Electronic Transactions on Numerical Analysis.
Volume 23, pp. 63-75, 2006.

ISSN 1068-9613.

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TOWARD THE SINC-GALERKIN METHOD FOR THE POISSON PROBLEM IN ONE TYPE OF CURVILINEAR COORDINATE DOMAIN*

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Abstract. This paper introduces the Sinc-Galerkin method for the Poisson problem in one type of curvilinear coordinate domain and shows an example of the numerical results. The method proposed in this paper transforms the domain of the Poisson problem designated by the curvilinear coordinates into a square domain. In this process, Poisson's equation is transformed into a more general two-variable second-order linear partial differential equation. Therefore, this paper also shows a unified solution for general two-variable second-order linear partial differential equations. The derived matrix equation is represented by a simple matrix equation by the use of the Kronecker product. However, the implementation for real applications requires a more efficient calculation of the matrix equation.

Key words. Sinc-Galerkin method, Sinc methods, Poisson problem, differential equations.

AMS subject classifications. 65N99

1. Introduction.

The Sinc-Galerkin method is the numerical method for solving differential equations introduced in [1], which proposes the solution for second-order differential equations. The solution for higher order differential equations is studied in [2, 3, 4]. The textbook [5] shows its applications for solving Poisson's equation, the heat equation and Burger's equation in a square domain. The textbook [6] collects a wide range of topics involving so-called "Sinc methods"—a class of numerical methods, including the Sinc-Galerkin method, based on the use of the Cardinal function, which is an expansion of a function using the Sinc basis functions. These topics are more recently summarized in [7]. The Sinc-Galerkin method for typical differential equations in a square domain has been well studied in these and other references. Also, the domain decomposition for rectangular domains and L-shaped domains is discussed in [8, 9, 10, 11, 12]. However, the case of more complicated domains has not been studied very much.

This paper introduces the solution of the Poisson problem in a certain type of curvilinear coordinate domain and points out problems with it. In this solution, Poisson's equation in the curvilinear coordinate domain is transformed into a more general two-variable second-order linear partial differential equation in a square domain. Therefore, this paper first shows a unified solution for general two-variable second-order linear partial differential equations in a square domain.

2. The Sinc-Galerkin Method for General Two-Variable Second-Order Linear Partial Differential Equations.

To solve a differential equation, the Sinc-Galerkin method derives a matrix equation which corresponds to the differential equation. Derivations of the matrix equation in response to the type of the differential equation have been introduced in [1, 2, 3, 4, 5, 6, 7]. This section shows a more unified derivation of the matrix equation for two-variable second-order linear partial differential equations using integration by parts.

*Received July 29, 2005. Accepted for publication December 31, 2005. Recommended by F. Stenger.
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Let us consider solving the two-variable second-order linear partial differential equation

$$(2.1)\quad \begin{aligned} Lu(x,y) \equiv a_1(x)\,b_1(y)\,\frac{\partial^2 u(x,y)}{\partial x^2} + a_2(x)\,b_2(y)\,\frac{\partial^2 u(x,y)}{\partial x \partial y} \\ + a_3(x)\,b_3(y)\,\frac{\partial^2 u(x,y)}{\partial y^2} + a_4(x)\,b_4(y)\,\frac{\partial u(x,y)}{\partial x} \\ + a_5(x)\,b_5(y)\,\frac{\partial u(x,y)}{\partial y} + a_6(x)\,b_6(y)\,u(x,y) = f(x,y), \end{aligned}$$

whose boundary conditions are

$$\begin{aligned} u(0,y) = u(1,y) = 0, &\qquad 0 \le y \le 1, \\ u(x,0) = u(x,1) = 0, &\qquad 0 \le x \le 1. \end{aligned}$$

In the Sinc-Galerkin method, by using the sinc function, which is defined by

$$\mathrm{sinc}(x) \equiv \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \ne 0, \\ 1, & x = 0, \end{cases}$$

the basis functions for approximating a one-variable function are defined by

$$S(k,h) \circ \phi(x) \equiv \mathrm{sinc}\left(\frac{\phi(x) - kh}{h}\right),$$

and the basis functions for approximating a two-variable function are defined by

$$S_{kl}(x,y) \equiv \{S(k,h_x) \circ \phi_x(x)\}\{S(l,h_y) \circ \phi_y(y)\},$$

where ϕ, ϕ_x and ϕ_y are conformal maps from the domain of the approximated function onto $(-\infty,\infty)$, k and l, the subscripts of the basis functions, are integers, and h, h_x and h_y, the step sizes for the discretization, are positive real numbers. By using the basis functions for a two-variable function, the approximate solution of (2.1) is represented by the truncated series

$$(2.2)\qquad u_{m_x m_y}(x,y) = \sum_{l=-M_y}^{N_y} \sum_{k=-M_x}^{N_x} u(x_k,y_l)\, S_{kl}(x,y),$$

where

$$(2.3)\qquad \begin{aligned} m_x &\equiv M_x + N_x + 1, \\ m_y &\equiv M_y + N_y + 1, \\ x_k &\equiv \phi_x^{-1}(kh_x), \\ y_l &\equiv \phi_y^{-1}(lh_y). \end{aligned}$$

As noticed from the notation, the coefficients of the basis functions correspond to the values of the approximated function at the discrete points. The selection of M_x, N_x, M_y, N_y, h_x and h_y for a good approximation is precisely discussed in [5, 6].

