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UNIFORM CONVERGENCE OF MONOTONE ITERATIVE METHODS FOR SEMILINEAR SINGULARLY PERTURBED PROBLEMS OF ELLIPTIC AND PARABOLIC TYPES *

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Abstract. This paper deals with discrete monotone iterative methods for solving semilinear singularly perturbed problems of elliptic and parabolic types. The monotone iterative methods solve only linear discrete systems at each iterative step of the iterative process. Uniform convergence of the monotone iterative methods are investigated and rates of convergence are estimated. Numerical experiments complement the theoretical results.

Key words. singular perturbation, reaction-diffusion problem, convection-diffusion problem, discrete monotone iterative method, uniform convergence

AMS subject classifications. 65M06, 65N06

1. Introduction.

We are interested in monotone iterative methods for solving nonlinear singularly perturbed problems of elliptic and parabolic types.

Firstly, introduce singularly perturbed problems which correspond to the reaction-diffusion and the convection-diffusion problems of the elliptic type

$$Lu = -f(x, y, u), \tag{1.1}$$

$$Lu = \begin{cases} L_\mu u \equiv -\mu^2 \left(u_{xx} + u_{yy}\right), & \text{or} \\ L_\varepsilon u \equiv -\varepsilon \left(u_{xx} + u_{yy}\right) + b_1 u_x + b_2 u_y, \end{cases} \tag{1.2}$$

$$(x, y) \in \omega, \quad \omega = \omega^x \times \omega^y = \{0 < x < 1\} \times \{0 < y < 1\},$$

$$f_u \geq c_* > 0, \quad (x, y, u) \in \overline{\omega} \times (-\infty, \infty), \quad f_u \equiv \partial f / \partial u,$$

$$b_1 \geq \beta_1 > 0, \quad b_2 \geq \beta_2 > 0 \text{ on } \overline{\omega},$$

$$u = g \text{ on } \partial\omega,$$

where μ and ε are small positive parameters, c_* and $\beta_{1,2}$ are constants. If f, g and $b_{1,2}$ are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (1.1) exists (see [10] for details).

For $\mu \ll 1$, the reaction-diffusion problem (1.1) with $L = L_\mu$ is singularly perturbed and characterized by the boundary layers (i.e., regions with rapid change of the solution) of width $O(\mu|\ln \mu|)$ near $\partial\omega$ (see [2] for details). For $\varepsilon \ll 1$, the convection-diffusion problem (1.1) with $L = L_\varepsilon$ is singularly perturbed and characterized by the regular boundary layers of width $O(\varepsilon|\ln \varepsilon|)$ at $x = 1$ and $y = 1$ (see [12] for details).

Secondly, introduce singularly perturbed problems which correspond to the reaction-diffusion and the convection-diffusion problems of the parabolic type

$$Lu + u_t = -f(x, y, t, u), \tag{1.3}$$

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$$(x,y,t) \in Q = \omega \times (0,T], \quad \omega = \{0 < x < 1\} \times \{0 < y < 1\},$$

$$f_u \geq 0, \quad (x,y,t,u) \in \overline{Q} \times (-\infty, \infty),$$

$$b_1 \geq \beta_1 > 0, \quad b_2 \geq \beta_2 > 0 \text{ on } \overline{\omega},$$

where L from (1.2). The initial-boundary conditions are defined by

$$u = g, \ (x,y,t) \in \partial\omega \times (0,T], \quad u(x,y,0) = u^0(x,y), \ (x,y) \in \overline{\omega}.$$

If f, g, $b_{1,2}$ and u^0 are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (1.3) exists (see [11] for details).

For $\mu \ll 1$, the reaction-diffusion problem (1.3) with $L = L_\mu$ is singularly perturbed and characterized by the boundary layers of width $O(\mu|\ln\mu|)$ near $\partial\omega$ (see [3] for details). For $\varepsilon \ll 1$, the convection-diffusion problem (1.3) with $L = L_\varepsilon$ is singularly perturbed and characterized by the regular boundary layers of width $O(\varepsilon|\ln\varepsilon|)$ at $x = 1$ and $y = 1$ (see [12] for details).

It is well-known that classical numerical methods for solving singularly perturbed problems are inefficient, since in order to resolve layers they require a fine mesh covering the whole domain. For constructing efficient numerical algorithms to handle these problems, there are two general approaches: the first one is based on layer-adapted meshes and the second is based on exponential fitting or on locally exact schemes. The basic property of the efficient numerical methods is uniform convergence with respect to the perturbation parameter. The three books [9], [12] and [16] develop these approaches and give comprehensive applications to wide classes of singularly perturbed problems.

In the study of numerical methods for nonlinear singularly perturbed problems, the two major points have to be developed: i) constructing parameter uniform difference schemes; ii) obtaining reliable and efficient computing algorithms for computing nonlinear discrete problems. A fruitful method for the treatment of these nonlinear systems is the monotone method (known as the method of lower and upper solutions, see [13] for details). The monotone method leads to iterative algorithms which converge globally and solve only linear discrete systems at each iterative step which is of great importance in practice. Since the initial iteration in the monotone iterative method is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

In this paper, we investigate uniform convergence properties of the monotone iterative methods constructed in [5]-[8].

The structure of the paper is as follows. In Section 1, we present differences schemes which approximate the nonlinear problems (1.1) and (1.3). In Section 3, we construct a monotone iterative method for solving the nonlinear difference schemes which approximate the nonlinear elliptic problems (1.1) and study convergence properties of the proposed method. Section 4 is devoted to the construction and investigation of a monotone iterative method for solving the nonlinear difference schemes which approximate the nonlinear parabolic problems (1.3). The final Section 5 presents results of numerical experiments.

2. Difference schemes.

2.1. Difference schemes for solving (1.1). On $\overline{\omega}$ introduce nonuniform mesh $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\}, \tag{2.1}$$

$$\overline{\omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\}.$$

For approximation (1.1), we use the classical difference scheme for the reaction-diffusion problem with $L = L_\mu$ and the upwind difference scheme for the convection-diffusion problem with $L = L_\varepsilon$:

$$\mathcal{L}^h U(P) + f(P, U) = 0,\ P \in \omega^h, \quad U = g \text{ on } \partial\omega^h, \tag{2.2}$$

$$\mathcal{L}^h U = \left\{ \begin{array}{l} \mathcal{L}^h_\mu U = -\mu^2 \left(\mathcal{D}^2_x + \mathcal{D}^2_y\right) U, \quad \text{or} \\ \mathcal{L}^h_\varepsilon U = -\varepsilon \left(\mathcal{D}^2_x + \mathcal{D}^2_y\right) U + b_1 \mathcal{D}^-_x U + b_2 \mathcal{D}^-_y U. \end{array} \right. \tag{2.3}$$

$\mathcal{D}^2_x U$, $\mathcal{D}^2_y U$ and $\mathcal{D}^-_x U$, $\mathcal{D}^-_y U$ are the central difference and the backward difference approximations to the second and first derivatives, respectively,

$$\mathcal{D}^2_x U_{ij} = (\hbar_{xi})^{-1} \left[(U_{i+1,j} - U_{ij})\,(h_{xi})^{-1} - (U_{ij} - U_{i-1,j})\,(h_{x,i-1})^{-1}\right],$$

$$\mathcal{D}^2_y U_{ij} = (\hbar_{yj})^{-1} \left[(U_{i,j+1} - U_{ij})\,(h_{yj})^{-1} - (U_{ij} - U_{i,j-1})\,(h_{y,j-1})^{-1}\right],$$

$$\hbar_{xi} = 2^{-1}\,(h_{x,i-1} + h_{xi}), \quad \hbar_{yj} = 2^{-1}\,(h_{y,j-1} + h_{yj}),$$

$$\mathcal{D}^-_x U_{ij} = (h_{x,i-1})^{-1}\,(U_{ij} - U_{i-1,j}), \quad \mathcal{D}^-_y U_{ij} = (h_{y,j-1})^{-1}\,(U_{ij} - U_{i,j-1}),$$

where $P = (x_i, y_j) \in \omega^h$ and $U_{ij} = U\,(x_i, y_j)$.

