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ETNA
Kent State University
etna@mcs.kent.edu

FRACTAL TRIGONOMETRIC APPROXIMATION *

M. A. NAVASCUES†

Abstract. A general procedure to define nonsmooth fractal versions of classical trigonometric approximants is proposed. The systems of trigonometric polynomials in the space of continuous and periodic functions $\mathcal{C}(2\pi)$ are extended to bases of fractal analogues. As a consequence of the process, the density of trigonometric fractal functions in $\mathcal{C}(2\pi)$ is deduced. We generalize also some classical results (Dini-Lipschitz's Theorem, for instance) concerning the convergence of the Fourier series of a function of $\mathcal{C}(2\pi)$. Furthermore, a method for real data fitting is proposed, by means of the construction of a fractal function proceeding from a classical approximant.

Key words. iterated function systems, fractal interpolation functions, trigonometric approximation

AMS subject classifications. 37M10, 58C05

1. Introduction.

A classical approach to handle real experimental recordings consists in their decomposition in signal content and noise. The first component is considered as deterministic and the noise is studied from a statistical point of view. We give here a global deterministic method to model both, signal and noise, by means of fractal interpolation. This method was introduced by M. Barnsley and others ([1], [2], [3], [4], [10]) in the eighties and provides good techniques for the construction of not necessarily smooth interpolants to real data. The procedure is based on the theory of Iterated Function Systems and their associated attractors ([2], [8]).

In former papers, we have proved that Barnsley's method is a general theory which contains other interpolation techniques as particular cases (see for instance [12], [13]). Another important fact is that the graph of these interpolants possesses a fractal dimension, and this number can be used to measure the complexity of a signal, allowing an automatic comparison of recordings, electroencephalographic for instance ([14]).

A general procedure to define nonsmooth fractal versions of classical trigonometric approximants is proposed. The fractal trigonometric polynomials defined here do not share in general the properties of differentiability of classical trigonometric functions and they preserve some others like closeness to continuous periodic functions. The systems of trigonometric polynomials, in the space of continuous and periodic functions $\mathcal{C}(2\pi)$,are extended to bases of fractal analogues. As a consequence of the process, the density of trigonometric fractal functions in $\mathcal{C}(2\pi)$ is deduced. This result illustrates the fact that, fractal interpolation functions are everywhere in the space of continuous functions, in a metric sense. In the reference [14], for instance, we have proved the density of affine fractal functions in $\mathcal{C}([a,b])$.

We generalize also some classical theorems (Dini-Lipschitz's Theorem, for instance) concerning the convergence of the Fourier series of a function of $\mathcal{C}(2\pi)$. Furthermore, a method for real data fitting is proposed, by means of the construction of a fractal function proceeding from a classical approximant.

From an applied point of view, the trigonometric approximants display the spectral content of a signal, providing a representation in the frequency domain which allows its processing and filtering. On the other hand, Besicovitch and Ursell, in the reference [5], proved that the graph of a smooth function has a fractal dimension of one. As a consequence, the nonsmoothnes is a required condition in order to obtain an approximation of the geometrical complexity of arbitrary signals. A conventional interpolant excludes the possibility of using

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†Dept. Matemática Aplicada, Centro Politécnico Superior de Ingenieros, Universidad de Zaragoza, C/ María de Luna 3, 50018 Zaragoza, Spain (manavas@unizar.es).

this parameter for numerical characterizations of experimental signals ([16]).

2. α-Fractal Functions. Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ be the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R : n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n,\ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n \tag{2.1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l\,|c_1 - c_2| \quad \forall\, c_1, c_2\ \in I \tag{2.2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1; n = 1, 2, ..., N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings, $F_n : F \to R$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1},\ \ F_n(t_N, x_N) = x_n,\ \ n = 1, 2, ..., N \tag{2.3}$$

$$|F_n(t, x) - F_n(t, y)| \leq \alpha_n |x - y|,\quad t \in I,\quad x, y \in R \tag{2.4}$$

Now define functions, $w_n(t, x) = (L_n(t), F_n(t, x)),\ \forall\, n = 1, 2, ..., N$.