The values of the approximate solution at the discrete points, $u_{m_x m_y}(x_k, y_l)$, are determined by the orthogonal conditions

$$(2.4)\qquad (Lu_{m_x m_y}, S_{kl}) = (f, S_{kl}), \qquad \begin{aligned} k &= -M_x, -M_x+1, \ldots, N_x, \\ l &= -M_y, -M_y+1, \ldots, N_y. \end{aligned}$$

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TABLE 2.1
Values of s_n and t_n.

n	1	2	3	4	5	6
s_n	2	1	0	1	0	0
t_n	0	1	2	0	1	0

The inner product is, with its interval adjusted to that of the problem, defined by

$$(f,g) = \int_0^1 \int_0^1 f(x,y)\, g(x,y)\, v(x)\, w(y)\, dxdy, \tag{2.5}$$

where v and w are weight functions.

For the differential equation (2.1), the left-hand side of the orthogonal conditions (2.4) comprises the terms

$$\left(a_n(x)\, b_n(y)\, \frac{\partial^{s_n+t_n} u_{m_x m_y}(x,y)}{\partial x^{s_n} \partial y^{t_n}}, S_{kl} \right), \tag{2.6}$$

where s_n and t_n are set to be $0, 1$ or 2 according to $n = 1, 2, \ldots, 6$ (Table 2).

From the definition of the inner product (2.5), the terms (2.6) are represented by the integral

$$\int_0^1 \int_0^1 a_n(x)\, b_n(y)\, \frac{\partial^{s_n+t_n} u_{m_x m_y}(x,y)}{\partial x^{s_n} \partial y^{t_n}} \{S(k,h_x) \circ \phi_x(x)\} \{S(l,h_y) \circ \phi_y(y)\}\, v(x)\, w(y)\, dxdy,$$

which can be transformed into

$$\int_0^1 \int_0^1 u_{m_x m_y}(x,y)\, (-1)^{s_n} \frac{\partial^{s_n}}{\partial x^{s_n}} [a_n(x) \{S(k,h_x) \circ \phi_x(x)\}\, v(x)] (-1)^{t_n} \frac{\partial^{t_n}}{\partial y^{t_n}} [b_n(y) \{S(l,h_y) \circ \phi_y(y)\}\, w(y)]\, dxdy, \tag{2.7}$$

by using integration by parts multiple times so as to eliminate the derivatives of the approximate solution $u_{m_x m_y}$. Note that the boundary terms generated by integration by parts can be assumed to vanish when the weight functions v and w are appropriately set because $u_{m_x m_y}$, $S(k,h_x) \circ \phi_x$ and $S(l,h_y) \circ \phi_y$ are 0 at the boundaries. To make this assumption hold, appropriate functions are selected to be the weight functions v and w so as to make these terms converge. The most important problem is about the derivatives of the Sinc basis functions

$$\{S(k,h) \circ \phi(x)\}' = S'(k,h) \circ \phi(x) \cdot \phi'(x),$$

because the map ϕ from the finite domain (a,b) onto $(-\infty,\infty)$ is usually set to be

$$\phi(x) = \log\left(\frac{x-a}{b-x}\right),$$

and its derivative is

$$\phi'(x) = \frac{b-a}{(x-a)(b-x)},$$

which has singularities at the boundaries. It means that the derivatives of the Sinc basis functions have singularities at the boundary. To avoid this problem, it is convenient to set the weight function to be

$$v(x) = \frac{1}{\phi'(x)},$$

so that the terms converge.