2.2. Difference schemes for solving (1.3). On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, where $\overline{\omega}^h$ is defined in (2.1) and

$$\overline{\omega}^\tau = \{t_k = k\tau,\ 0 \le k \le N_\tau,\ N_\tau \tau = T\}.$$

For approximation of problem (1.3), we use the implicit difference scheme

$$\mathcal{L}^h U(P,t) + \frac{1}{\tau}\,[U(P,t) - U(P,t-\tau)] = -f(P,t,U), \tag{2.4}$$

$$U(P,t) = g(P,t),\ (P,t) \in \partial\omega^h \times \omega^\tau, \quad U(P,0) = u^0(P),\ P \in \overline{\omega}^h,$$

where on each time level $\mathcal{L}^h U$ is defined in (2.3) and $U^k_{ij} \equiv U\,(x_i, y_j, t_k)$.

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2.3. The maximum principle. On $\overline{\omega}^h$, we represent a difference scheme in the following canonical form

$$d(P)W(P) = \sum_{P' \in S(P)} e\left(P,P'\right)W\left(P'\right) + F(P),\ P \in \omega^h, \tag{2.5}$$

$$W(P) = W^0(P),\ P \in \partial\omega^h,$$

and suppose that

$$d(P) > 0,\ e\left(P,P'\right) \geq 0,\ c(P) = d(P) - \sum_{P' \in S'(P)} e\left(P,P'\right) > 0,\ P \in \omega^h,$$

where $S'(P) = S(P) \setminus \{P\}$, $S(P)$ is a stencil of the difference scheme. Now, we formulate a discrete maximum principle and give an estimate on the solution to (2.5).

LEMMA 2.1. *Let the positive property of the coefficients of the difference scheme (2.5) be satisfied.*

(i) If $W(P)$ satisfies the conditions

$$d(P)W(P) - \sum_{P' \in S(P)} e\left(P,P'\right)W\left(P'\right) - F(P) \geq 0 (\leq 0),\ P \in \omega^h,$$

$$W(P) \geq 0 (\leq 0),\ P \in \partial\omega^h,$$

then $W(P) \geq 0 (\leq 0),\ P \in \overline{\omega}^h$.

(ii) The following estimate on the solution to (2.5) holds true

$$\|W\|_{\overline{\omega}^h} \leq \max\left[\left\|W^0\right\|_{\partial\omega^h}; \|F/c\|_{\omega^h}\right], \tag{2.6}$$

where

$$\|W\|_{\overline{\omega}^h} = \max_{P \in \overline{\omega}^h} |W(P)|, \quad \left\|W^0\right\|_{\partial\omega^h} = \max_{P \in \partial\omega^h} \left|W^0(P)\right|.$$

The proof of the lemma can be found in [17].

3. Monotone iterative method for the elliptic problems.

3.1. Monotone convergence. For solving the nonlinear difference scheme (2.2), we investigate uniform convergence of the monotone iterative methods constructed in [5] and [7].

Additionally, we assume that f from (1.1) satisfies the two-sided constraints

$$0 < c_* \leq f_u \leq c^*, \quad c_*, c^* = \text{const}. \tag{3.1}$$

We say that $\overline{U}(P)$ is an upper solution of (2.2) if it satisfies the inequalities

$$\mathcal{L}^h\overline{U} + f(P,\overline{U}) \geq 0,\ P \in \omega^h, \quad \overline{U} \geq g \text{ on } \partial\omega^h.$$

Similarly, $\underline{U}(P)$ is called a lower solution if it satisfies all the reversed inequalities.

The iterative sequence $\left\{U^{(n)}\right\}$ is constructed using the following recurrence formulas

$$U^{(0)}(P) = \text{fixed}, \quad U^{(0)}(P) = g(P),\ P \in \partial\omega^h, \tag{3.2}$$

$$\left(\mathcal{L}^h + c^*\right) Z^{(n+1)} = -G^{(n)}(P), \quad P \in \omega^h,$$

$$G^{(n)}(P) \equiv \mathcal{L}U^{(n)} + f\left(P, U^{(n)}\right), \quad Z^{(n+1)}(P) = 0,\ P \in \partial\omega^h,$$

$$U^{(n+1)}(P) = U^{(n)}(P) + Z^{(n+1)}(P),\ P \in \overline{\omega}^h.$$

The following proposition gives the monotone property of the iterative method (3.2).

PROPOSITION 3.1. *Let $\overline{U}^{(0)}, \underline{U}^{(0)}$ be upper and lower solutions of problem (2.2) and let f satisfy (3.1). Then the upper sequence $\left\{\overline{U}^{(n)}\right\}$ generated by (3.2) converges monotonically from above to the unique solution U of (2.2), the lower sequence $\left\{\underline{U}^{(n)}\right\}$ generated by (3.2) converges monotonically from below to U:*

$$\underline{U}^{(0)} \le \underline{U}^{(n)} \le \underline{U}^{(n+1)} \le U \le \overline{U}^{(n+1)} \le \overline{U}^{(n)} \le \overline{U}^{(0)}, \text{ on } \overline{\omega}^h,$$

and the sequences converge with the linear rate $q = 1 - c_*/c^*$.
The proof of the proposition can be found in [5], [7].

REMARK 3.2. *Consider the following approach for constructing initial upper and lower solutions $\overline{U}^{(0)}$ and $\underline{U}^{(0)}$. Suppose that a mesh function $R(P)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $R = g$ on $\partial\omega^h$. Introduce the following difference problems*

$$\left(\mathcal{L}^h + c_*\right) Z_\nu^{(0)} = \nu \left|\mathcal{L}R + f(P, R)\right|,\ P \in \omega^h, \tag{3.3}$$

$$Z_\nu^{(0)}(P) = 0,\ P \in \partial\omega^h, \quad \nu = 1, -1.$$

Then the functions $\overline{U}^{(0)} = R + Z_1^{(0)}$, $\underline{U}^{(0)} = R + Z_{-1}^{(0)}$ are upper and lower solutions, respectively.
The proof of this result can be found in [5], [7].

REMARK 3.3. *Since the initial iteration in the monotone iterative method (3.2) is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in the previous remark, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.*

REMARK 3.4. *We can modify the iterative method (3.2) in the following way. Proposition 3.1 still holds true if the coefficient c^* in the difference equation from (3.2) is replaced by*

$$c^{(n)}(P) = \max f_u(P, U),\ \underline{U}^{(n)}(P) \le U(P) \le \overline{U}^{(n)}(P),\ P = \mathit{fixed}.$$

To perform the modified algorithm we have to compute two sequences of upper and lower solutions simultaneously. But, on the other hand, this modification increases significantly the rate of the convergence of the iterative method.