THEOREM 2.1. (Barnsley [1]) *The Iterated Function System (IFS) $\{F,\ w_n : n = 1, 2, ..., N\}$ defined above admits a unique attractor G. G is the graph of a continuous function $f : I \to R$ which obeys $f(t_n) = x_n$, for $n = 0, 1, 2, ..., N$.*

The previous function is called a Fractal Interpolation Function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$.

Let $\mathcal{G}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. $\mathcal{G}$ is a complete metric space respect to the uniform norm. Define a mapping $T : \mathcal{G} \to \mathcal{G}$ by:

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall\, t \in [t_{n-1}, t_n],\ \ n = 1, 2, ..., N \tag{2.5}$$

T is a contraction mapping on the metric space $(\mathcal{G}, \|\cdot\|_\infty)$:

$$\|Tf - Tg\|_\infty \leq |\alpha|_\infty \|f - g\|_\infty \tag{2.6}$$

where $|\alpha|_\infty = \max\ \{|\alpha_n|;\ n = 1, 2, ..., N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point on $\mathcal{G}$, that is to say, there is $f \in \mathcal{G}$ such that $(Tf)(t) = f(t)\ \ \forall\ t \in [t_0, t_N]$. This function is the FIF corresponding to w_n and it is the unique $f \in \mathcal{G}$ satisfying the functional equation ([1]):

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)),\quad n = 1, 2, ..., N,\ \ t \in I_n = [t_{n-1}, t_n] \tag{2.7}$$

The most widely studied fractal interpolation functions so far, are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t, x) = \alpha_n x + q_n(t) \end{cases} \tag{2.8}$$

α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. Following the equalities (2.1)

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0} \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0} \tag{2.9}$$

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Kent State University
etna@mcs.kent.edu

Let $f \in \mathcal{C}(I)$ be a continuous function. We consider here ,the case

$$q_n(t) = f \circ L_n(t) - \alpha_n b(t) \tag{2.10}$$

where b is continuous, such that $b(t_0) = x_0$, $b(t_N) = x_N$ and $b \neq f$.

This case is proposed by Barnsley, in the reference ([1]), as generalization of any continuous function. It is easy to check that the condition (2.3) is fulfilled. By this method one can define fractal analogues of any continuous function (see Fig. 2.1).

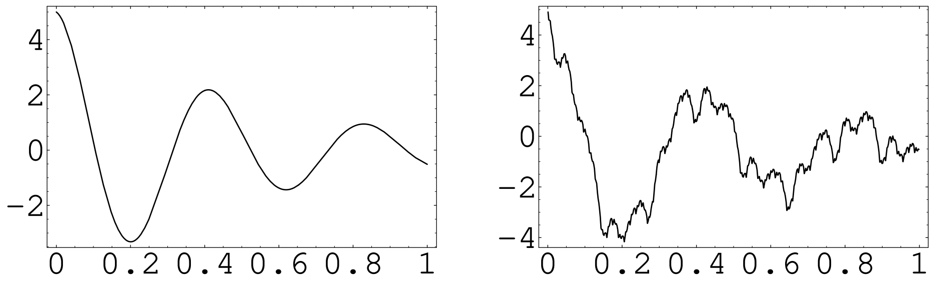

FIG. 2.1. *The left figure represents the graph of the function $f(t) = 5e^{-2t}cos(15t)$. The right graph represents the corresponding α-fractal, with $\Delta : 0 < 1/8 < 2/8 < \ldots < 1$, b a line in the interval $[0, 1]$ and $\alpha_n = 0.2 \quad \forall n = 1, \ldots, 8$.*

DEFINITION 2.2. *Let f^α be the continuous function defined by the IFS (2.8), (2.9) and (2.10). f^α is the α-fractal function associated to f with respect to b and the partition Δ.*