In the Sinc-Galerkin method, the integrals of the orthogonal conditions (2.4) are approximated by the trapezoidal rule using variable transformations. To compute these integrals by the use of the values of the integrands at the discrete points defined in (2.3), the integrals with respect to x from 0 to 1 are transformed into the integrals with respect to $\xi = \phi_x(x)$ and approximated by

$$\int_0^1 u(x)\,dx = \int_{-\infty}^{\infty} u(x)\,\frac{1}{\phi_x'(x)}\,d\xi \approx h_x \sum_{p=-M_x}^{N_x} \frac{u(x_p)}{\phi_x'(x_p)}.$$

Applying the two-variable trapezoidal rule to (2.7) results in

$$\begin{aligned}(2.8)\quad & h_x h_y \sum_{q=-M_y}^{N_y} \sum_{p=-M_x}^{N_x} u_{m_x m_y}(x_p, y_q) \\ & \frac{(-1)^{s_n}}{\phi_x'(x_p)} \frac{\partial^{s_n}}{\partial x^{s_n}}\left[a_n(x)\left\{S(k,h_x)\circ\phi_x(x)\right\}v(x)\right]\Big|_{x=x_p} \\ & \cdot \frac{(-1)^{t_n}}{\phi_y'(y_q)} \frac{\partial^{t_n}}{\partial y^{t_n}}\left[b_n(y)\left\{S(l,h_y)\circ\phi_y(y)\right\}w(y)\right]\Big|_{y=y_q}.\end{aligned}$$

Here, the Sinc basis functions and their first-order and second-order derivatives have the following properties:

$$\delta_{kp}^{(0)} \equiv \left\{S(k,h)\circ\phi(x)\right\}\Big|_{x=x_p} = \begin{cases} 1, & k=p, \\ 0, & k\neq p, \end{cases}$$

$$\delta_{kp}^{(1)} \equiv h\frac{d}{d\phi}\left\{S(k,h)\circ\phi(x)\right\}\Big|_{x=x_p} = \begin{cases} 0, & k=p, \\ \dfrac{(-1)^{p-k}}{p-k}, & k\neq p, \end{cases}$$

$$\delta_{kp}^{(2)} \equiv h^2\frac{d^2}{d\phi^2}\left\{S(k,h)\circ\phi(x)\right\}\Big|_{x=x_p} = \begin{cases} -\dfrac{\pi^2}{3}, & k=p, \\ -\dfrac{2\,(-1)^{p-k}}{(p-k)^2}, & k\neq p. \end{cases}$$

Therefore, the differentiation parts of x of (2.8) can be represented by

$$\left[a_n(x)\left\{S(k,h_x)\circ\phi_x(x)\right\}v(x)\right]\Big|_{x=x_p} = \delta_{kp}^{(0)}\,a_n(x_p)\,v(x_p),$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[a_n(x)\left\{S(k,h_x)\circ\phi_x(x)\right\}v(x)\right]\Big|_{x=x_p} \\ &\qquad = \frac{\delta_{kp}^{(1)}}{h_x}\left[\phi_x'(x_p)\,a_n(x_p)\,v(x_p)\right] + \delta_{kp}^{(0)}\left[\frac{\partial}{\partial x}\left\{a_n(x)\,v(x)\right\}\Big|_{x=x_p}\right],\end{aligned}$$

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$$\frac{\partial^2}{\partial x^2}[a_n(x)\{S(k,h_x)\circ\phi_x(x)\}v(x)]\Big|_{x=x_p}$$
$$= \frac{\delta^{(2)}_{kp}}{h_x^2}\left[\{\phi'_x(x_p)\}^2 a_n(x_p)\,v(x_p)\right]$$
$$+ \frac{\delta^{(1)}_{kp}}{h_x}\left[\phi''_x(x_p)\,a_n(x_p)\,v(x_p) + 2\phi'_x(x_p)\frac{\partial}{\partial x}\{a_n(x)\,v(x)\}\Big|_{x=x_p}\right]$$
$$+ \delta^{(0)}_{kp}\left[\frac{\partial^2}{\partial x^2}\{a_n(x)\,v(x)\}\Big|_{x=x_p}\right],$$

for $s_n = 0, 1, 2$ respectively. The differentiation parts of y are obtained just by replacing the functions and variable names in the above.

Applying the two-variable trapezoidal rule to the right-hand side of the orthogonal conditions (2.4) yields

$$h_x h_y \sum_{q=-M_y}^{N_y}\sum_{p=-M_x}^{N_x}\frac{\delta^{(0)}_{kp}\,\delta^{(0)}_{lq}\,f(x_p,y_q)\,v(x_p)\,w(y_q)}{\phi'_x(x_p)\,\phi'_y(y_q)} = h_x h_y\frac{f(x_k,y_l)\,v(x_k)\,w(y_l)}{\phi'_x(x_k)\,\phi'_y(y_l)}.$$

Let us define the symbol

$$\boldsymbol{A} = [a_{ij}]_{ij}$$

to be the matrix which has a_{ij}, a value related to i and j, as its $(i+M+1)$,$(j+M+1)$-th element $(i, j = -M, -M+1, \ldots, N)$.