Without loss of generality, we assume that the boundary condition in (1.1) is zero, i.e. $g(P) = 0$. This assumption can always be obtained via a change of variables. Let the initial function $U^{(0)}$ be chosen in the form of (3.3), i.e. $U^{(0)}$ is the solution of the following difference problem

$$\left(\mathcal{L}^h + c_*\right) U^{(0)} = \nu \left|f(P, 0)\right|, \quad P \in \omega^h, \tag{3.4}$$

$$U^{(0)}(P)=0,\quad P\in\partial\omega^h,\quad \nu=1,-1,$$

where $R(P)=0$. Then the functions $\overline{U}^{(0)}(P)$, $\underline{U}^{(0)}(P)$ corresponding to $\nu=1$ and $\nu=-1$ are upper and lower solutions, respectively.

THEOREM 3.5. *Suppose that the initial upper or lower solution $U^{(0)}$ is chosen in the form of (3.4). Then the monotone iterative method (3.2) converges uniformly in the perturbation parameters μ and ε:*

$$\left\|U^{(n+1)}-U^{(n)}\right\|_{\overline{\omega}^h}\le c_0q^n\,\|f(P,0)\|_{\overline{\omega}^h}\,,\quad c_0=\frac{3c_*+c^*}{c_*c^*},\tag{3.5}$$

where $q=1-c_*/c^*$.

Proof. Using the mean-value theorem and (3.2), we obtain

$$\left(\mathcal{L}^h+c^*\right)Z^{(n+1)}=\left[c^*-f_u^{(n)}(P)\right]Z^{(n)}(P),\quad P\in\omega^h,$$

$$Z^{(n+1)}(P)=0,\quad P\in\partial\omega^h,$$

where $f_u^{(n)}(P)\equiv f_u\left[P,U^{(n-1)}(P)+\Theta^{(n)}(P)Z^{(n)}(P)\right]$, $0<\Theta^{(n)}(P)<1$. By (2.6) and (3.1),

$$\left\|Z^{(n+1)}\right\|_{\overline{\omega}^h}\le q^n\left\|Z^{(1)}\right\|_{\overline{\omega}^h}.\tag{3.6}$$

Applying (2.6) to (3.2) for $n=1$ and taking into account (3.4), we have

$$\left\|Z^{(1)}\right\|_{\overline{\omega}^h}\le\frac{1}{c^*}\left\|G^{(0)}\right\|_{\omega^h}\le\frac{1}{c^*}\left\|\mathcal{L}^hU^{(0)}\right\|_{\omega^h}+\frac{1}{c^*}\left\|f\left(P,U^{(0)}\right)\right\|_{\overline{\omega}^h}.\tag{3.7}$$

Estimating $U^{(0)}$ from (3.4) by (2.6), we get

$$\left\|U^{(0)}\right\|_{\overline{\omega}^h}\le\frac{1}{c_*}\|f(P,0)\|_{\overline{\omega}^h}.$$

From here and (3.4), it follows that

$$\left\|\mathcal{L}^hU^{(0)}\right\|_{\omega^h}\le c_*\left\|U^{(0)}\right\|_{\overline{\omega}^h}+\|f(P,0)\|_{\overline{\omega}^h}\le 2\,\|f(P,0)\|_{\overline{\omega}^h}.$$

Using the mean-value theorem, (3.1) and the estimate on $U^{(0)}$, we conclude that

$$\left\|f(P,U^{(0)})\right\|_{\overline{\omega}^h}\le\|f(P,0)\|_{\overline{\omega}^h}+c^*\left\|U^{(0)}\right\|_{\overline{\omega}^h}\le\left(1+\frac{c^*}{c_*}\right)\|f(P,0)\|_{\overline{\omega}^h}.$$

Substituting the above estimates in (3.7), we estimate $Z^{(1)}$ in the form

$$\left\|Z^{(1)}\right\|_{\overline{\omega}^h}\le c_0\,\|f(P,0)\|_{\overline{\omega}^h},$$

where c_0 is defined in (3.5). Thus, from here and (3.6), we conclude the uniform estimate (3.5). □

3.2. Uniform convergence of the monotone iterative method (3.2). Here we analyze a convergence rate of the monotone iterative method (3.2) defined on meshes of the general type introduced in [15].

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3.2.1. Layer-adapted meshes. The reaction-diffusion problem (1.1). For the reaction-diffusion problem (1.1), a layer-adapted mesh from [15] is formed in the following manner. We divide each of the intervals $\overline{\omega}^x = [0,1]$ and $\overline{\omega}^y = [0,1]$ into three parts $[0,\sigma_x]$, $[\sigma_x, 1-\sigma_x]$, $[1-\sigma_x, 1]$, and $[0,\sigma_y]$, $[\sigma_y, 1-\sigma_y]$, $[1-\sigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0,\sigma_x]$, $[1-\sigma_x, 1]$ and $[0,\sigma_y]$, $[1-\sigma_y, 1]$ we allocate $N_x/4+1$ and $N_y/4+1$ mesh points, respectively, and in the parts $[\sigma_x, 1-\sigma_x]$ and $[\sigma_y, 1-\sigma_y]$ we allocate $N_x/2+1$ and $N_y/2+1$ mesh points, respectively. Points σ_x, $(1-\sigma_x)$ and σ_y, $(1-\sigma_y)$ correspond to transition to the boundary layers. We consider meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ which are equidistant in $\left[x_{N_x/4}, x_{3N_x/4}\right]$ and $\left[y_{N_y/4}, y_{3N_y/4}\right]$ but graded in $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$. On $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$ let our mesh be given by a mesh generating function ϕ with $\phi(0) = 0$ and $\phi(1/4) = 1$ which is supposed to be continuous, monotonically increasing, and piecewise continuously differentiable. Then our mesh is defined by

$$x_i = \begin{cases} \sigma_x \phi\left(\xi_i\right), & \xi_i = i/N_x,\ i = 0,\dots,N_x/4; \\ ih_x, & i = N_x/4+1,\dots,3N_x/4-1; \\ 1-\sigma_x\left(1-\phi\left(\xi_i\right)\right), & \xi_i = \left(i-3N_x/4\right)/N_x,\ i = 3N_x/4+1,\dots,N_x, \end{cases}$$

$$y_j = \begin{cases} \sigma_y \phi\left(\xi_j\right), & \xi_j = j/N_y,\ j = 0,\dots,N_y/4; \\ jh_y, & j = N_y/4+1,\dots,3N_y/4-1; \\ 1-\sigma_y\left(1-\phi\left(\xi_j\right)\right), & \xi_j = \left(j-3N_y/4\right)/N_y,\ j = 3N_y/4+1,\dots,N_y, \end{cases}$$

$$h_x = 2\left(1-2\sigma_x\right)N_x^{-1}, \quad h_y = 2\left(1-2\sigma_y\right)N_y^{-1}.$$