Following (2.7) and (2.10), f^α verifies the fixed point equation:

$$f^\alpha(t) = f(t) + \alpha_n (f^\alpha - b) \circ L_n^{-1}(t) \qquad \forall t \in I_n \tag{2.11}$$

f^α interpolates to f at t_n as, using (2.1), (2.11) and Barnsley's Theorem:

$$f^\alpha(t_n) = f(t_n) + \alpha_n (f^\alpha - b) \circ (t_N) = f(t_n) \qquad \forall n = 0, 1, \ldots, N \tag{2.12}$$

From (2.11) it is easy to deduce that:

$$\|f^\alpha - f\|_\infty \leq |\alpha|_\infty \|f^\alpha - b\|_\infty \leq |\alpha|_\infty (\|f^\alpha - f\|_\infty + \|f - b\|_\infty)$$

and

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty \tag{2.13}$$

If $\alpha = 0$, then (2.11) $f^\alpha = f$.
Let $\mathcal{B}^\alpha$ be the operator of $\mathcal{G}$; $\mathcal{B}^\alpha = \mathcal{B}^\alpha_{\Delta,b}$:

$$\begin{array}{rccc} \mathcal{B}^\alpha_{\Delta,b}: & \mathcal{G} & \rightarrow & \mathcal{G} \\ & f & \hookrightarrow & f^\alpha \end{array}$$

$\mathcal{B}^\alpha_{\Delta,b}$ depends on b and Δ but sometimes we will omit the subindices in order to simplify the notation.

PROPOSITION 2.3. *For fixed Δ and b, $\mathcal{B}^\alpha$ satisfies the Lipschitz condition:*

$$\|\mathcal{B}^\alpha(f) - \mathcal{B}^\alpha(g)\|_\infty \leq \frac{1}{1 - |\alpha|_\infty} \|f - g\|_\infty \tag{2.14}$$

Proof. Let $\mathcal{B}^\alpha(f) = f^\alpha$, $\mathcal{B}^\alpha(g) = g^\alpha$. By the equation (2.11), $\forall t \in I_n$

$$f^\alpha(t) = f(t) + \alpha_n(f^\alpha - b) \circ L_n^{-1}(t)$$
$$g^\alpha(t) = g(t) + \alpha_n(g^\alpha - b) \circ L_n^{-1}(t)$$

and

$$f^\alpha(t) - g^\alpha(t) = f(t) - g(t) + \alpha_n(f^\alpha - g^\alpha) \circ L_n^{-1}(t)$$

then

$$sup_{t\in I_n}|f^\alpha(t) - g^\alpha(t)| \leq \|f - g\|_\infty + |\alpha|_\infty \|f^\alpha - g^\alpha\|_\infty$$

and

$$\|f^\alpha - g^\alpha\|_\infty \leq \|f - g\|_\infty + |\alpha|_\infty \|f^\alpha - g^\alpha\|_\infty$$

from which the result is deduced. ☐

THEOREM 2.4. *$\mathcal{B}^\alpha = \mathcal{B}^\alpha_{\Delta,b}$ is a continuous operator of $\mathcal{G}$.*

Proof. It is an inmediate consequence of the former proposition. ☐

PROPOSITION 2.5. *Let $\alpha_m \in R^N$ be such that $|\alpha_m|_\infty < 1$ and $\alpha_m \to 0$ as m tends to infinity. Then $\mathcal{B}^{\alpha_m}(f) \to f$ uniformly as m tends to infinity.*

Proof. By the inequality (2.13):

$$\|\mathcal{B}^{\alpha_m}(f) - f\|_\infty \leq \frac{|\alpha_m|_\infty}{1 - |\alpha_m|_\infty}\|f - b\|_\infty$$

from which the result is deduced. ☐

To construct non-smooth interpolating functions one can proceed in the following way. Let f be a classical (smooth) interpolant of the data. Choose a nowhere differentiable function b (for instance, a Weierstrass function ([9])) and α_n non-null $\forall n$. As f is smooth, f^α can not be differentiable in every point because if it were, for any $t \in I$, $L_n(t) \in I_n$ and the equation (2.11) can be written as

$$b(t) = f^\alpha(t) + \frac{1}{\alpha_n}(f - f^\alpha) \circ L_n(t)$$

As a consequence, b would be differentiable at t (see Fig. 2.2).