The orthogonal conditions (2.4) of the problem (2.1) can be represented by the linear matrix equation

$$\boldsymbol{A}_1\boldsymbol{U}\boldsymbol{B}_1^T + \boldsymbol{A}_2\boldsymbol{U}\boldsymbol{B}_2^T + \cdots + \boldsymbol{A}_6\boldsymbol{U}\boldsymbol{B}_6^T = \boldsymbol{G}, \tag{2.9}$$

where

$$\boldsymbol{U} = [u_{m_x m_y}(x_p,y_q)]_{pq},$$
$$\boldsymbol{A}_n = \left[\frac{(-1)^{s_n}}{\phi'_x(x_p)}\frac{\partial^{s_n}}{\partial x^{s_n}}[a_n(x)\{S(k,h_x)\circ\phi_x(x)\}v(x)]\Big|_{x=x_p}\right]_{kp},$$
$$\boldsymbol{B}_n = \left[\frac{(-1)^{t_n}}{\phi'_y(y_q)}\frac{\partial^{t_n}}{\partial y^{t_n}}[b_n(y)\{S(l,h_y)\circ\phi_y(y)\}w(y)]\Big|_{y=y_q}\right]_{lq},$$
$$\boldsymbol{G} = \left[\frac{f(x_k,y_l)\,v(x_k)\,w(y_l)}{\phi'_x(x_k)\,\phi'_y(y_l)}\right]_{kl}.$$

Note that in (2.9) the matrix $\boldsymbol{U}$ is multiplied by $\boldsymbol{A}_n$, matrices of x, from the left and multiplied by $\boldsymbol{B}_n$, matrices of y, from the right $(n = 1, 2, \ldots, 6)$. Therefore, the coefficient functions of the differential equation to be solved must be able to be separated with respect to the variables like (2.1).

The linear matrix equation (2.9) can be transformed into a simple matrix equation by the use of the Kronecker product and vec-function [13]. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times m$ matrices. The Kronecker product of the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by

$$\boldsymbol{A}\otimes\boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1m}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \cdots & a_{2m}\boldsymbol{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \cdots & a_{mm}\boldsymbol{B} \end{bmatrix}.$$

Let $\boldsymbol{x}_j$ $(j = 1, 2, \ldots, m)$ be the j-th column vector of the matrix $\boldsymbol{X}$. The matrix $\boldsymbol{X}$ can be denoted by $\boldsymbol{X} = [\boldsymbol{x}_1\, \boldsymbol{x}_2\, \cdots\, \boldsymbol{x}_m]$. The operator vec, concatenating these column vectors, transforms $\boldsymbol{X}$ into the m^2-element column vector which is defined by

$$\mathrm{vec}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix}.$$

By using the Kronecker product and vec-function, the linear matrix equation (2.9) can be transformed into

$$\boldsymbol{C}\boldsymbol{u} = \boldsymbol{g}, \tag{2.10}$$

where

$$\begin{aligned} \boldsymbol{C} &= \sum_{n=1}^{6} (\boldsymbol{B}_n \otimes \boldsymbol{A}_n), \\ \boldsymbol{u} &= \mathrm{vec}\, \boldsymbol{U}, \\ \boldsymbol{g} &= \mathrm{vec}\, \boldsymbol{G}. \end{aligned}$$

To solve (2.10), we can use a solution for simultaneous linear equations. Then, we get approximate values of the solution at the discrete points, $u_{m_x m_y}(x_k, y_l)$. If you want to know the values at other points, you can exploit (2.2) as an interpolation formula, using $u_{m_x m_y}(x_k, y_l)$ as its coefficients since $u_{m_x m_y}(x_k, y_l) \approx u(x_k, y_l)$.

When the number of the used bases is rather big, the coefficient matrix of the derived matrix equation (2.10) becomes immense. Then, the implementation for real applications requires a more efficient matrix calculation of (2.9). In the case of Poisson's equation, the derived matrix equation can be transformed into the Sylvester equation. The numerical solution of the Sylvester equation has been well studied [14, 15] et al. and implemented in the matrix computation package LAPACK [16]. Also, the textbooks [5, 6] show a solution of the Sylvester equation using the diagonalization of the coefficient matrices. However, the general solution of the matrix equation (2.9) has not been studied enough.

3. The Sinc-Galerkin Method for The Poisson Problem in One Type of Curvilinear Coordinate Domain. This section introduces the Sinc-Galerkin method for the Poisson problem in the curvilinear coordinate domain which is proposed in [6, 7] but has not been discussed in detail for the Sinc-Galerkin method.

Let us consider solving the two-variable Poisson problem

$$\Delta u(\xi, \eta) = f(\xi, \eta), \tag{3.1}$$

in a domain designated by the curvilinear coordinates in which the range of ξ is determined by the constants a_1 and b_1, and the range of η is determined by $a_2(\xi)$ and $b_2(\xi)$, the functions of ξ (Figure 3.1). For the sake of simplicity, assume $u(\xi, \eta) = 0$ at the boundary.

