) \\ &F_{m+1} = \left[-1+\sqrt{1+\left[4x\cdot(1-x)\,/\,(1-2x)^2\right]}\right]/2 \end{aligned} \tag{3.5}$$

The starting data for (3.5), collected in Table 3.2, are valid for any n, as confirmed by numerical experiments.
The continued fraction $F[n,x]$ is also convergent in the half-planes $x<0$ and $x>1$, according to the possibility of analytical extension of $N_n(x)$ as shown in the following expressions:

$$\begin{aligned} N_n(x) &= \frac{1}{\sqrt{1-x}} \cdot \int_0^{\pi/2} \frac{\sin^{2n}(\Phi)}{\sqrt{1-x\cdot\sin^2(\Phi)/(x-1)}} \cdot d\Phi \\ &= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cdot \frac{1}{\sqrt{1-x}} \cdot F\left[n+\frac{1}{2}, n+\frac{1}{2}; 1; \frac{x}{1-x}\right] \quad (x<0) \\ N_n(1-x) &= \frac{1}{\sqrt{x}} \cdot \int_0^{\pi/2} \frac{\sin^{2n}(\Phi)}{\sqrt{1-(x-1)\cdot\sin^2(\Phi)/x}} \cdot d\Phi \\ &= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cdot \frac{1}{\sqrt{x}} \cdot F\left[n+\frac{1}{2}, n+\frac{1}{2}; 1; \frac{x-1}{x}\right] \quad (x>1) \end{aligned}$$

directly deriving from (1.1) by easy transformation.
The efficiency data of $F[n,x]$ for $n = 1$ and related to the intervals $(-\infty < x < 0)$ and $(1 < x < \infty)$, expressed in terms of the variable $y = 4x\cdot(1-x)\,/\,(1-2x)^2$ for $(-1 < y < 0)$, are shown in Table 3.3. Also in this case the efficiency data are almost independent on the values of variable n, as verified by numerical experiments.

TABLE 3.3
Numerical efficiency of the continued fraction $F[n,x]$ for $n = 1$ and $x > 1$ and $x < 0$, expressed in terms of the variable $y = 4x\cdot(1-x)\,/\,(1-2x)^2$ for $(-1 < y < 0)$.

h	4	6	8	10	12	14
$y = 4x\cdot(1-x)\,/\,(1-2x)^2$			m			
0.00	1	1	1	1	1	1
-0.10	3	5	6	7	8	9
-0.20	4	6	7	9	10	12
-0.30	5	7	8	10	12	14
-0.40	5	8	10	12	14	16
-0.50	6	9	11	14	17	19
-0.60	7	10	13	16	19	22
-0.70	8	12	16	20	23	27
-0.80	10	15	20	25	30	34
-0.90	15	22	29	36	43	49
-1.00	-	-	-	-	-	-

4. Computation of the initial conditions. For the numerical computation of the initial conditions (2.4) the known power series expansions and polynomial approximations of the functions $K(x)$ and $E(x)$ [4] [5] can be used. To reach a high accuracy we have used the following power series expansions, obtained by the quadratic and linear transformations of the hypergeometric functions describing $K(x)$ and $E(x)$ [6]:

- $(0 \leq x < 0.8)$:

$$
\begin{aligned}
K(x) &= \frac{\pi}{1+\sqrt{1-x}} \cdot \left[1 + \sum_{n=1}^{\infty} g_n^2 \cdot \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{2n}\right] \\
R_0(x) \cdot K(x) &= \pi \cdot \left[\frac{1-\sqrt{1-x}}{2} + \sum_{n=1}^{\infty} g_n^2 \cdot \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{2n}\right] - \\
&\quad -\pi \cdot \left(1+\sqrt{1-x}\right) \cdot \sum_{n=1}^{\infty} g_n^2 \cdot \frac{4n-1}{4n-2} \cdot \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{2n}
\end{aligned}
\tag{4.1}
$$

