Electronic Transactions on Numerical Analysis
Volume 22, 2006

Contents

Abstract.
A two-level additive Schwarz preconditioning scheme for solving Ciarlet-Raviart, Hermann-Miyoshi, and Hellan-Hermann-Johnson mixed method equations for the biharmonic Dirichlet problem is presented. Using suitably defined mesh-dependent forms, a unified approach, with ties to the work of Brenner for nonconforming methods, is provided. In particular, optimal preconditioning of a Schur complement formulation for these equations is proved on polygonal domains without slits, provided the overlap between subdomains is sufficiently large.

Key Words.
additive Schwarz preconditioner, mixed finite elements, biharmonic equation, domain decomposition, mesh dependent norms

AMS(MOS) Subject Classifications.
65F10, 65N30, 65N55

Abstract.
Mixed-hybrid finite element discretization of Darcy's law and the continuity equation that describe the potential fluid flow problem in porous media leads to symmetric indefinite saddle-point problems. In this paper we consider solution techniques based on the computation of a null-space basis of the whole or of a part of the left lower off-diagonal block in the system matrix and on the subsequent iterative solution of a projected system. This approach is mainly motivated by the need to solve a sequence of such systems with the same mesh but different material properties. A fundamental cycle null-space basis of the whole off-diagonal block is constructed using the spanning tree of an associated graph. It is shown that such a basis may be theoretically rather ill-conditioned. Alternatively, the orthogonal null-space basis of the sub-block used to enforce continuity over faces can be easily constructed. In the former case, the resulting projected system is symmetric positive definite and so the conjugate gradient method can be applied. The projected system in the latter case remains indefinite and the preconditioned minimal residual method (or the smoothed conjugate gradient method) should be used. The theoretical rate of convergence for both algorithms is discussed and their efficiency is compared in numerical experiments.

Key Words.
saddle-point problem, preconditioned iterative methods, sparse matrices, finite element method

AMS(MOS) Subject Classifications.
65F05, 65F50

41 A network programming approach in solving Darcy's equations by mixed finite-element methods. *M. Arioli and G. Manzini*.

Abstract.
We use the null space algorithm approach to solve the augmented systems produced by the mixed finite-element approximation of Darcy's laws. Taking into account the properties of the graph representing the triangulation, we adapt the null space technique proposed in [M. ARIOLI AND L. BALDINI, *A backward error analysis of a null space algorithm in sparse quadratic programming*, SIAM J. Matrix Anal. and Applics., 23 (2001), pp. 425–442], where an iterative-direct hybrid method is described. In particular, we use network programming techniques to identify the renumbering of the triangles and the edges, which enables us to compute the null space without floating-point operations. Moreover, we extensively take advantage of the graph properties to build efficient preconditioners for the iterative algorithm. Finally, we present the results of several numerical tests.

Key Words.
augmented systems, sparse matrices, mixed finite-element, graph theory

AMS(MOS) Subject Classifications.
65F05, 65F10, 64F25, 65F50, 65G05

71 An augmented Lagrangian approach to the numerical solution of the Dirichlet problem for the elliptic Monge-Ampère equation in two dimensions. *E. J. Dean and R. Glowinski*.

Abstract.
In this article, we discuss the numerical solution of the Dirichlet problem for the real elliptic Monge-Ampère equation, in two dimensions, by an augmented Lagrangian based iterative method. To derive the above algorithm, we take advantage of a reformulation of the Monge-Ampère problem as a saddle-point one, for a well-chosen augmented Lagrangian functional over the product of appropriate primal and dual sets. The convergence of the finite element approximation and of the iterative methods described in this article still has to be proved, however, on the basis of numerical experiments reported in this article, it is safe to say that: (i) The augmented Lagrangian methodology discussed here provides a sequence converging to a solution of the Monge-Ampère problem under consideration, if such a solution exists in the space $H^2(\Omega)$. (ii) If, despite the smoothness of its data, the above problem has no solution, the above augmented Lagrangian method solves it in a least-squares sense, to be precisely defined in this article.

Key Words.
elliptic Monge-Ampère equation, augmented Lagrangian algorithms, mixed finite element approximations

AMS(MOS) Subject Classifications.
35J60, 65F10, 65N30

97 Regularization and stabilization of discrete saddle-point variational problems. *P. B. Bochev and R. B. Lehoucq*.

Abstract.
Our paper considers parameterized families of saddle-point systems arising in the finite element solution of PDEs. Such saddle point systems are ubiquitous in science and engineering. Our motivation is to explain how these saddle-point systems can be modified to avoid onerous stability conditions and to obtain linear systems that are amenable to iterative methods of solution. In particular, the algebraic and variational perspectives of regularization and stabilization are explained.