We also assume that ϕ' does not decrease. This condition implies that

$$h_{xi} \le h_{x,i+1},\ i = 1,\dots,N_x/4-1, \quad h_{xi} \ge h_{x,i+1},\ i = 3N_x/4+1,\dots,N_x-1,$$

$$h_{yj} \le h_{y,j+1},\ j = 1,\dots,N_y/4-1, \quad h_{yj} \ge h_{y,j+1},\ j = 3N_y/4+1,\dots,N_y-1.$$

The convection-diffusion problem (1.1). For the convection-diffusion problem (1.1), a layer-adapted mesh from [15] is formed in the following manner. We divide each of the intervals $\overline{\omega}^x = [0,1]$ and $\overline{\omega}^y = [0,1]$ into two parts $[0, 1-\sigma_x]$, $[1-\sigma_x, 1]$, and $[0, 1-\sigma_y]$, $[1-\sigma_y, 1]$, respectively. Assuming that N_x, N_y are even, in each part we allocate $N_x/2+1$ and $N_y/2+1$ mesh points in the x- and y-directions, respectively. Points $(1-\sigma_x)$ and $(1-\sigma_y)$ correspond to transition to the boundary layers. We consider meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ which are equidistant in $\left[0, x_{N_x/2}\right]$ and $\left[0, y_{N_y/2}\right]$ but graded in $\left[x_{N_x/2}, 1\right]$ and $\left[y_{N_y/2}, 1\right]$. On $\left[x_{N_x/2}, 1\right]$ and $\left[y_{N_y/2}, 1\right]$ let our mesh be given by a mesh generating function $\varphi(\xi)$ with $\varphi(0) = 1$ and $\varphi(1/2) = 0$ which is supposed to be continuous, monotonically decreasing, and piecewise continuously differentiable. Then our mesh is defined by

$$x_i = \begin{cases} ih_x, & i = 0,1,\dots,N_x/2; \\ 1-\sigma_x\varphi\left(\xi_i\right), & \xi_i = \left(i-N_x/2\right)/N_x,\ i = N_x/2+1,\dots,N_x, \end{cases}$$

$$y_j = \begin{cases} jh_y, & j = 0,1,\dots,N_y/2; \\ 1-\sigma_y\varphi\left(\xi_j\right), & \xi_j = \left(j-N_y/2\right)/N_y,\ j = N_y/2+1,\dots,N_y, \end{cases}$$

$$h_x = 2\left(1-\sigma_x\right)N_x^{-1}, \quad h_y = 2\left(1-\sigma_y\right)N_y^{-1}.$$

We also assume that φ' does not decrease. This condition implies that

$$h_{xi} \geq h_{x,i+1}, \quad i = N_x/2+1, \ldots, N_x - 1,$$

$$h_{yj} \geq h_{y,j+1}, \quad j = N_y/2+1, \ldots, N_y - 1.$$

3.2.2. Shishkin-type mesh. The reaction-diffusion problem (1.1). For the reaction-diffusion problem (1.1), we choose the transition points σ_x, $(1-\sigma_x)$ and σ_y, $(1-\sigma_y)$ as in [12]:

$$\sigma_x = \min\left\{4^{-1}, (1/\sqrt{c_*})\,\mu \ln N_x\right\}, \quad \sigma_y = \min\left\{4^{-1}, (1/\sqrt{c_*})\,\mu \ln N_y\right\}.$$

If $\sigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ. In this case, the difference scheme (2.2) can be analyzed using standard techniques. We therefore assume that

$$\sigma_x = (1/\sqrt{c_*})\,\mu \ln N_x, \quad \sigma_y = (1/\sqrt{c_*})\,\mu \ln N_y.$$

Consider the mesh generating function ϕ in the form

$$\phi(\xi) = 4\xi. \tag{3.8}$$

In this case the meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are piecewise equidistant with the step sizes

$$N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = 4\,(1/\sqrt{c_*})\,\mu N_x^{-1} \ln N_x,$$

$$N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = 4\,(1/\sqrt{c_*})\,\mu N_y^{-1} \ln N_y.$$

The difference scheme (2.2) on the piecewise uniform mesh (3.8) converges μ-uniformly to the solution of (1.1):

$$\|U - u\|_{\overline{\omega}^h} \leq CN^{-2}\ln^2 N, \quad N = \min\{N_x, N_y\}, \tag{3.9}$$

where C (sometimes subscripted) denotes a generic constant that is independent of μ or ε and N. The proof of this result can be found in [12].

The convection-diffusion problem (1.1). For the convection-diffusion problem (1.1), we choose the transition points $(1-\sigma_x)$ and $(1-\sigma_y)$ as in [12]:

$$\sigma_x = \min\left\{2^{-1}, (2/\beta_1)\,\varepsilon \ln N_x\right\}, \quad \sigma_y = \min\left\{2^{-1}, (2/\beta_2)\,\varepsilon \ln N_y\right\}.$$

If $\sigma_{x,y} = 1/2$, then $N_{x,y}^{-1}$ are very small relative to ε. In this case, the difference scheme (2.2) can be analyzed using standard techniques. We therefore assume that

$$\sigma_x = (2/\beta_1)\,\varepsilon \ln N_x, \quad \sigma_y = (2/\beta_2)\,\varepsilon \ln N_y.$$

Consider the mesh generating function φ in the form

$$\varphi(\xi) = 1 - 2\xi. \tag{3.10}$$

In this case the meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are piecewise equidistant with the step sizes

$$N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\varepsilon} = (4/\beta_1)\,\varepsilon N_x^{-1} \ln N_x,$$

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$$N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\varepsilon} = (4/\beta_2)\,\varepsilon N_y^{-1} \ln N_y.$$

The upwind difference scheme (2.2) on the piecewise uniform mesh converges ε-uniformly to the solution of (1.1):

$$\|U - u\|_{\overline{\omega}^h} \le CN^{-1} \ln^2 N, \quad N = \min\{N_x, N_y\}, \tag{3.11}$$

where constant C is independent of ε and N. The proof of this result can be found in [12].

THEOREM 3.6. *Suppose that the initial upper or lower solution $U^{(0)}$ is chosen in the form of (3.4). Then the monotone iterative method (3.2) on the piecewise uniform meshes (3.8) and (3.10) converges parameter-uniformly to the solution of problem (1.1):*

$$\left\|U^{(n)} - u\right\|_{\overline{\omega}^h} \le \begin{cases} C\left(N^{-2}\ln^2 N + q^n\right), & \text{for } L = L_\mu, \\ C\left(N^{-1}\ln^2 N + q^n\right), & \text{for } L = L_\varepsilon, \end{cases}$$

where $q = 1 - c_*/c^*$ and constant C is independent of μ or ε and N.