- $(0.80 \leq x < 1)$:

$$
\begin{aligned}
K(x) &= \ln \frac{4}{\sqrt{1-x}} \cdot \left[1 + \sum_{n=1}^{\infty} g_n^2 \cdot (1-x)^n\right] - \sum_{n=1}^{\infty} g_n^2 \cdot \nu_n \cdot (1-x)^n \\
R_0(x) \cdot K(x) &= \frac{1-x}{4} \cdot \ln \frac{1-x}{16} \cdot \left(1 + \sum_{n=1}^{\infty} \frac{g_n^2}{n+1} \cdot (1-x)^n\right) + \\
&\quad + \frac{3+x}{4} + \frac{1-x}{2} \cdot \sum_{n=1}^{\infty} \frac{g_n^2 \cdot \mu_n}{n+1} \cdot (1-x)^n
\end{aligned}
\tag{4.2}
$$

where

$$
g_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}; \qquad \nu_n = \sum_{m=1}^{n} \frac{1}{m \cdot (2m-1)};
$$

$$
\mu_n = \nu_n - \frac{1}{2(n+1)}.
$$

The numerical efficiency of the power series expansions (4.1), (4.2), expressed in terms of least number m of partial sums of the power series expansions necessary to reach an accuracy not greater than $\epsilon = 0.5 \cdot 10^{-h}$ $\quad (h = 4, 6, 8, 10, 12, 14)$, is shown in the Tables 4.1, 4.2 respectively.

5. Final remarks. The algorithm described in section 2 is specific for the numerical calculation of the integral function. The advantage of the proposed algorithm, instead of using a generalized method to compute the hypergeometric function $F[a, b; c; x]$, such as that proposed by Forrey [7], is particularly evident in the domain $(0.5 \leq x < 1)$ and in the case of $c - a - b =$ integer, that is the case of the we aim to investigate. In fact the Forrey method requires a complex mechanism based on finite differences in order to solve the singularities occurring in the linear transformation, whereas the use of the proposed simple recursive relations (2.2) and (2.3) is undoubtedly to be preferred.
To show the efficiency of the proposed algorithm, we have developed a numerical tabulation

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TABLE 4.1
Numerical efficiency of $K(x)$ *and* $R_0(x)\cdot K(x)$ *) for* $(0 \leq x \leq 0.90)$ *(4.1).*

	$K(x)$						$R_0(x)\cdot K(x)$)					
h	4	6	8	10	12	14	4	6	8	10	12	14
x	m						m					
0.00	1	1	1	1	1	1	1	1	1	1	1	1
0.10	1	1	2	2	3	4	1	2	2	3	4	4
0.20	1	2	2	3	4	5	1	2	3	4	4	5
0.30	1	2	3	4	5	6	2	3	3	4	5	6
0.40	1	2	4	5	6	7	2	3	4	5	6	7
0.50	2	3	4	5	7	8	2	3	5	6	7	8
0.60	2	4	5	6	8	9	3	4	5	7	8	10
0.70	3	4	6	8	10	11	3	5	6	8	10	12
0.80	3	6	8	10	12	14	4	6	8	10	13	15
0.90	5	8	11	15	18	22	5	8	12	15	18	22

TABLE 4.2
Numerical efficiency of $K(x)$ *and* $R_0(x)\cdot K(x)$ *for* $(0.75 \leq x \leq 0.99)$ *(4.2)*.

	$K(x)$						$R_0(x)\cdot K(x)$)					
h	4	6	8	10	12	14	4	6	8	10	12	14
x	m						m					
0.75	4	7	10	13	16	20	3	5	8	11	14	17
0.80	4	6	9	11	14	17	2	5	7	9	12	15
0.85	3	5	7	10	12	14	2	4	6	8	10	12
0.90	3	4	6	8	10	12	1	3	5	6	8	10
0.95	2	3	5	6	8	9	1	2	3	5	6	8
0.96	2	3	4	6	7	8	1	2	3	4	6	7
0.97	2	3	4	5	7	8	1	2	3	4	5	6
0.98	1	2	4	5	6	7	1	1	2	4	5	6
0.99	1	2	3	4	5	6	1	1	2	3	4	5

of the integral function $N_n(x)$ for $x = 0(0.02)1$ and $n = 1(1)20$, with twelve right digits $(12.D)$ (cutting off the results at 13th digit), as shown in Tables 5.1, 5.2, 5.3, 5.4, 5.5.
The recursive relations (2.2) (2.3) and (2.5) and (2.6)(2.7) respectively in the ranges $(0.45 \leq x < 1)$ and $(0 \leq x < 0.45)$ have been adopted. The associated continued fraction $F[n, x]$ has been calculated by means of the ascending method described in (3.5).