Key Words.
constrained minimization, saddle point systems, mixed finite elements, regularization, stabilization, penalty, Stokes problem, Darcy flow problem

AMS(MOS) Subject Classifications.
65F15, 65N25, 65N30, 65N22, 65M60, 65N55, 65M55

114 Preconditioners for saddle point linear systems with highly singular (1,1) blocks. *Chen Greif and Dominik Schötzau*.

Abstract.
We introduce a new preconditioning technique for the iterative solution of saddle point linear systems with (1,1) blocks that have a high nullity. The preconditioners are block diagonal and are based on augmentation, using symmetric positive definite weight matrices. If the nullity is equal to the number of constraints, the preconditioned matrices have precisely two distinct eigenvalues, giving rise to immediate convergence of preconditioned MINRES. Numerical examples illustrate our analytical findings.

Key Words.
saddle point linear systems, high nullity, augmentation, block diagonal preconditioners, Krylov subspace iterative solvers

AMS(MOS) Subject Classifications.
65F10

122 Combinatorial algorithms for computing column space bases that have sparse inverses. *Ali Pinar, Edmond Chow, and Alex Pothen*.

Abstract.
This paper presents a new combinatorial approach towards constructing a sparse, implicit basis for the null space of a sparse, under-determined matrix A. Our approach is to compute a column space basis of A that has a sparse inverse, which could be used to represent a null space basis in implicit form. We investigate three different algorithms for computing column space bases: two greedy algorithms implemented using graph matchings, and a third, which employs a divide and conquer strategy implemented with hypergraph partitioning followed by a matching. Our results show that for many matrices from linear programming, structural analysis, and circuit simulation, it is possible to compute column space bases having sparse inverses, contrary to conventional wisdom. The hypergraph partitioning method yields sparser basis inverses and has low computational time requirements, relative to the greedy approaches. We also discuss the complexity of selecting a column

space basis when it is known that such a basis exists in block diagonal form with a given small block size.

Key Words.
sparse column space basis, sparse null space basis, block angular matrix, block diagonal matrix, matching, hypergraph partitioning, inverse of a basis

AMS(MOS) Subject Classifications.
65F50, 68R10, 90C20

Abstract.
We study some parallel overlapping Schwarz preconditioners for solving Stokes-like problems arising from finite element discretization of incompressible flow problems. Most of the existing methods are based on the splitting of the velocity and pressure variables. With the splitting, fast solution methods are often constructed using various fast Poisson solvers for one of the variables. More recently, several papers have investigated the so-called fully coupled approaches in which the variables are not separated. The fully coupled approach has some advantages over the variable splitting method when solving Stokes-like equations with many variables, where a good splitting may be hard to obtain. In this paper we systematically study the parallel scalability of several versions of the fully coupled Schwarz method for both symmetric and nonsymmetric Stokes-like problems. We show numerically that the performance of a two-level method with a multiplicative iterative coarse solver is superior to the other variants of Schwarz preconditioners.

Key Words.
saddle point problem, two-level Schwarz preconditioning, fully coupled methods, finite element, parallel processing

AMS(MOS) Subject Classifications.
65F10, 65N30, 65N55

Abstract.
Several Schur complement-based preconditioners have been proposed for solving (generalized) saddle-point problems. We consider matrices where the Schur complement has rapid decay over some graph known *a priori*. This occurs for many matrices arising from the discretization of systems of partial differential equations, and this graph is then related to the mesh. We propose the use of probing methods to approximate these Schur complements in preconditioners for saddle-point problems. We demonstrate these techniques for the block-diagonal and constraint preconditioners proposed by [Murphy, Golub and Wathen '00], [de Sturler and Liesen '04] and [Siefert and de Sturler '05]. However, these techniques are applicable to many other preconditioners as well. We discuss the implementation of probing methods, and we consider the application of those approximations in preconditioners for Navier-Stokes problems and metal deformation problems. Finally, we study eigenvalue clustering for the preconditioned matrices, and we present convergence and timing results for various problem sizes. These results demonstrate the effectiveness of the proposed preconditioners with probing-based approximate Schur complements.

Key Words.
saddle-point systems, constraint preconditioners, Krylov methods, Schur complements, probing

AMS(MOS) Subject Classifications.
65F10, 65F50, 05C15