By using the variable transformations

$$\begin{cases} \xi = a_1 + (b_1 - a_1)x, \\ \eta = a_2(\xi) + \{b_2(\xi) - a_2(\xi)\}\, y, \end{cases} \tag{3.2}$$

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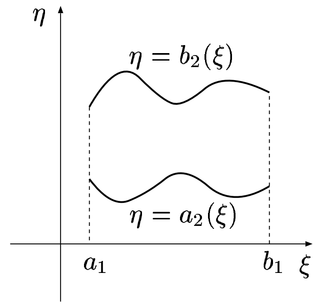

FIG. 3.1. *One type of curvilinear coordinate domain.*

the domain designated by ξ and η is mapped onto the square domain $(x, y) = (0, 1) \times (0, 1)$. Also, the derivatives with respect to ξ and η transform into

$$\begin{cases} \dfrac{\partial}{\partial \xi} = \dfrac{1}{b_1 - a_1} \dfrac{\partial}{\partial x} - \dfrac{a_2'(\xi) + \{b_2'(\xi) - a_2'(\xi)\}\, y}{b_2(\xi) - a_2(\xi)} \dfrac{\partial}{\partial y}, \\ \dfrac{\partial}{\partial \eta} = \dfrac{1}{b_2(\xi) - a_2(\xi)} \dfrac{\partial}{\partial y}. \end{cases}$$

Therefore, the problem mapped onto the new domain is not of the form of Poisson's equation but a little more complex two-variable second-order linear differential equation, which requires the solution introduced in Section 2. For the sake of simplicity, the range of ξ is fixed to $(0, 1)$, and the range of η is described by $(a(\xi), b(\xi))$ in the remainder of this paper. This simplification allows the transformations (3.2) to be represented by

$$\begin{cases} \xi = x, \\ \eta = a(x) + \{b(x) - a(x)\}y. \end{cases} \tag{3.3}$$

By using the transformations (3.3), Poisson's equation (3.1) is transformed into

$$\frac{\partial^2 \tilde{u}(x,y)}{\partial x^2} + C_1(x,y)\frac{\partial^2 \tilde{u}(x,y)}{\partial x \partial y} + C_2(x,y)\frac{\partial^2 \tilde{u}(x,y)}{\partial y^2} + C_3(x,y)\frac{\partial \tilde{u}(x,y)}{\partial y} = \tilde{f}(x,y), \tag{3.4}$$

where

$$\begin{aligned} u(\xi,\eta) &= \tilde{u}(x,y), \\ f(\xi,\eta) &= \tilde{f}(x,y), \\ C_1(x,y) &= -2 \cdot \frac{a'(x) + \{b'(x) - a'(x)\}\, y}{b(x) - a(x)}, \\ C_2(x,y) &= \frac{[a'(x) + \{b'(x) - a'(x)\}y]^2 + 1}{\{b(x) - a(x)\}^2}, \end{aligned}$$

$$C_3(x,y) = \frac{a'(x)\{b'(x) - a'(x)\} + \{b'(x) - a'(x)\}^2 y}{\{b(x) - a(x)\}^2} - \frac{\partial}{\partial x}\left(\frac{a'(x) + \{b'(x) - a'(x)\}y}{b(x) - a(x)}\right).$$

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The solution of general two-variable second-order linear differential equations in Section 2 is applied to this transformed equation. However, since the coefficient function of each term must be separated with respect to the variables x and y, by expanding the coefficient functions, (3.4) is transformed into

$$\begin{aligned}(3.5)\quad \frac{\partial^2 \tilde{u}(x,y)}{\partial x^2} &+ \{C_{1\alpha}(x) + C_{1\beta}(x)\cdot y\}\frac{\partial^2 \tilde{u}(x,y)}{\partial x \partial y} \\ &+ \{C_{2\alpha}(x) + C_{2\beta}(x)\cdot y + C_{2\gamma}(x)\cdot y^2\}\frac{\partial^2 \tilde{u}(x,y)}{\partial y^2} \\ &+ \{C_{3\alpha}(x) + C_{3\beta}(x)\cdot y\}\frac{\partial \tilde{u}(x,y)}{\partial y} = \tilde{f}(x,y),\end{aligned}$$