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TABLE 5.1
Tabulation of $N_n(x)$ for $x = 0(0.02)1$ and $n = 1(1)4$ $12D$.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
0.00	0.785398163397	0.589048622548	0.490873852123	0.429514620607
0.02	0.787376540324	0.590035939077	0.491490224948	0.429945755569
0.04	0.789385317163	0.591034583130	0.492112237553	0.430380170463
0.06	0.791425485963	0.592044848588	0.492740011293	0.430817925558
0.08	0.793498090584	0.593067042028	0.493373671988	0.431259083056
0.10	0.795604230495	0.594101483517	0.494013350166	0.431703707180
0.12	0.797745064957	0.595148507464	0.494659181326	0.432151864280
0.14	0.799921817599	0.596208463561	0.495311306213	0.432603622929
0.16	0.802135781461	0.597281717788	0.495969871123	0.433059054043
0.18	0.804388324561	0.598368653523	0.496635028225	0.433518230999
0.20	0.806680896037	0.599469672739	0.497306935913	0.433981229763
0.22	0.809015032955	0.600585197325	0.497985759186	0.434448129028
0.24	0.811392367873	0.601715670522	0.498671670055	0.434919010364
0.26	0.813814637257	0.602861558501	0.499364847990	0.435393958375
0.28	0.816283690869	0.604023352107	0.500065480406	0.435873060874
0.30	0.818801502291	0.605201568768	0.500773763188	0.436356409065
0.32	0.821370180720	0.606396754613	0.501489901266	0.436844097746
0.34	0.823991984263	0.607609486816	0.502214109244	0.437336225522
0.36	0.826669334951	0.608840376197	0.502946612092	0.437832895046
0.38	0.829404835766	0.610090070122	0.503687645899	0.438334213265
0.40	0.832201290006	0.611359255743	0.504437458708	0.438840291707
0.42	0.835061723414	0.612648663623	0.505196311439	0.439351246777
0.44	0.837989409560	0.613959071822	0.505964478902	0.439867200095
0.46	0.840987899080	0.615291310498	0.506742250935	0.440388278849
0.48	0.844061053535	0.616646267126	0.507529933656	0.440914616205
0.50	0.847213084793	0.618024892433	0.508327850876	0.441446351738
0.52	0.850448601114	0.619428207178	0.509136345674	0.441983631920
0.54	0.853772661373	0.620857309933	0.509955782175	0.442526610659
0.56	0.857190839279	0.622313386065	0.510786547557	0.443075449894
0.58	0.860709299943	0.623797718155	0.511629054326	0.443630320265
0.60	0.864334891874	0.625311698151	0.512483742923	0.444191401859
0.62	0.868075258389	0.626856841657	0.513351084695	0.444758885948
0.64	0.871938973766	0.628434804827	0.514231585335	0.445332971440
0.66	0.875935711265	0.630047404512	0.515125788868	0.445913874956
0.68	0.880076452748	0.631696642493	0.516034282302	0.446501823059
0.70	0.884373753368	0.633384734896	0.516957701104	0.447097058162
0.72	0.888842080365	0.635114148303	0.517896735700	0.447699839260
0.74	0.893498253355	0.636887644592	0.518852139252	0.448310443826
0.76	0.898362026539	0.638708337394	0.529824737076	0.448929170028
0.78	0.903456874016	0.640579764283	0.520815438165	0.449556339368
0.80	0.908811073704	0.642505980754	0.521825249508	0.450192299818
0.82	0.914459244384	0.644491685142	0.522855294129	0.450837429631
0.84	0.920444596457	0.646542388865	0.523906834288	0.451492142005
0.86	0.926822358748	0.648664655444	0.524981301942	0.452156890897
0.88	0.933665253836	0.650866448719	0.526080339883	0.452832178425
0.90	0.941072801521	0.653157664358	0.527205859201	0.453518564499
0.92	0.949188468037	0.655550992321	0.528360123158	0.454216679759
0.94	0.958235085176	0.658063439615	0.529545877187	0.454927243682
0.96	0.968601799255	0.660719378586	0.530766568564	0.455651091397
0.98	0.981134377670	0.663558106816	0.532026774048	0.456389217084
1.00	1.000000000000	0.666666666666	0.533333333333	0.457142857142

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TABLE 5.2
Tabulation of $N_n(x)$ for $x = 0(0.02)1$ and $n = 5(1)8$ $12D$.