3. Fractal Linear Operator. If we choose $b = f \circ c$ where c is continuous, increasing and such that $c(t_0) = t_0$ and $c(t_N) = t_N$ (for instance, $c(t) = (e^{\lambda t} - 1)/(e^\lambda - 1)$ for $\lambda > 0$ in the interval $[0, 1]$), then the operator of $\mathcal{C}(I)$ which assigns f^α (α-fractal of f respect to $f \circ c$ and Δ) to the function f

$$\mathcal{F}^\alpha = \mathcal{F}^\alpha_{\Delta,c}$$

is linear as, by (2.11) $\forall t \in I_n$:

$$f^\alpha(t) = f(t) + \alpha_n(f^\alpha - f \circ c) \circ L_n^{-1}(t) \qquad g^\alpha(t) = g(t) + \alpha_n(g^\alpha - g \circ c) \circ L_n^{-1}(t)$$

Multiplying the first equation by λ and the second by μ, the uniqueness of the solution of the fixed point equation defining the FIF gives:

$$(\lambda f + \mu g)^\alpha = \lambda f^\alpha + \mu g^\alpha \qquad \forall \lambda, \mu \in R$$

ETNA
Kent State University
etna@mcs.kent.edu

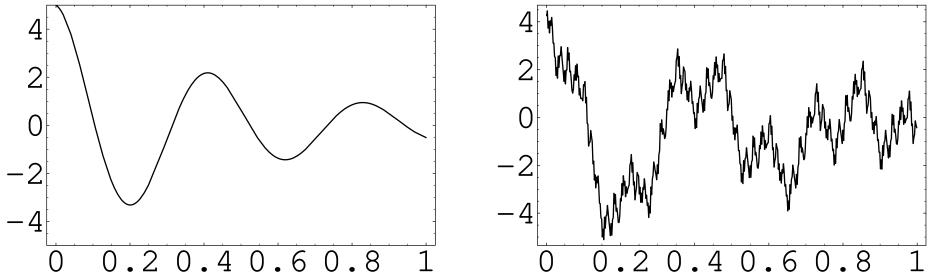

FIG. 2.2. *The left figure represents the graph of the function* $f(t) = 5e^{-2t}cos(15t)$. *The right graph represents the corresponding* α*-fractal, with* $\Delta : 0 < 1/8 < 2/8 < \ldots < 1$, b *the Weierstrass function* $b(t) = k_1 + k_2 \sum_{k=1}^{\infty} \frac{1}{2^k} sin(6^k t)$ *(with suitable* k_1, k_2 *in order to verify the hypotheses) and* $\alpha_n = 0.1 \quad \forall n = 1, \ldots, 8$.

Besides, applying the equation (2.13) for $b = f \circ c$, one has

$$\|\mathcal{F}^\alpha(f) - f\|_\infty \leq \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty} \|f\|_\infty \tag{3.1}$$

from which it is clear that,

$$\|\mathcal{F}^\alpha(f)\|_\infty \leq \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty} \|f\|_\infty + \|f\|_\infty \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \|f\|_\infty$$

and as a consequence,

$$\|\mathcal{F}^\alpha\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \tag{3.2}$$

and so $\mathcal{F}^\alpha$ is a linear and bounded operator. From here on we consider this particular case ($b = f \circ c$).

4. Fractal Trigonometric Polynomials. We consider here the space of 2π-periodic continuous functions

$$\mathcal{C}(2\pi) = \{f : [-\pi, \pi] \to R; f continuous, f(-\pi) = f(\pi)\}$$

Let τ_m be the set of trigonometric polynomials of degree (or order) at most m, linearly spanned by the set $\{1, sin(x), cos(x), sin(2x), cos(2x), \cdots, sin(mx), cos(mx)\}$.

$$\tau_m = < \{1, sin(x), cos(x), sin(2x), cos(2x), \cdots, sin(mx), cos(mx)\} > \tag{4.1}$$

This family constitutes a basis for τ_m. This system is orthogonal with respect to the inner product (5.1). In fact it is a complete system in $\mathcal{L}^2(2\pi)$ ([7]).