Proof. Using (3.6), we obtain

$$\begin{aligned}\left\|U^{(n+p)} - U^{(n)}\right\|_{\overline{\omega}^h} &\le \sum_{i=n}^{n+p-1} \left\|U^{(i+1)} - U^{(i)}\right\|_{\overline{\omega}^h} = \sum_{i=n}^{n+p-1} \left\|Z^{(i+1)}\right\|_{\overline{\omega}^h} \\ &\le \frac{q}{1-q}\left\|Z^{(n)}\right\|_{\overline{\omega}^h} \le \frac{c_0 q^n}{1-q}\|f(P,0)\|_{\overline{\omega}^h},\end{aligned}$$

where c_0 is defined in (3.5). Taking into account that $\lim U^{(n+p)} = U$ as $p \to \infty$, where U is the solution to (2.2), we conclude the estimate

$$\left\|U^{(n)} - U\right\|_{\overline{\omega}^h} \le \frac{c_0 q^n}{1-q}\|f(P,0)\|_{\overline{\omega}^h}.$$

From here, it follows that

$$\left\|U^{(n)} - u\right\|_{\overline{\omega}^h} \le \|U - u\|_{\overline{\omega}^h} + \frac{c_0 q^n}{1-q}\|f(P,0)\|_{\overline{\omega}^h}.$$

From here and (3.9) for the reaction-diffusion problem, and (3.11) for the convection-diffusion problem, we prove the theorem. □

3.2.3. Bakhvalov-type mesh. The reaction-diffusion problem (1.1). For the reaction-diffusion problem (1.1), we choose the transition points σ_x, $(1-\sigma_x)$ and σ_y, $(1-\sigma_y)$ in Bakhvalov's sense (see [2] for details), i.e.

$$\sigma_x = (1/\sqrt{c_*})\,\mu \ln(1/\mu), \quad \sigma_y = (1/\sqrt{c_*})\,\mu \ln(1/\mu),$$

and the mesh generating function ϕ is given in the form

$$\phi(\xi) = \frac{\ln[1 - 4(1-\mu)\xi]}{\ln \mu}. \tag{3.12}$$

The difference scheme (2.2) on the Bakhvalov-type mesh converges μ-uniformly to the solution of (1.1):

$$\|U - u\|_{\overline{\omega}^h} \le CN^{-1}, \quad N = \min\{N_x, N_y\},$$

where constant C is independent of μ and N. The proof of this result can be found in [2].

The convection-diffusion problem (1.1). For the convection-diffusion problem (1.1), we choose the transition points $(1-\sigma_x)$ and $(1-\sigma_y)$ in Bakhvalov's sense (see [15] for details), i.e.

$$\sigma_x = (2/\beta_1)\varepsilon \ln(1/\varepsilon), \quad \sigma_y = (2/\beta_2)\varepsilon \ln(1/\varepsilon),$$

and the mesh generating function ϕ is given in the form

$$\phi(\xi) = \frac{\ln[1-(1-\varepsilon)(1-2\xi)]}{\ln \varepsilon}. \tag{3.13}$$

The upwind difference scheme (2.2) on the Bakhvalov-type mesh converges ε-uniformly to the solution of (1.1):

$$\|U-u\|_{\overline{\omega}^h} \leq CN^{-1}, \quad N = \min\{N_x, N_y\},$$

where constant C is independent of ε and N. The proof of this result can be found in [15].

Similar to Theorem 3.6, for the monotone iterative method (3.2) on the log-meshes (3.12) and (3.13), we can prove the following theorem.

THEOREM 3.7. *Suppose that the initial upper or lower solution $U^{(0)}$ is chosen in the form of (3.4). Then the monotone iterative method (3.2) on the log-meshes (3.12) and (3.13) converges parameter-uniformly to the solution of problem (1.1):*

$$\left\|U^{(n)}-u\right\|_{\overline{\omega}^h} \leq C\left(N^{-1}+q^n\right), \quad N = \min\{N_x, N_y\},$$

where $q = 1 - c_*/c^*$ and constant C is independent of μ or ε and N.

4. Monotone iterative method for the parabolic problems.

4.1. Monotone convergence. For solving the nonlinear difference scheme (2.4), we investigate uniform convergence of the monotone iterative methods constructed in [6] and [8].

Represent the difference equation from (2.4) in the equivalent form

$$\mathcal{L}U(P,t) = -f(P,t,U) + \frac{U(P,t-\tau)}{\tau}, \quad \mathcal{L} \equiv \left(\mathcal{L}^h + \frac{1}{\tau}\right).$$

We say that on a time level $t \in \omega^\tau$, $\overline{V}(P,t)$ is an upper solution with a given function $V(P,t-\tau)$, if it satisfies

$$\mathcal{L}\overline{V}(P,t) + f\left(P,t,\overline{V}\right) - \tau^{-1}V(P,t-\tau) \geq 0, \ P \in \omega^h,$$

$$\overline{V}(P,t) \geq g(P,t), \ P \in \partial\omega^h.$$

Similarly, $\underline{V}(P,t)$ is called a lower solution on a time level $t \in \omega^\tau$ with a given function $V(P,t-\tau)$, if it satisfies all the reversed inequalities.

Additionally, we assume that f from (1.3) satisfies the two-sided constraints

$$0 \leq f_u \leq c^*, \quad c^* = \text{const}. \tag{4.1}$$

An iterative solution $V(P,t)$ to (2.4) is constructed in the following way. On each time level $t \in \omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t)$, $P \in \overline{\omega}^h$, $n = 1,\dots,n_*$ using the recurrence formulas

$$(\mathcal{L}+c^*)\, Z^{(n+1)}(P,t) = -G^{(n)}(P,t), \quad P \in \omega^h, \tag{4.2}$$

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$$G^{(n)}(P,t) \equiv \mathcal{L}V^{(n)}(P,t) + f\left(P,t,V^{(n)}\right) - \tau^{-1}V(P,t-\tau),$$

$$Z^{(n+1)}(P,t) = 0,\ P \in \partial\omega^h, \quad n = 0,\dots,n_* - 1,$$

$$V^{(n+1)}(P,t) = V^{(n)}(P,t) + Z^{(n+1)}(P,t),\ P \in \overline{\omega}^h,$$

$$V(P,t) \equiv V^{(n_*)}(P,t),\ P \in \overline{\omega}^h, \quad V(P,0) = u^0(P),\ P \in \overline{\omega}^h,$$

where an initial guess $V^{(0)}(P,t)$ satisfies the boundary condition

$$V^{(0)}(P,t) = g(P,t),\ P \in \partial\omega^h.$$

PROPOSITION 4.1. *Let $V^{(0)}(P,t)$ be an upper or a lower solution of problem (2.4) and let f satisfy (4.1). If on each time level the number of iterates n_* in the iterative method (4.2) satisfies $n_* \geq 2$, then the following estimate on convergence rate of the iterative method (4.2) holds*

$$\max_{1\leq k\leq N_\tau} \|V(t_k) - U(t_k)\|_{\overline{\omega}^h} \leq C\rho^{n_*-1}, \quad \rho = c^*/(c^* + \tau^{-1}), \tag{4.3}$$

where $U(P,t)$ is the solution to (2.4), and constant C is independent of τ. Furthermore, on each time level the sequence $\left\{V^{(n)}(P,t)\right\}$ converges monotonically.
The proof of the theorem for the reaction-diffusion problem (2.4) can be found in [6], the result for the convection-diffusion problem (2.4) may be proved in a similar way.