x	$N_5(x)$	$N_6(x)$	$N_7(x)$	$N_8(x)$
0.00	0.386563158547	0.354349562001	0.329038879001	0.308473949063
0.02	0.386886335379	0.354603384010	0.329245044578	0.308645710540
0.04	0.387211615966	0.354858649806	0.329452251424	0.308818252071
0.06	0.387539034032	0.355115379913	0.329660512833	0.308991582695
0.08	0.387868624258	0.355373595376	0.329869842409	0.309165711641
0.10	0.388200422325	0.355633317784	0.330080254068	0.309340648337
0.12	0.388534464952	0.355894569289	0.330291762061	0.309516402412
0.14	0.388870789947	0.356157372630	0.330504380974	0.309692983706
0.16	0.389209436255	0.356421751155	0.330718125752	0.309870402276
0.18	0.389550444008	0.356687728850	0.330933011702	0.310048668404
0.20	0.389893854584	0.356955330360	0.331149054516	0.310227792605
0.22	0.390239710661	0.357224581023	0.331366270278	0.310407785635
0.24	0.390588056285	0.357495506896	0.331584675486	0.310588658499
0.26	0.390938936935	0.357768134787	0.331804287063	0.310770422459
0.28	0.391292399591	0.358042492293	0.332025122378	0.310953089049
0.30	0.391648492819	0.358318607828	0.332247199265	0.311136670078
0.32	0.392007266846	0.358596510670	0.332470536036	0.311321177646
0.34	0.392368773650	0.358876230995	0.332695151511	0.311506624151
0.36	0.392733067058	0.359157799924	0.332921065032	0.311693022304
0.38	0.393100202845	0.359441249566	0.333148296489	0.311880385141
0.40	0.393470238845	0.359726613074	0.333376866347	0.312068726034
0.42	0.393843235070	0.360013924689	0.333606795668	0.312258058708
0.44	0.394219253840	0.360303219808	0.333838106142	0.312448397255
0.46	0.394598359920	0.360594535034	0.334070820116	0.312639756146
0.48	0.394980620676	0.360887908252	0.334304960626	0.312832150254
0.50	0.395366106237	0.361183378695	0.334540551431	0.313025594869
0.52	0.395754889675	0.361480987022	0.334777617049	0.313220105713
0.54	0.396147047207	0.361780775403	0.335016182800	0.313415698943
0.56	0.396542658497	0.362082787608	0.335256274842	0.313612391292
0.58	0.396941806448	0.362387069110	0.335497920223	0.313810199843
0.60	0.397344578363	0.362693667184	0.335741146928	0.314009142306
0.62	0.397751065337	0.363002631037	0.335985983932	0.314209236918
0.64	0.398161363033	0.363314011926	0.336232461260	0.314410502495
0.66	0.398575571952	0.363627863307	0.336480610047	0.314612958467
0.68	0.398993797838	0.363944240993	0.336730462610	0.314816624910
0.70	0.399416152133	0.364263203320	0.336982052525	0.315021522580
0.72	0.399842752495	0.364584811353	0.337235414704	0.315227672954
0.74	0.400273723383	0.364909129089	0.337490585492	0.315435098273
0.76	0.400709196726	0.365236223712	0.337747602765	0.315643821589
0.78	0.401149312698	0.365566165857	0.338006506044	0.315838668158
0.80	0.401594220616	0.365899029928	0.338267336619	0.316065258784
0.82	0.402044079989	0.366234894449	0.338530137687	0.316278023308
0.84	0.402499061760	0.366573842474	0.338794954513	0.316492187247
0.86	0.402959349802	0.366915962062	0.339061834607	0.316707778589
0.88	0.403425142718	0.367261346834	0.339330827930	0.316924826533
0.90	0.403896656078	0.367610096634	0.339601987125	0.317143361587
0.92	0.404374125235	0.367962318320	0.339875367799	0.317363415679
0.94	0.404857808976	0.368318126749	0.340151028837	0.317585022279
0.96	0.405347994431	0.368677645991	0.340429032796	0.317808216550
0.98	0.405845004033	0.369041010927	0.340709446371	0.318033035515
1.00	0.406349206349	0.369408369408	0.340992340992	0.318259518259