where

$$\begin{aligned}C_{1\alpha}(x) &= \frac{-2a'(x)}{b(x)-a(x)},\\ C_{1\beta}(x) &= \frac{-2\{b'(x)-a'(x)\}}{b(x)-a(x)},\\ C_{2\alpha}(x) &= \frac{\{a'(x)\}^2+1}{\{b(x)-a(x)\}^2},\\ C_{2\beta}(x) &= \frac{2a'(x)\{b'(x)-a'(x)\}}{\{b(x)-a(x)\}^2},\\ C_{2\gamma}(x) &= \frac{\{b'(x)-a'(x)\}^2}{\{b(x)-a(x)\}^2},\\ C_{3\alpha}(x) &= \frac{a'(x)\{b'(x)-a'(x)\}}{\{b(x)-a(x)\}^2} - \frac{\partial}{\partial x}\left(\frac{a'(x)}{b(x)-a(x)}\right),\\ C_{3\beta}(x) &= \frac{\{b'(x)-a'(x)\}^2}{\{b(x)-a(x)\}^2} - \frac{\partial}{\partial x}\left(\frac{b'(x)-a'(x)}{b(x)-a(x)}\right).\end{aligned}$$

Notice that the coefficient function of each term is represented as a product of two parts of x and y. The transformed equation is treated as an eight-term differential equation. Therefore, the derived matrix equation becomes like

$$\boldsymbol{A}_1\boldsymbol{U}\boldsymbol{B}_1^T + \boldsymbol{A}_2\boldsymbol{U}\boldsymbol{B}_2^T + \cdots + \boldsymbol{A}_8\boldsymbol{U}\boldsymbol{B}_8^T = \boldsymbol{G},$$

where the n-th term in the equation above corresponds to the n-th term in (3.5).

Note that the coefficient matrices which depend on the functions a and b are only $\boldsymbol{A}_2, \boldsymbol{A}_3, \dots, \boldsymbol{A}_8$ and that $\boldsymbol{B}_4$, $\boldsymbol{B}_7$ and $\boldsymbol{B}_8$ are identical to $\boldsymbol{A}_1$, $\boldsymbol{B}_2$ and $\boldsymbol{B}_3$ respectively.

This paper shows only the case of the type of curvilinear coordinate domain described above, but the procedure in this section can be applied to other types of coordinate domains if the coefficient functions of the transformed equation can be separated with respect to the variables.

4. Numerical Experiment. This section shows an example of the numerical results for the Poisson problem (3.1) in a curvilinear coordinate domain solved by a C language program.

The functions $a(x)$ and $b(x)$ were set to be

$$a(x) = 40x(x-1/4)(x-3/4)(x-1) - 1,$$

$$b(x) = -\frac{2}{3}\sqrt{1-(x-1)^2} + \frac{7}{6}.$$

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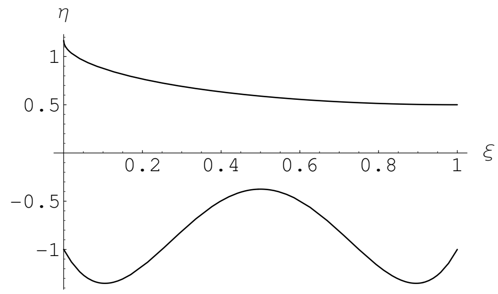

FIG. 4.1. *The domain of the example.*

TABLE 4.1
Maximum errors of Sinc-Galerkin solution.

M_x	Max Error
2	7.259×10^{-2}
5	1.063×10^{-2}
10	2.579×10^{-3}
20	2.971×10^{-4}

The shape of the domain is shown in Figure 4.1.

A certain function whose value is 0 at the boundary was chosen as the solution $u(\xi, \eta)$ in advance, and the nonhomogeneous term $f(\xi, \eta)$ was obtained by calculating $\Delta u(\xi, \eta)$. In this example, we selected

$$\{40x(x-1/4)(x-3/4)(x-1)-1-y\}\{y+2/3\sqrt{1-(x-1)^2}-7/6\}x(1-x)$$

as the solution. The numbers of the basis functions were set to be $M_x = M_y = N_x = N_y =$ 20 and the step sizes h_x and h_y were set to be different values, $h_x = h_y = h$. Figures 4.2–4.7 show the shapes of the approximate solution obtained by the procedure introduced in this paper with different step sizes h. Table 4.1 shows the maximum errors at the discrete points (x_k, y_l) in the cases $M_x = 2, 5, 10, 20$ with $h = \sqrt{\pi/M_x}$, which gives the best result for the above selected h's.

Since the solutions of Poisson's equation in non-smooth boundary domains have singularities at the angular points [6, 17, 18], a bad selection of the step sizes h_x and h_y sometimes results in a fatal error. The step size smaller than the optimal value makes the discrete points get closer to the center of the domain, while the step size larger than the optimal value makes the discrete points get closer to the boundaries [6]. Therefore, a large step size gives rise to the occurrence of corner singularities and drastically decreases the accuracy. Figures 4.6 and 4.7 show the shapes which have corner singularities.