REMARK 4.2. *Consider the following approach for constructing initial upper and lower solutions $\overline{V}^{(0)}(P,t)$ and $\underline{V}^{(0)}(P,t)$. Suppose that for t fixed, a mesh function $R(P,t)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $R(P,t) = g(P,t)$ on $\partial\omega^h$. Introduce the following difference problems*

$$\mathcal{L}Z_\nu^{(0)}(P,t) = \nu \left|\mathcal{L}R(P,t) + f(P,t,R) - \tau^{-1}V(P,t-\tau)\right|,\ P \in \omega^h, \tag{4.4}$$

$$Z_\nu^{(0)}(P,t) = 0,\ P \in \partial\omega^h, \quad \nu = 1,-1.$$

Then the functions $\overline{V}^{(0)}(P,t) = R(P,t) + Z_1^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t) = R(P,t) + Z_{-1}^{(0)}(P,t)$ are upper and lower solutions, respectively.
The proof of this result for (2.4) with $\mathcal{L} = \mathcal{L}_\mu^h + \tau^{-1}$ can be found in [6] and this result for (2.4) with $\mathcal{L} = \mathcal{L}_\varepsilon^h + \tau^{-1}$ may be proved in a similar way.

REMARK 4.3. *On each time level the initial iteration in the monotone iterative method (4.2) is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in the previous remark, hence, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.*

Without loss of generality, we assume that the boundary condition $g = 0$. This assumption can always be obtained via a change of variables. On each time level, let $V^{(0)}(P,t)$ be chosen in the form of (4.4), i.e. $V^{(0)}(P,t)$ is the solution of the following difference problem

$$\mathcal{L}V_\nu^{(0)}(P,t) = \nu \left|f(P,t,0) - \tau^{-1}V(P,t-\tau)\right|,\ P \in \omega^h, \tag{4.5}$$

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$$V_\nu^{(0)}(P,t) = 0,\ P \in \partial\omega^h, \quad \nu = 1, -1,$$

where $R(P,t) = 0$. Then the functions $V_1^{(0)}(P,t)$, $V_{-1}^{(0)}(P,t)$ are the upper and the lower solutions.

THEOREM 4.4. *Let initial upper or lower solutions be chosen in the form of (4.5), and let f satisfy (4.1). Suppose that on each time level the number of iterates $n_* \geq 2$. Then the monotone iterative method (4.2) converges parameter-uniformly, and the estimate (4.3) holds true with constant C which is independent of τ, the perturbation parameter (μ or ε) and the nonuniform mesh.*

Proof. Using the mean-value theorem and the equation for $Z^{(n)}$, we have

$$\mathcal{L}V^{(n)}(P,t) + f\left(P,t,V^{(n)}\right) - \frac{V(P,t-\tau)}{\tau} = -\left[c^* - f_u^{(n)}(P,t)\right] Z^{(n)}(P,t), \tag{4.6}$$

where

$$f_u^{(n)}(P,t) \equiv f_u\left[P,t,V^{(n-1)}(P,t) + \theta^{(n)}(P,t)Z^{(n)}(P,t)\right],$$

and $0 < \theta^{(n)}(P,t) < 1$. From here and (4.2), it follows that $Z^{(n+1)}(P,t)$ satisfies the difference equation

$$(\mathcal{L} + c^*)\, Z^{(n+1)}(P,t) = \left(c^* - f_u^{(n)}\right) Z^{(n)}(P,t),\ P \in \omega^h.$$

Using (2.6) and (4.1), we conclude

$$\left\| Z^{(n+1)}(t) \right\|_{\overline{\omega}^h} \leq \rho^n \left\| Z^{(1)}(t) \right\|_{\overline{\omega}^h}, \quad \rho = \frac{c^*}{c^* + \tau^{-1}}. \tag{4.7}$$

Introduce the notation

$$W(P,t) = U(P,t) - V(P,t),$$

where $V(P,t) \equiv V^{(n_*)}(P,t)$. Using the mean-value theorem, from (2.4) and (4.6), conclude that $W(P,\tau)$ satisfies

$$\mathcal{L}W(P,\tau) + f_u(P,\tau)W(P,\tau) = \left[c^* - f_u^{(n_*)}(P,\tau)\right] Z^{(n_*)}(P,\tau),\ P \in \omega^h,$$

$$W(P,\tau) = 0,\ P \in \partial\omega^h,$$

where $f_u^{(n_*)}(P,\tau) \equiv f_u\left[P,\tau,U(P,\tau) + \theta(P,\tau)W(P,\tau)\right],\ 0 < \theta(P,\tau) < 1$, and we have taken into account that $V(P,0) = U(P,0)$. By (2.6), (4.1) and (4.7),

$$\|W(\tau)\|_{\overline{\omega}^h} \leq c^*\tau\rho^{n_*-1} \left\| Z^{(1)}(\tau) \right\|_{\overline{\omega}^h}.$$

Using (4.5) and the mean-value theorem, estimate $Z^{(1)}(P,\tau)$ from (4.2) by (2.6),

$$\begin{aligned}
\left\| Z^{(1)}(\tau) \right\|_{\overline{\omega}^h} &\leq \tau \left\| \mathcal{L}V^{(0)}(\tau) \right\|_{\overline{\omega}^h} + c^*\tau \left\| V^{(0)}(\tau) \right\|_{\overline{\omega}^h} \\
&\quad + \tau \left\| f(P,\tau,0) - \tau^{-1}u^0 \right\|_{\overline{\omega}^h} \\
&\leq \left(2\tau + c^*\tau^2\right) \left\| f(P,\tau,0) - \tau^{-1}u^0 \right\|_{\overline{\omega}^h} \\
&\leq (2 + c^*\tau)\left[\tau \|f(P,\tau,0)\|_{\overline{\omega}^h} + \left\|u^0\right\|_{\overline{\omega}^h}\right] \leq C_1,
\end{aligned}$$

where C_1 is independent of τ, the perturbation parameter (μ or ε) and the nonuniform mesh. Thus,

$$\|W(\tau)\|_{\overline{\omega}^h} \leq c^* C_1 \tau \rho^{n_*-1}. \tag{4.8}$$

Similarly, from (2.4) and (4.6), it follows that

$$\mathcal{L}W(P,2\tau) + f_u(P,2\tau)W(P,2\tau) = \frac{W(P,\tau)}{\tau} + \left[c^* - f_u^{(n_*)}(P,2\tau)\right] Z^{(n_*)}(P,2\tau).$$

Using (4.7), by (2.6),

$$\|W(2\tau)\|_{\overline{\omega}^h} \leq \|W(\tau)\|_{\overline{\omega}^h} + c^* \tau \rho^{n_*-1} \left\| Z^{(1)}(2\tau) \right\|_{\overline{\omega}^h}. \tag{4.9}$$