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TABLE 5.3
Tabulation of $N_n(x)$ for $x = 0(0.02)1$ and $n = 9(1)12$ $12D$.

x	$N_9(x)$	$N_{10}(x)$	$N_{11}(x)$	$N_{12}(x)$
0.00	0.291336507449	0.276769682076	0.264189241982	0.253181356899
0.02	0.291482474700	0.276895723120	0.264299511923	0.253278891280
0.04	0.291629044143	0.277022240513	0.264410166341	0.253376741292
0.06	0.291776222165	0.277149238914	0.264521208724	0.253474909604
0.08	0.291924015279	0.277276723064	0.264632642617	0.253573398927
0.10	0.292072430122	0.277404697792	0.264744471626	0.253672212016
0.12	0.292221473466	0.277533168014	0.264856699419	0.253771351670
0.14	0.292371152217	0.277662138738	0.264969329726	0.253870822073
0.16	0.292521473423	0.277791615067	0.265082366343	0.253970622090
0.18	0.292672444274	0.277921600219	0.265195813133	0.254070758684
0.20	0.292824072111	0.278052105431	0.265309674026	0.254171233498
0.22	0.292976364428	0.278183130165	0.265423953026	0.254272049569
0.24	0.293129328879	0.278314681906	0.265538654205	0.254373209984
0.26	0.293282973283	0.278446766270	0.265653781714	0.254474717883
0.28	0.293437305626	0.278579388985	0.265769339778	0.254576576460
0.30	0.293592334074	0.278712555894	0.265885332703	0.254678788965
0.32	0.293748066972	0.278846272961	0.266001764875	0.254781358706
0.34	0.293904512856	0.278980546273	0.266118640766	0.254884289047
0.36	0.294061680454	0.279115382043	0.266235964932	0.254987583416
0.38	0.294219578700	0.279250786620	0.266353742019	0.255091245301
0.40	0.294378216737	0.279386766486	0.266471976768	0.255195278255
0.42	0.294537603924	0.279523328266	0.266590674012	0.255299685896
0.44	0.294697749848	0.279660478731	0.266709838682	0.255404471913
0.46	0.294858664332	0.279798224803	0.266829475813	0.255509640060
0.48	0.295020357441	0.279936573562	0.266949590543	0.255615194169
0.50	0.295182839498	0.280075532251	0.267070188117	0.255721138142
0.52	0.295346121087	0.280215108279	0.267191273896	0.255827475960
0.54	0.295510213071	0.280355309235	0.267312853353	0.255934211684
0.56	0.295675126600	0.280496142887	0.267434932084	0.256041349455
0.58	0.295840873124	0.280637617194	0.267557515809	0.256148893501
0.60	0.296007464407	0.280779740311	0.267680610376	0.256256848135
0.62	0.296174912540	0.280922520599	0.267804221768	0.256365217764
0.64	0.296343229961	0.281065966634	0.267928356108	0.256474006887
0.66	0.296512429464	0.281210087213	0.268053019661	0.256583220100
0.68	0.296682524223	0.281354891368	0.268178218846	0.256692862101
0.70	0.296853527804	0.281500388372	0.268303960235	0.256802937690
0.72	0.297025454195	0.281646587755	0.268430250566	0.256913451780
0.74	0.297198317818	0.281793499311	0.268557096744	0.257024409382
0.76	0.297372133558	0.281941133115	0.268684505854	0.257135815667
0.78	0.297546916788	0.282089499535	0.268812485163	0.257247675867
0.80	0.297722683396	0.282238609246	0.268941042135	0.257359995381
0.82	0.297899444981	0.282388473247	0.269070184433	0.257472779729
0.84	0.298077233059	0.282539102879	0.269199919935	0.257586034570
0.86	0.298256050754	0.282690509841	0.269330256740	0.257699765707
0.88	0.298435921184	0.282842796316	0.269461203180	0.257813979093
0.90	0.298616863334	0.282995704485	0.269592767836	0.257928680837
0.92	0.298798896936	0.283149517563	0.269724959544	0.258043877216
0.94	0.298982042531	0.283304158814	0.269857787417	0.258159574676
0.96	0.299166321525	0.283459642090	0.269991260854	0.258275779847
0.98	0.299351756267	0.283615981760	0.270125389562	0.258392499551
1.00	0.299538370126	0.283773192751	0.270260183572	0.258509740808