5. Conclusion. This paper introduced the Sinc-Galerkin method for the Poisson problem in one type of curvilinear coordinate domain. In this procedure, the derived matrix equation can be represented by the simple matrix equation (2.10), but the coefficient matrix of (2.10) becomes immense when the number of the bases used is rather big. Therefore, the implementation for real applications requires a more efficient matrix calculation of (2.9).

This paper shows only the case of one type of curvilinear coordinate domain, but the procedure in this paper can be applied to other types of coordinate domains if the coefficient functions of the transformed equation can be separated with respect to the variables.

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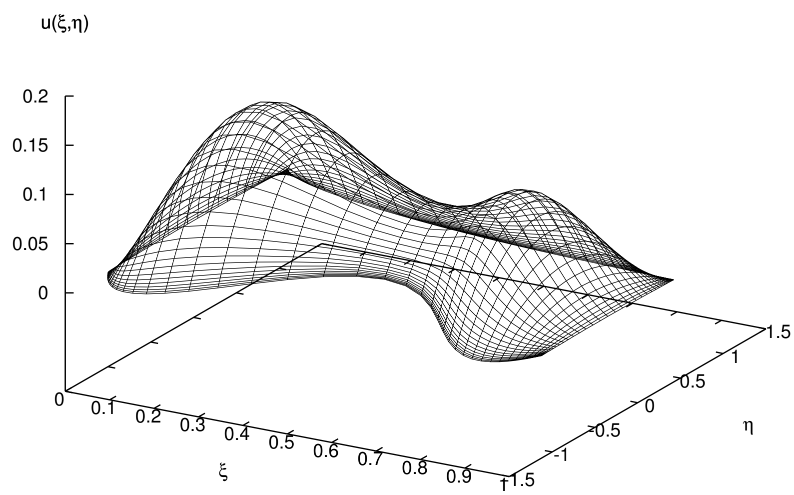

FIG. 4.2. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = 1/\sqrt{M_x}$.
Max Error=1.056×10^{-2}

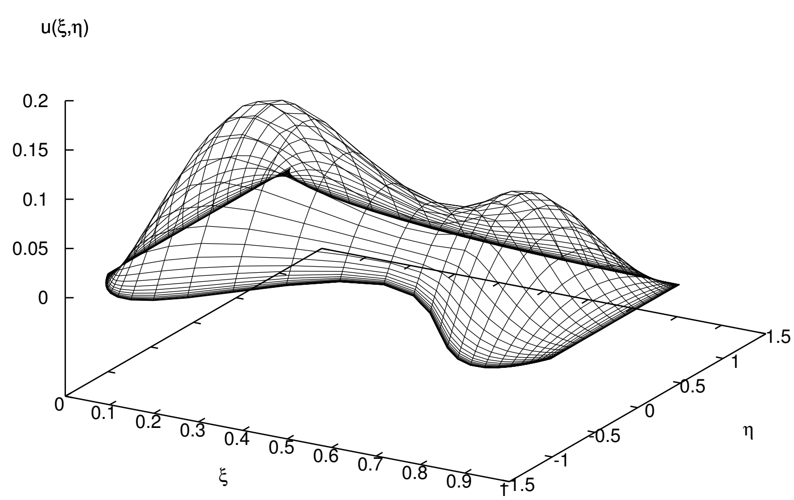

FIG. 4.3. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = 1.5/\sqrt{M_x}$.
Max Error=1.049×10^{-3}

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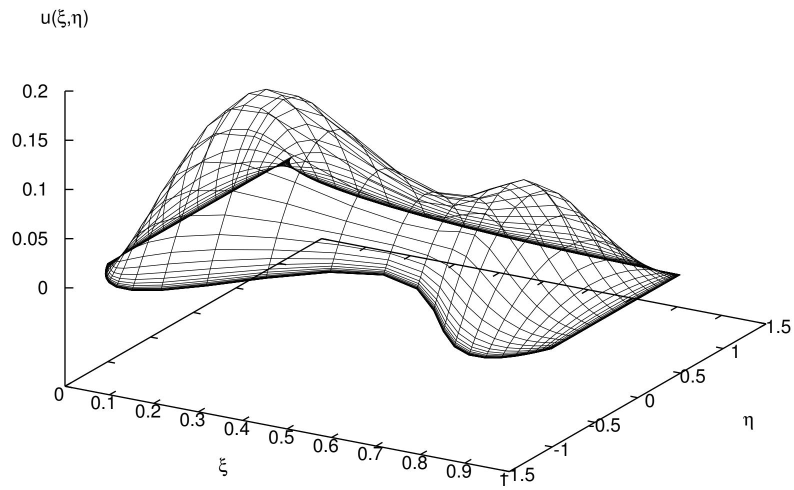