Using (4.5), estimate $Z^{(1)}(P,2\tau)$ from (4.2) by (2.6),

$$\left\| Z^{(1)}(2\tau) \right\|_{\overline{\omega}^h} \leq (2 + c^*\tau) \left[\tau \left\| f(P,2\tau,0) \right\|_{\overline{\omega}^h} + \|V(\tau)\|_{\overline{\omega}^h}\right] \leq C_2,$$

where $V(P,\tau) = V^{(n_*)}(P,\tau)$. As follows from [6], the monotone sequences $\left\{\overline{V}^{(n)}(P,\tau)\right\}$ and $\left\{\underline{V}^{(n)}(P,\tau)\right\}$ are bounded from above by $\overline{V}^{(0)}(P,\tau)$ and from below by $\underline{V}^{(0)}(P,\tau)$. Applying (2.6) to the problem (4.5) at $t = \tau$, we have

$$\left\| V^{(0)}(\tau) \right\|_{\overline{\omega}^h} \leq \tau \left\| f(P,\tau,0) - \tau^{-1} u^0(P) \right\|_{\overline{\omega}^h} \leq K_1,$$

where constant K_1 is independent of τ, the perturbation parameter (μ or ε) and the nonuniform mesh. Thus, we prove that C_2 is independent of τ, the perturbation parameter (μ or ε) and the nonuniform mesh. From (4.8) and (4.9), we conclude

$$\|W(2\tau)\|_{\overline{\omega}^h} \leq c^* (C_1 + C_2) \tau \rho^{n_*-1}.$$

By induction on k, we prove

$$\|W(t_k)\|_{\overline{\omega}^h} \leq c^* \left(\sum_{l=1}^{k} C_l\right) \tau \rho^{n_*-1}, \; k = 1, \ldots, N_\tau,$$

where all constants C_l are independent of τ, the perturbation parameter (μ or ε) and the nonuniform mesh. Taking into account that $N_\tau \tau = T$, we prove the estimate (4.3) with $C = c^* T \max_{1 \leq l \leq N_\tau} C_l$. ∎

REMARK 4.5. *The implicit two-level difference schemes (2.4) are of the first order with respect to τ. From here and since $\rho \leq c^*\tau$, one may choose $n_* = 2$ to keep the global error of the monotone iterative method (4.2) consistent with the global error of the difference schemes (2.4).*

4.2. Uniform convergence of the monotone iterative method (4.2). Here we analyze a convergence rate of the monotone iterative method (4.2) defined on the spatial meshes of Shishkin-type (3.8), (3.10) and on the spatial meshes of Bakhvalov-type (3.12), (3.13).

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4.2.1. Shishkin-type mesh. The difference scheme (4.2) on the spatial meshes of Shishkin-type (3.8), (3.10) converges parameter-uniformly to the solution of problem (1.3):

$$\max_{1\le k\le N_\tau} \|U(t_k)-u(t_k)\|_{\overline{\omega}^h} \le \begin{cases} C\left(N^{-2}\ln^2 N+\tau\right), & \text{for } L=L_\mu,\\ C\left(N^{-1}\ln^2 N+\tau\right), & \text{for } L=L_\varepsilon,\end{cases}$$

where constant C is independent of μ or ε, N and τ. The proof of these results can be found in [12]. From here and Theorem 4.4, we conclude the following theorem.

THEOREM 4.6. *Let initial upper or lower solutions $V^{(0)}(P,t_k)$ be chosen in the form of (4.5). Suppose that on each time level the number of iterates $n_*\ge 2$. Then the monotone iterative method (4.2) on the piecewise uniform meshes (3.8) and (3.10) converges parameter-uniformly to the solution of problem (1.3):*

$$\max_{1\le k\le N_\tau} \|V(t_k)-u(t_k)\|_{\overline{\omega}^h} \le \begin{cases} C\left(N^{-2}\ln^2 N+\tau+\rho^{n_*-1}\right), & \text{for } L=L_\mu,\\ C\left(N^{-1}\ln^2 N+\tau+\rho^{n_*-1}\right), & \text{for } L=L_\varepsilon,\end{cases}$$

where $\rho=c^*/\left(c^*+\tau\right)$ and constant C is independent of μ or ε, N and τ.

REMARK 4.7. *In the case of the parabolic reaction-diffusion problem (4.2), Theorem 4.6 holds true on the piecewise uniform mesh (3.8) with an arbitrary fixed constant $m>0$ instead of $\sqrt{c_*}$ in the transition points.*

4.2.2. Bakhvalov-type mesh. The difference scheme (4.2) on the spatial meshes of Bakhvalov-type (3.12), (3.13) converges parameter-uniformly to the solution of problem (1.3):

$$\max_{1\le k\le N_\tau} \|U(t_k)-u(t_k)\|_{\overline{\omega}^h} \le C\left(N^{-1}+\tau\right),$$

where constant C is independent of μ or ε, N and τ. The proof of this result for the reaction-diffusion problem can be found in [4] and for the convection-diffusion problem in [3]. From here and Theorem 4.4, we conclude the following theorem.

THEOREM 4.8. *Let initial upper or lower solutions $V^{(0)}(P,t_k)$ be chosen in the form of (4.5). Suppose that on each time level the number of iterates $n_*\ge 2$. Then the monotone iterative method (4.2) on the log-meshes (3.12) and (3.13) converges parameter-uniformly to the solution of problem (1.3):*

$$\max_{1\le k\le N_\tau} \|V(t_k)-u(t_k)\|_{\overline{\omega}^h} \le C\left(N^{-1}+\tau+\rho^{n_*-1}\right),$$

where $\rho=c^*/\left(c^*+\tau\right)$ and constant C is independent of μ or ε, N and τ.

REMARK 4.9. *In the case of the parabolic reaction-diffusion problem (4.2), Theorem 4.8 holds true on the log-mesh (3.12) with an arbitrary fixed constant $m>0$ instead of $\sqrt{c_*}$ in the transition points.*

5. Numerical experiments. It is found that in all the numerical experiments the basic feature of monotone convergence of the upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from the analytical consideration.

5.1. The elliptic problems. The stopping criterion for the monotone iterative method (3.2) is defined by

$$\left\|U^{(n+1)}-U^{(n)}\right\|_{\overline{\omega}^h} \le \eta.$$

In our numerical experiments we use $\eta = 10^{-5}$ and $N_x = N_y$.

The reaction-diffusion problem. Consider the problem (1.1) with $L = L_\mu$, $f = (u-4)/(5-u)$ and $g = 1$. We mention that $u_r(P) = 4$ is the solution to the reduced problem. This problem gives $c_* = 1/25$, $c^* = 1$,

$$\underline{U}^{(0)}(P) = 1,\ P \in \overline{\omega}^h, \quad \overline{U}^{(0)}(P) = \begin{cases} 4, & P \in \omega^h; \\ 1, & P \in \partial\omega^h, \end{cases} \tag{5.1}$$

where $\underline{U}^{(0)}(P)$ and $\overline{U}^{(0)}(P)$ are the lower and upper solutions to (2.2).

All the discrete linear systems are solved by GMRES-solver [1].

Introduce the notation: $\underline{n}$ and $\overline{n}$ are numbers of iterative steps required for the monotone iterative method (3.2) to reach the prescribed accuracy η with the initial guesses $\underline{U}^{(0)}(P)$ and $\overline{U}^{(0)}(P)$, respectively.

TABLE 5.1
Numbers of iterations for method (3.2) on the piecewise uniform mesh (3.8).

μ	$\underline{n}$; $\overline{n}$			
10^{-1}	20; 15	20; 15	20; 15	20; 15
$\leq 10^{-2}$	20; 15	20; 15	20; 15	20; 14
N_x	64	128	256	≥ 512

In Tables 5.1 and 5.2, for various numbers of N_x and μ, we give the numbers of iterations $\underline{n}$ and $\overline{n}$, required to satisfy the stopping criterion, for the monotone method (3.2) on the piecewise uniform mesh (3.8) and on the log-mesh (3.12), respectively. From the data, we conclude that the numbers of iterations are independent of the perturbation parameter μ. These numerical results confirm our theoretical results stated in Theorems 3.6 and 3.7.

TABLE 5.2
Numbers of iterations for method (3.2) on the log-mesh (3.12).