Let $\Delta : -\pi = t_0 < t_1 < \cdots < t_N = \pi$ be a partition of the interval $[-\pi, \pi]$.

DEFINITION 4.1. $\tau_m^\alpha = \mathcal{F}^\alpha(\tau_m)$ *is the set of* α*-fractal trigonometric polynomials of degree at most* m.

PROPOSITION 4.2. τ_m^α *is spanned by* $\{1, cos^\alpha(t), sin^\alpha(t), \cdots, sin^\alpha(mt), cos^\alpha(mt)\}$ *where* $cos^\alpha(jt) = \mathcal{F}^\alpha cos(jt)$ *and* $sin^\alpha(jt) = \mathcal{F}^\alpha sin(jt)$.

Proof. The constant functions are fixed points of $\mathcal{F}^\alpha$,as the equation (2.11), for $f(t) = k, \quad \forall t \in I$

$$f^\alpha(t) = k + \alpha_n(f^\alpha - f \circ c) \circ L_n^{-1}(t)$$

ETNA
Kent State University
etna@mcs.kent.edu

is satisfied by $f^\alpha(t) = k \quad \forall t \in I$ and by the uniqueness of the solution $\mathcal{F}^\alpha(f) = f$.

If $t_m^\alpha \in \tau_m^\alpha$, then $t_m^\alpha = \mathcal{F}^\alpha(t_m)$ where $t_m \in \tau_m$. By the linearity of $\mathcal{F}^\alpha$, t_m^α is a linear combination of $\{1, cos^\alpha(t), sin^\alpha(t), \cdots, sin^\alpha(mt), cos^\alpha(mt)\}$. ☐

CONSEQUENCE 4.3. $dim(\tau_m^\alpha) < +\infty$.

This fact allows the existence of a finite uniform distance from $f \in \mathcal{C}(2\pi)$ to τ_m^α:

$$d_m^{*\alpha} = d(f, \tau_m^\alpha) = inf\{\|f - t_m^\alpha\|_\infty; t_m^\alpha \in \tau_m^\alpha\} \tag{4.2}$$

In §6, we approach the problem of finding a basis for τ_m^α. The case $\alpha = 0$ gives the classical case of smooth sin and cos functions.

The Theorem of Uniform Approximation (Weierstrass) for 2π-periodic functions asserts that, any $f \in \mathcal{C}(2\pi)$ can be uniformly approximated by trigonometric polynomials (see for instance [11]).

THEOREM 4.4. *Let $f \in \mathcal{C}(2\pi)$ be given. For all $\epsilon > 0$, any partition Δ of the interval $I = [-\pi, \pi]$ with $N+1$ points ($N > 1$) and any function c verifying the conditions prescribed, there exists an α-fractal trigonometric polynomial $s^\alpha(t)$ with $\alpha \neq 0$ in R^N such that*

$$|f(t) - s^\alpha(t)| < \epsilon$$

Proof. For any $\epsilon > 0$, let us consider $\epsilon/2 > 0$. Applying the theorem of uniform approximation of $\mathcal{C}(2\pi)$, $\exists s(t) \in \tau_m$,such that,

$$|f(t) - s(t)| < \epsilon/2 \qquad t \in I \tag{4.3}$$

For a partition Δ we choose $\alpha \in R^N$, $\alpha \neq 0$ small enough to verify

$$|s(t) - s^\alpha(t)| \le \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\|s\|_\infty < \epsilon/2 \tag{4.4}$$

Then, by (4.3) and (4.4) we obtain the result. ☐

THEOREM 4.5. *For fixed Δ and c, the set of fractal trigonometric polynomials*

$$\bigcup\{t_m^\alpha; t_m \in \tau_m; \alpha \in (R^N)^*; |\alpha|_\infty < 1; m \in N\}$$

is dense in $\mathcal{C}(2\pi)$.

