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WEIERSTRASS' THEOREM IN WEIGHTED SOBOLEV SPACES WITH K DERIVATIVES: ANNOUNCEMENT OF RESULTS*

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Abstract. We characterize the set of functions which can be approximated by smooth functions and by polynomials with the norm

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty(w_j)},$$

for a wide range of (even non-bounded) weights w_j's. We allow a great deal of independence among the weights w_j's.

Key words. Weierstrass' theorem, weight, Sobolev spaces, weighted Sobolev spaces

AMS subject classifications. 41A10, 46E35, 46G10

1. Introduction. If I is any compact interval, Weierstrass' Theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere.

In [29] and [24] we study the same problem with the norm $L^\infty(w)$ defined by

$$\|f\|_{L^\infty(w)} := \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|w(x)\,,$$

where w is a (bounded or unbounded) weight, i.e. a non-negative measurable function.

Considering weighted norms $L^\infty(w)$, has been proved to be interesting mainly because of two reasons: on the one hand, it allows to wider the set of approximable functions (since the functions in $L^\infty(w)$ can have singularities where the weight tends to zero); and, on the other hand , it is possible to find functions which approximate f whose qualitative behaviour is similar to the one of f at those points where the weight tends to infinity.

If $w = (w_0, \dots, w_k)$ is a vectorial weight, we study this approximation problem with the Sobolev norm $W^{k,\infty}(w)$ defined by

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^{k} \left\|f^{(j)}\right\|_{L^\infty(w_j)}.$$

The papers [27], [28], [29], [30], [31], [1], [32], [24] and [25] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \le p \le \infty$. This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [19], [20], [28] and [30]). The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [19]).

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2. Main results. The fundamental results of this paper guarantee that a function f belongs to the closure of the space of polynomials (respectively, smooth functions) in the norm $W^{k,\infty}(w)$,if and only if, $f^{(j)}$ belongs to the closure of polynomials (respectively, smooth functions) in the norm $L^\infty(w_j)$, for every $0 \le j \le k$.

This article is an abridged version of the paper [26].

THEOREM 2.1. *Let be a vectorial weight $w = (w_0, \dots, w_k)$ on $[\alpha, \beta]$ satisfying:*

(i) $\int_\alpha^\beta 1/w_k < \infty$.

(ii) $w_j \in L^\infty_{loc}([\alpha,\beta] \setminus \{a_1^j, \dots, a_{m_j}^j\})$, *for every* $0 \le j < k$.

(iii) $w_j(x) \left| \int_{a_i^j}^x 1/(1+w_{j+1}) \right| \le c$, *a.e. in some neighborhood of* a_i^j, *for every* $1 \le i \le m_j$, $0 \le j \le k-2$, *and* $w_{k-1}(x) \left| \int_{a_i^{k-1}}^x 1/w_k \right| \le c$, *a.e. in some neighborhood of* a_i^{k-1}, *for every* $1 \le i \le m_{k-1}$.

Then the closure of the space of polynomials in $W^{k,\infty}(w)$ is

$$H_1 := \left\{ f \in W^{k,\infty}(w) : \, f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(w_k)}^{L^\infty(w_k)} \right\} .$$

REMARK 1.

(i) *We observe that this theorem characterizes the closure of $\mathbb{P} \cap W^{k,\infty}(w)$ in $W^{k,\infty}(w)$, in terms of the similar problem in $L^\infty(w_k)$. This question of approximation in $L^\infty(w_k)$ is solved in [24].*

(ii) *The hypothesis* (ii) *is not restrictive at all, since if* $\operatorname{ess\,lim\,sup}_{x\to a} w_j(x) = \infty$ *for an infinite number of points $a \in \mathbb{R}$, for some $0 \le j < k$, then 0 is the only polynomial in $L^\infty(w_j)$, and it is trivial to find the closure of the space of polynomials in $W^{k,\infty}(w)$.*

(iii) *Notice that hypothesis,* $w_j(x) \left| \int_{a_i^j}^x 1/(1+w_{j+1}) \right| \le c$, *is much weaker than* $w_j(x) \left| \int_{a_i^j}^x 1/w_{j+1} \right| \le c$, *since some w_{j+1} are allowed to be 0.*

(iv) *The possibility of some w_j to be bounded is, naturally, allowed. That is to say, $\{a_1^j, \dots, a_{m_j}^j\}$ might be the empty set.*

Sketch of the proof. It is obvious that the closure of the space of polynomials in $W^{k,\infty}(w)$ is contained in H_1.