FIG. 4.4. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = \sqrt{\pi/M_x}$.
Max Error=2.971×10^{-4}

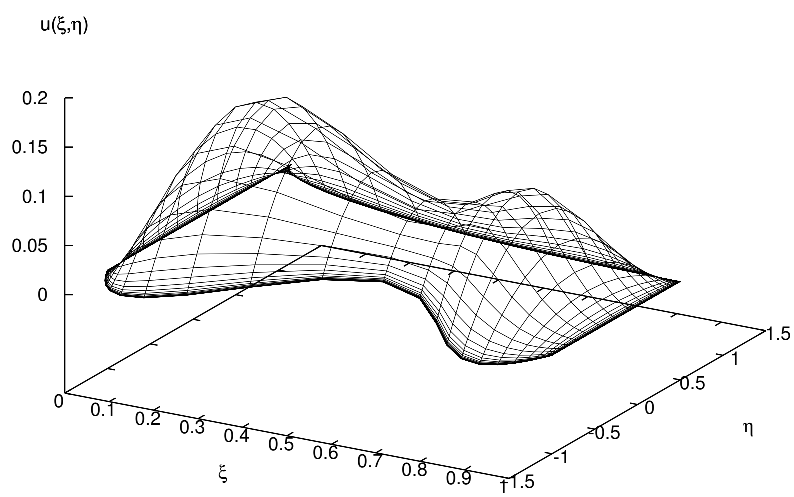

FIG. 4.5. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = 2/\sqrt{M_x}$.
Max Error=5.897×10^{-4}

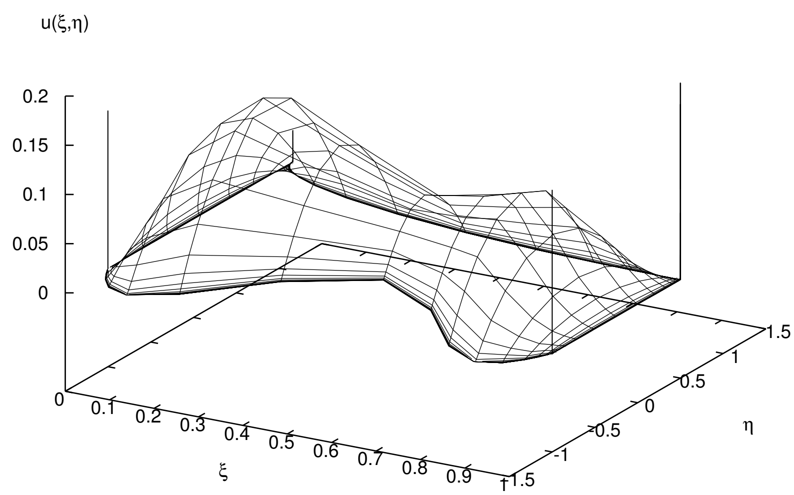

FIG. 4.6. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = \pi/\sqrt{M_x}$.
Max Error=1.149×10^3

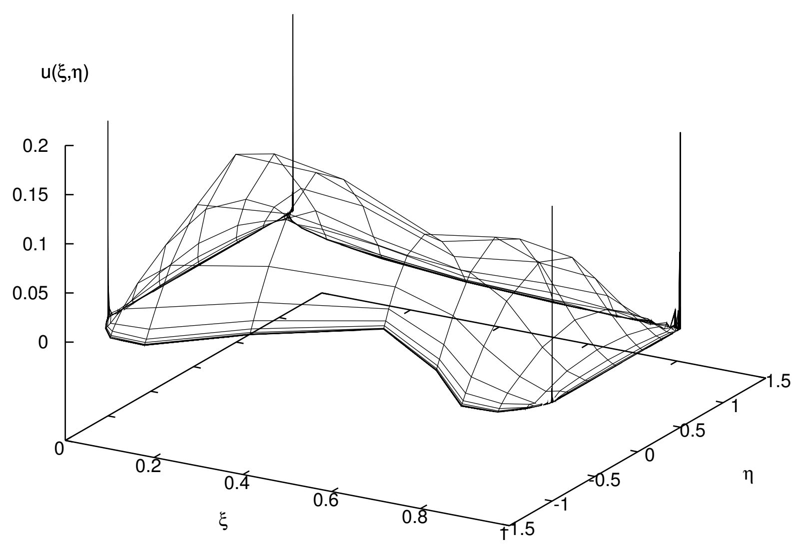

FIG. 4.7. *Sinc-Galerkin solution of the example with* $M_x = 20$ *and* $h = 4/\sqrt{M_x}$.
Max Error=2.969×10^7

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