Proof. It is an inmediate consequence of the former theorem. ☐

We exclude $\alpha = 0$ because in this case $t_m^\alpha = t_m$ and the fact is known. This result confirms that it is possible to choose $\alpha \neq 0$ and $m \in N$ such that there exists a fractal trigonometric polynomial s_m^α arbitrarily close to any $f \in \mathcal{C}(2\pi)$. That is to say, the set

$$\bigcup_{\alpha \in J}\{1, cos^\alpha(x), sin^\alpha(x), \cdots, sin^\alpha(mx), cos^\alpha(mx)\cdots\}$$

where $J = \{\alpha \in R^N; |\alpha|_\infty < 1; \alpha \neq 0\}$, is fundamental respect to the uniform norm. ([6]).

5. Fractal Fourier Series. We consider here $\mathcal{C}(2\pi)$ with the inner product

$$< f, g >= \int_{-\pi}^{\pi} f(t)g(t)dt \tag{5.1}$$

The system

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}sin(t), \frac{1}{\sqrt{\pi}}cos(t), \frac{1}{\sqrt{\pi}}sin(2t), \frac{1}{\sqrt{\pi}}cos(2t), \cdots\} \tag{5.2}$$

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Kent State University
etna@mcs.kent.edu

is orthonormal and complete ([7], [15]). The Fourier series of f is

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{+\infty}(a_k cos(kt) + b_k sin(kt)) \tag{5.3}$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(kt)dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(kt)dt$$

In general, the Fourier series of an element is merely the sum of its projections on a system of orthonormal elements. Fourier expansions converge in the mean of order 2 ($\mathcal{L}^2$-norm) to the elements that give rise to them. That is to say, if

$$S_m(t) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k cos(kt) + b_k sin(kt)) \tag{5.4}$$

then,

$$\|f - S_m\|_2^2 = \int_{-\pi}^{\pi}(f - S_m)^2 dt \to 0$$

as $m \to \infty$.
Pointwise and uniform convergence of the Fourier series of f is not verified in general. A collection of theorems concerning this topic can be consulted in ([7], [6], [15]). For instance, we remark the following result.

THEOREM 5.1. (Dini-Lipschitz) *If $f(t) \in \mathcal{C}(2\pi)$, and if $\omega(\delta)log(\delta) \to 0$ as $\delta \to 0$, then the Fourier series of f converges uniformly to f. (ω is the modulus of continuity of f).*

Let us consider the operator $\mathcal{S}_m : \mathcal{C}(2\pi) \to \tau_m$,such that $\mathcal{S}_m(f)$ is defined by

$$\mathcal{S}_m(f)(t) = S_m(t) \qquad \forall t \in [-\pi, \pi],$$

where S_m is the Fourier sum of order m of f (5.4). In the reference ([6]), it is proved that $\mathcal{S}_m$ is a bounded operator and the following inequality holds:

$$\frac{4}{\pi^2}log(m) < \|\mathcal{S}_m\| \leq 3 + log(m) \tag{5.5}$$

$\mathcal{S}_m$ is a projection on τ_m as $\mathcal{S}_m \circ \mathcal{S}_m = \mathcal{S}_m$.

The error committed by the finite Fourier sum can be bounded in several ways. For instance, if

$$d_m^*(f) = d(f, \tau_m) = inf\{\|f - s\|_\infty, s \in \tau_m\}$$

it can be proved that (see [17])

$$\|\mathcal{S}_m f - f\|_\infty \leq (4 + log(m))d_m^*(f) \tag{5.6}$$

The theorems of Jackson give upper bounds for the quantity $d_m^*(f)$. For instance:

THEOREM 5.2. (Jackson's Theorem [6]) *For all $f \in \mathcal{C}(2\pi)$*

$$d_m^*(f) \leq \omega(\frac{\pi}{m+1})$$

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Kent State University
etna@mcs.kent.edu

where ω is the modulus of continuity of f. The coefficient 1 of $\omega(\frac{\pi}{m+1})$ is the best possible one independent of f and m.