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TABLE 5.4
Tabulation of $N_n(x)$ for $x = 0(0.02)1$ and $n = 13(1)16$ $12D$.

x	$N_{13}(x)$	$N_{14}(x)$	$N_{15}(x)$	$N_{16}(x)$
0.00	0.233443612403	0.234749197674	0.226924224419	0.219832842405
0.02	0.243530687307	0.234827557716	0.226995232313	0.219897580123
0.04	0.243618025099	0.234906139467	0.227066429234	0.219962480552
0.06	0.243705627861	0.234984944578	0.227137816511	0.220027544779
0.08	0.243793497707	0.235063974724	0.227209395490	0.220092773898
0.10	0.243881636781	0.235143231603	0.227281167537	0.220158169020
0.12	0.243970047260	0.235222716939	0.227353134033	0.220223731272
0.14	0.244058731355	0.235302432478	0.227425296381	0.220289461790
0.16	0.244147691308	0.235382379991	0.227497656000	0.220355361731
0.18	0.244236929400	0.235462561278	0.227570214332	0.220421432261
0.20	0.244326447942	0.235542978161	0.227642972835	0.220487674565
0.22	0.244506335817	0.235704526149	0.227789096299	0.220620679311
0.26	0.244596709963	0.235785661040	0.227862464282	0.220687444199
0.28	0.244687374187	0.235867039099	0.227936038485	0.220754385758
0.30	0.244778330994	0.235948662291	0.228009820473	0.220821505251
0.32	0.244869582929	0.236030532613	0.228083811836	0.220888803963
0.34	0.244961132581	0.236112652091	0.228158014185	0.220956283193
0.36	0.245052982582	0.236195022784	0.228232429156	0.221023944261
0.38	0.245145135607	0.236277646781	0.228307058411	0.221091788503
0.40	0.245237594378	0.236360526209	0.228381903634	0.221159817276
0.42	0.245330361665	0.236443663225	0.228456966537	0.221228031956
0.44	0.245423440284	0.236527060024	0.228532248857	0.221296433938
0.46	0.245516833103	0.236610718837	0.228607752357	0.221365024638
0.48	0.245610543041	0.236694641932	0.228683478831	0.221433805493
0.50	0.245704573068	0.236778831613	0.228759430097	0.221502777960
0.52	0.245798926210	0.236863290227	0.228835608006	0.221571943521
0.54	0.245893605549	0.236948020158	0.228912014436	0.221641303677
0.56	0.245988614223	0.237033023835	0.228988651296	0.221710859954
0.58	0.246083955431	0.237118303726	0.229065520528	0.221780613900
0.60	0.246179632434	0.237203862346	0.229142624105	0.221850567088
0.62	0.246275648553	0.237289702254	0.229219964032	0.221920721117
0.64	0.246372007178	0.237375826056	0.229297542352	0.221991077610
0.66	0.246468711765	0.237462236405	0.229375361138	0.222061638215
0.68	0.246565765839	0.237548936005	0.229453422504	0.222132404609
0.70	0.246663172997	0.237635927611	0.229531728597	0.222203378497
0.72	0.246760936913	0.237723214030	0.229610281605	0.222274561609
0.74	0.246859061337	0.237810798125	0.229689083755	0.222345955707
0.76	0.246957550097	0.237898682813	0.229768137313	0.222417562583
0.78	0.247056407107	0.237986871071	0.229847444589	0.222489384058
0.80	0.247155636365	0.238075365935	0.229927007935	0.222561421986
0.82	0.247255241961	0.238164170506	0.230006829747	0.222633678253
0.84	0.247355228074	0.238253287946	0.230086912470	0.222706154780
0.86	0.247455598982	0.238342721487	0.230167258594	0.222778853521
0.88	0.247556359065	0.238432474428	0.230247870658	0.222851776467
0.90	0.247657512804	0.238522550141	0.230328751254	0.222924925645
0.92	0.247759064793	0.238612952073	0.230409903025	0.222998303122
0.94	0.247861019736	0.238703683750	0.230491328671	0.223071911003
0.96	0.247963382459	0.238794748776	0.230573030945	0.223145751434
0.98	0.248066157912	0.238861508419	0.230655012663	0.223219826603
1.00	0.248169351176	0.238977893725	0.230737276700	0.223294138742