μ	$\underline{n}$; $\overline{n}$					
10^{-2}	20; 15	20; 15	20; 14	20; 15	20; 15	20; 15
10^{-3}	20; 15	20; 15	20; 15	20; 14	20; 14	20; 14
10^{-4}	20; 14	20; 15	20; 15	20; 15	20; 14	20; 15
N_x	64	128	256	512	1024	2048

TABLE 5.3
Numbers of iterations for the Newton method on the piecewise uniform mesh (3.8).

μ	n_0^{nm}; n_2^{nm}; n_4^{nm}					
10^{-1}	38; 10; 13	36; 68; 18	58; 187; *	*; *; *	*; *; *	*; *; *
10^{-2}	8; 8; 8	7; 15; 7	15; 14; 6	10; 23; 6	13; 35; *	13; *; *
10^{-3}	8; 8; 8	7; 11; 7	7; 9; 6	7; 11; 6	7; 10; 8	8; 15; *
10^{-4}	6; 8; 8	6; 8; 7	6; 8; 6	6; 9; 6	7; 8; 6	7; 9; *
N_x	64	128	256	512	1024	2048

Table 5.3 presents the number of iterations n^{nm} for solving the test problem by the Newton iterative method with the initial guesses $U^{(0)}(P) = 0, 2, 4,\ P \in \omega^h$. We denote by an '*' if more then 200 iterations is needed to satisfy the stopping criterion, or if the

Newton method diverge. The experimental results show that the Newton method cannot be successfully used for this test problem.

The convection-diffusion problem. Consider the problem (1.1) with $L = L_\varepsilon$, $b_{1,2} = 0.1$, $f = (u-4)/(5-u)$ and $g = 1$. We mention that $u_r(P) = 4$ is the solution to the reduced problem. This problem gives $c_* = 1/25$, $c^* = 1$, and the initial lower and upper solutions are defined by (5.1).

All the discrete linear systems are solved by GMRES-solver [1] with the diagonal preconditioner as in [14].

In Tables 5.4 and 5.5, for various numbers of N_x and ε, we present the numbers of iterations $\underline{n}$ and $\overline{n}$ for the monotone method (3.2) on the piecewise uniform mesh (3.10) and on the log-mesh (3.13), respectively. From the data, we conclude that the numbers of iterations are independent of the perturbation parameter ε. These numerical results confirm our theoretical results stated in Theorems 3.6 and 3.7.

TABLE 5.4
Numbers of iterations for method (3.2) on the piecewise uniform mesh (3.10).

ε	$\underline{n}$; $\overline{n}$					
10^{-1}	16; 13	16; 16	16; 16	16; 16	16; 16	16; 16
10^{-2}	22; 20	22; 20	22; 20	22; 20	22; 20	22; 20
10^{-3}	20; 19	19; 18	18; 18	17; 18	17; 17	17 ; 17
10^{-4}	19; 19	18; 18	17; 17	16; 17	16; 17	16 ; 16
N_x	64	128	256	512	1024	2048

TABLE 5.5
Numbers of iterations for method (3.2) on the log-mesh (3.13).

ε	$\underline{n}$; $\overline{n}$					
10^{-1}	16; 16	16; 16	16; 16	16; 16	16; 16	16; 16
10^{-2}	23; 20	22; 20	22; 20	22; 20	22; 20	22; 20
10^{-3}	20; 20	19; 19	18; 18	17; 18	17; 17	17 ; 17
10^{-4}	19; 19	18; 18	17; 17	16; 17	16; 17	16 ; 16
N_x	64	128	256	512	1024	2048

Similar to Table 5.3, the numerical results presented in Table 5.6 indicate that the Newton method cannot be successfully used for this test problem.

TABLE 5.6
Numbers of iterations for the Newton method on the piecewise uniform mesh (3.10).

ε	n_0^{nm}; n_2^{nm}; n_4^{nm}					
10^{-1}	4; 3; 5	4; 3; 5	4; 4; 5	4; 4; 5	5; 4; 6	6; 5; 7
10^{-2}	9; 6; 6	8; 6; 6	9; 7; 5	42; *; *	*; *; *	*; *; *
10^{-3}	9; 8; 6	9; 11; 6	13; 13; 7	10; 24; 16	39; 34; 31	*; *; *
10^{-4}	7; 9; 6	9; 12; 6	13; 9; 6	8; 12; 7	30; 20; *	27; *; *
N_x	64	128	256	512	1024	2048

5.2. The parabolic problems. On each time level t_k, the stopping criterion is chosen in the form

$$\left\| V^{(n)}(t_k) - V^{(n-1)}(t_k) \right\|_{\overline{\omega}^h} \leq \eta,$$

where $\eta = 10^{-5}$. All the discrete linear systems ($N_x = N_y$) in the algorithm (4.2) are solved by GMRES-solver [1] for the reaction-diffusion problem and by GMRES-solver with the diagonal preconditioner as in [14] for the convection-diffusion problem.

The reaction-diffusion problem. Consider the problem (1.3) with $L = L_\mu$, $f = 1 - \exp(-u)$, $g = 1$ and $u^0 = 1$. This problem gives $c^* = 1$.

In Table 5.7, for $\tau_1 = 10^{-1}$, $\tau_2 = 5\times10^{-2}$ and $\tau_3 = 10^{-2}$ and for various values of μ and N_x, we give the average (over ten time levels) numbers of iterations n_{τ_1}, n_{τ_2}, n_{τ_3}, required to satisfy the stopping criterion, for the monotone method (4.2) on the piecewise uniform mesh (3.8). From the data, we conclude that the numbers of iterations are independent of the perturbation parameter μ. We mention that the numerical experiments with the monotone method (4.2) on the log-mesh (3.12) give the same numerical results as in Table 5.7. These numerical results confirm our theoretical results stated in Theorems 4.6 and 4.8.

TABLE 5.7
Numbers of iterations for method (4.2) on the piecewise uniform mesh (3.8).

μ	$\overline{n}_{0.1}$; $\overline{n}_{0.05}$; $\overline{n}_{0.01}$	
10^{-1}	4; 4; 3	4.1; 4; 3
$\leq 10^{-2}$	4.1; 4; 3	4.1; 4; 3
N_x	64	≥ 128

The convection-diffusion problem. Consider the problem (1.3) with $L = L_\varepsilon$, $b_{1,2} = 1$, $f = 1 - \exp(-u)$, $g = 1$ and $u^0 = 1$. This problem gives $c^* = 1$.

TABLE 5.8
Numbers of iterations for method (4.2) on the piecewise uniform mesh (3.8).

ε	$\overline{n}_{0.1}$; $\overline{n}_{0.05}$; $\overline{n}_{0.01}$	
10^{-1}	4; 3.8; 3	4; 3.9; 3
$\leq 10^{-2}$	4; 4; 3	4; 4; 3
N_x	64	≥ 128

Similar to Table 5.7, Table 5.8 presents the numerical results for the monotone method (4.2) on the piecewise uniform mesh (3.8). From the data, we conclude that the numbers of iterations are independent of the perturbation parameter ε. We mention that the numerical experiments with the monotone method (4.2) on the log-mesh (3.12) give the same numerical results as in Table 5.8. These numerical results confirm our theoretical results stated in Theorems 4.6 and 4.8.

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