As a consequence, if $f \in \mathcal{C}(2\pi)$

$$\|\mathcal{S}_m f - f\|_\infty \leq (4 + log(m))\omega(\frac{\pi}{m+1}) \tag{5.7}$$

It can be observed that, from this inequality, the Dini-Lipschitz theorem is deduced.
Let $\mathcal{S}_m^\alpha = \mathcal{F}^\alpha \circ \mathcal{S}_m$ be the operator such that

$$\mathcal{S}_m^\alpha(f) = \mathcal{F}^\alpha(\mathcal{S}_m(f))$$

(α-fractal Fourier finite sum of f). $\mathcal{S}_m^\alpha$ is a linear and bounded operator and by (3.2) and (5.5)

$$\|\mathcal{S}_m^\alpha\| \leq \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}(3 + log(m))$$

$$\|\mathcal{S}_m^\alpha f - f\|_\infty \leq \|\mathcal{S}_m^\alpha f - f^\alpha\|_\infty + \|f^\alpha - f\|_\infty \leq$$

$$\leq \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\|\mathcal{S}_m f - f\|_\infty + \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|f\|_\infty$$

and applying (5.7)

$$\|\mathcal{S}_m^\alpha f - f\|_\infty \leq \frac{1}{1-|\alpha|_\infty}(((1+|\alpha|_\infty)(4+log(m))\omega(\frac{\pi}{m+1}) + 2|\alpha|_\infty\|f\|_\infty) \tag{5.8}$$

In this way, the theorem of Dini-Lipschitz can be generalized to the fractal series.

THEOREM 5.3. *Let $f \in \mathcal{C}(2\pi)$ such that $\omega(\delta)log(\delta) \to 0$ as $\delta \to 0$ and let α_m be a sequence of scale vectors, such that $\alpha_m \to 0$ as $m \to \infty$,then the α_m-fractal Fourier series of f converges uniformly to f as $m \to \infty$.*

Proof. As f is continuous on $[-\pi, \pi]$, $\omega(h) \to 0$ as $h \to 0$ ([6]). The hypotheses and (5.8) give the result. ☐

6. Trigonometric Fractal Interpolation. We approach here the problem of trigonometric interpolation by means of fractal techniques. Let $\Delta : -\pi = t_0 < t_1 < \cdots < t_{2m} < t_{2m+1} = \pi$ be given correponding to the data $\{(t_k, x_k)\}_{k=0}^{2m+1}$ where $x_0 = x_{2m+1}$. Let us consider the fundamental functions of interpolation ([17], [7])

$$\varphi_j(t) = \frac{\prod_{i=0,i\neq j}^{2m} sin(\frac{1}{2}(t-t_i))}{\prod_{i=0,i\neq j}^{2m} sin(\frac{1}{2}(t_j-t_i))} \tag{6.1}$$

For $j, k = 0, 1, \cdots, 2m, \quad \varphi_j(t_k) = \delta_{kj}$. Each function φ_j ,is a linear combination of

$$1, cos(t), \cdots, cos(mt), sin(t), \cdots, sin(mt)$$

and hence is an element of τ_m ([7]).

The function,

$$\varphi(t) = \sum_{k=0}^{2m} x_k \varphi_k(t) \tag{6.2}$$

,is also an element of τ_m and is the unique solution in this space for the interpolation problem

$$\varphi(t_k) = x_k \qquad k = 0, 1, \cdots, 2m.$$

If T_m represents the trigonometric interpolation operator, which assigns to a function f its trigonometric interpolant with respect to $\{(t_k, f(t_k))\}_{k=0}^{2m}$, then by (6.2)

$$\|T_m f\|_\infty \leq \|f\|_\infty \|l_m\|_\infty \tag{6.3}$$

where

$$l_m(t) = \sum_{k=0}^{2m} |\varphi_k(t)|$$

We define the α-fractal trigonometric interpolant as

$$\varphi^\alpha(t) = \mathcal{F}^\alpha(\varphi)(t) = \sum_{k=0}^{2m} x_k \varphi_k^\alpha(t)$$

φ_k^α is the α-fractal function of φ_k with respect to Δ.