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TABLE 5.5
Tabulation of $N_n(x)$ for $x = 0(0.02)1$ and $n = 17(1)20$ $12D$.

x	$N_{17}(x)$	$N_{18}(x)$	$N_{19}(x)$	$N_{20}(x)$
0.00	0.213367177057	0.207440304721	0.201981349333	0.196931815600
0.02	0.213426509562	0.207494955810	0.202031898876	0.196978752169
0.04	0.213485989810	0.207549730462	0.202082557241	0.197025785170
0.06	0.213545612208	0.207604629419	0.202133325054	0.197072915134
0.08	0.213605377658	0.207659653435	0.202184202946	0.197120142594
0.10	0.213665287076	0.207714803269	0.202235191556	0.197167468092
0.12	0.213725341387	0.207770079690	0.202286291528	0.197214892173
0.14	0.213785541527	0.207825483477	0.202337503517	0.197262415390
0.16	0.213845888444	0.207881015418	0.202388828181	0.197310038299
0.18	0.213906383099	0.207936676308	0.202440266188	0.197357761465
0.20	0.213967026463	0.207992466953	0.202491818212	0.197405585458
0.22	0.214027819520	0.208048388170	0.202543484937	0.197453510854
0.24	0.214088763266	0.208104440783	0.202595267051	0.197501538234
0.26	0.214149858710	0.208160625627	0.202647165254	0.197549668188
0.28	0.214211106874	0.208216943548	0.202699180252	0.197597901311
0.30	0.214272508793	0.208273395402	0.202751312759	0.197646238204
0.32	0.214334065515	0.208329982056	0.202803563498	0.197694679477
0.34	0.214395778103	0.208386704385	0.202855933201	0.197743225744
0.36	0.214457647633	0.208443563279	0.202908422607	0.197791877629
0.38	0.214519675196	0.208500559637	0.202961032467	0.197840635761
0.40	0.214581861896	0.208557694370	0.203013763537	0.197889500777
0.42	0.214644208854	0.208614968399	0.203066616587	0.197938473321
0.44	0.214706717206	0.208672382660	0.203119592392	0.197987554045
0.46	0.214769388102	0.208729938100	0.203172691739	0.198036743609
0.48	0.214832222710	0.208787635677	0.203225915425	0.198086042680
0.50	0.214895222213	0.208845476363	0.203279264255	0.198135451934
0.52	0.214958387811	0.208903461143	0.203332739047	0.198184972054
0.54	0.215021720722	0.208961591015	0.203386340628	0.198234603731
0.56	0.215085222181	0.209019866991	0.203440069835	0.198284347668
0.58	0.215148893440	0.209078290097	0.203493927517	0.198334204572
0.60	0.215212735771	0.209136861372	0.203547914535	0.198384175161
0.62	0.215276750463	0.209195581870	0.203602031758	0.198434260162
0.64	0.215340938827	0.209254452662	0.203656280070	0.198484460313
0.66	0.215405302191	0.209313474832	0.203710660366	0.198534776357
0.68	0.215469841905	0.209372649478	0.203765173552	0.198585209050
0.70	0.215534559338	0.209431977719	0.203819820548	0.198635759158
0.72	0.215599455883	0.209491460686	0.203874602287	0.198686427455
0.74	0.215664532954	0.209551099528	0.203929519712	0.198737214727
0.76	0.215729791985	0.209610895412	0.203984573783	0.198788121768
0.78	0.215795234437	0.209670849522	0.204039765472	0.198839149387
0.80	0.215860861793	0.209730963060	0.204095095765	0.198890298400
0.82	0.215926675559	0.209791237247	0.204150565662	0.198941569636
0.84	0.215992677268	0.209851673322	0.204206176178	0.198992963935
0.86	0.216058868478	0.209912272545	0.204261928344	0.199044482149
0.88	0.216125250775	0.209973036196	0.204317823206	0.199096125143
0.90	0.216191825771	0.210033965575	0.204373861824	0.199147893792
0.92	0.216258595106	0.210095062003	0.204430045278	0.199199788985
0.94	0.216325560451	0.210156326823	0.204486374660	0.199251811625
0.96	0.216392723503	0.210217761401	0.204542851084	0.199303962627
0.98	0.216460085995	0.210279367127	0.204599475678	0.199356242920
1.00	0.216527649689	0.210341145412	0.204656249590	0.199408655344

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