The function φ^α passes through the points (t_k, x_k) as $\varphi_j^\alpha(t_k) = \varphi_j(t_k) = \delta_{kj}$ (see §2 (2.12)). Besides

$$\varphi^\alpha = \mathcal{F}^\alpha \circ T_m(f)$$

The functions $\{\varphi_j\}_{j=0}^{2m}$ are orthogonal with respect to the form:

$$(f, g) = \sum_{k=0}^{2m} f(t_k) g(t_k)$$

and hence a basis of τ_m. This property is inherited by $\{\varphi_j^\alpha\}$ as φ_j^α interpolates to φ_j at the nodes and so

$$(\varphi_i^\alpha, \varphi_j^\alpha) = \sum_{k=0}^{2m} \varphi_i(t_k)\varphi_j(t_k) = \sum_{k=0}^{2m} \delta_{ki}\delta_{kj}$$

where δ_{kj} is the delta of Kronecker. If $s^\alpha \in \tau_m^\alpha$, by the linearity of the operator $\mathcal{F}^\alpha$,

$$s^\alpha = \sum_{k=0}^{2m} \lambda_k \varphi_k^\alpha$$

the orthogonality of φ_k^α implies the linear independence and hence $\{\varphi_k^\alpha\}_{k=0}^{2m}$ constitutes a basis of τ_m^α of α-fractal trigonometric polynomials with respect to the partition Δ. For $\alpha = 0$, we retrieve the standard basis φ_k.

7. Fitting Method by Fractal Approximants. Let $\{(\bar{t}_j, \bar{x}_j), j = 0, 1, \ldots, p\}$ be a collection of data. Let us define a function of approximation to the data (for instance, a minimax approximation or a least squares fitting curve) $f(t)$. Consider an interval I containing the abscissas and let Δ be a partition of I, $\Delta : a = t_0 < t_1 < \ldots < t_N = b$ such that $t_i \neq \bar{t}_j \quad \forall i, j$, $t_0 < \bar{t}_0$ and $t_N > \bar{t}_p$. We look for an α-fractal function $f^\alpha(t)$. To choose α_n we consider all the abscissas $\bar{t}_{j_1}, \cdots, \bar{t}_{j_r}$ in I_n: $t_{n-1} \leq \bar{t}_{j_i} \leq t_n$ for $i = 1, 2, \cdots, r$. Then, by the equation (fixed point) (2.11):

$$\bar{x}_{j_i} = f(\bar{t}_{j_i}) + \alpha_n (f^\alpha - f \circ c) \circ L_n^{-1}(\bar{t}_{j_i})$$

Approximating f^α by f

$$\bar{x}_{j_i} \simeq f(\bar{t}_{j_i}) + \alpha_n (f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_i})$$

We choose α_n by a least squares procedure:

$$minE(\alpha_n) = \sum_{i=1}^{r} (f(\bar{t}_{j_i}) - \bar{x}_{j_i} + \alpha_n (f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_i}))^2$$

Differentiating the former expression we obtain:

$$\alpha_n = -\frac{\sum_{i=1}^{r} (f(\bar{t}_{j_i}) - \bar{x}_{j_i})(f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_i})}{\sum_{i=1}^{r} ((f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_i}))^2}$$

By the Schwartz's inequality

$$|\alpha_n| \leq \frac{||u||_2}{||v||_2}$$

where

$$u = (f(\bar{t}_{j_1}) - \bar{x}_{j_1}, \cdots, f(\bar{t}_{j_r}) - \bar{x}_{j_r})$$

$$v = ((f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_1}), \cdots, (f - f \circ c) \circ L_n^{-1}(\bar{t}_{j_r}))$$

We must choose the order of the approximant f in such a way that the differences between $f(\bar{t}_{j_i})$ and $\bar{x}_{j_i}$ are small enough to obtain $|\alpha_n| < 1$.

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