Then, it suffices to prove that every function in H_1 can be approximated by polynomials in the norm $W^{k,\infty}(w)$. Let us consider then, $f \in H_1$ and $\{p_n\}_n$ a sequence of polynomials converging to $f^{(k)}$ in the norm $L^\infty(w_k)$. From the sequence $\{p_n\}_n$ we will construct another one of polynomials converging to f in the norm $W^{k,\infty}(w)$.

The key idea in order to carry out such a process, is to find, from p_n, a polynomial $q_{n,k}$ in M, where M is the space of polynomials which have a primitive of order k in $W^{k,\infty}(w)$. If $\mathbb{P}$ were a Hilbert space and M a closed subspace, it would suffice to take as $q_{n,k}$ the orthogonal projection of p_n on M. However, since our norms do not come from an inner product, the problem is much more complicated; fortunately, we can find a finite set of polynomials B in $L^\infty(w_k)$, such that $q_{n,k}$ can be expressed as a linear combination of p_n and elements of B.

DEFINITION 2.2. *We say that a vectorial weight $w = (w_0, \dots, w_k)$ in $[a,b]$ is of* type 1 *if $1/w_k \in L^1([a,b])$ and $w_0, \dots, w_{k-1} \in L^\infty([a,b])$.*

In the following theorems we describe the closure of smooth functions in Sobolev spaces with weights.

THEOREM 2.3. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ of type 1 in a compact interval $I = [a,b]$. Then the closure of $\mathbb{P} \cap W^{k,\infty}(I,w)$, $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I,w)$ and $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ are, respectively,*

$$H_1 := \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\},$$
$$H_2 := \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{C^\infty(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\},$$
$$H_3 := \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{C(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}.$$

REMARK 2.

(i) *We observe that Theorem 2.3 characterizes the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$, $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I,w)$ and $\mathbb{P} \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$, in terms of the similar problem in $L^\infty(I,w_k)$. This question of approximation in $L^\infty(I,w_k)$ is solved in [24].*

(ii) *If $w_k \in L^\infty(I)$, then the closure of $C^k(\mathbb{R})$, $\mathbb{P}$ and $C^\infty(\mathbb{R})$ are the same. This is a consequence of Bernstein's proof of Weierstrass' Theorem (see e.g. [5, p. 113]), which gives a sequence of polynomials converging uniformly up to the k-th derivative for any function in $C^k(I)$.*

Sketch of the proof. The inclusion

$$\overline{C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)}^{W^{k,\infty}(I,w)} \subseteq H_3$$

is obvious. Let us now consider a function $f \in H_3$, and let $g \in C(\mathbb{R})$, be a function which approximates $f^{(k)}$ in $L^\infty(I,w_k)$ norm. If we consider the function

$$h(x) := \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{k-1}}{(k-1)!} \, dt \,,$$

it is possible to show that h approximates f in $W^{k,\infty}(I,w)$ norm.

DEFINITION 2.4. *We say that u, v, are* comparable *functions in the set A if there exists a positive constant c such that $c^{-1}u \le v \le cu$ a.e. in A.*

DEFINITION 2.5. *We say that a vectorial weight $w = (w_0, \dots, w_k)$ in $[a,b]$ is of* type 2 *if there exist real numbers $a \le a_1 < a_2 < a_3 < a_4 \le b$ such that*

(i) *$1/w_k \in L^1([a_1,a_4])$, and $w_0, \dots, w_{k-1} \in L^\infty([a,b])$,*

(ii) *if $a < a_1$, then w_j is comparable to a finite non-decreasing weight in $[a,a_2]$, for $0 \le j \le k$,*

(iii) *if $a_4 < b$, then w_j is comparable to a finite non-increasing weight in $[a_3,b]$, for $0 \le j \le k$.*

THEOREM 2.6. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ of type 2 in a compact interval $I = [a,b]$. Then the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ is*

$$H_4 := \left\{ f \in W^{k,\infty}(I,w) : \, f^{(j)} \in \overline{C(I) \cap L^\infty(I,w_j)}^{L^\infty(I,w_j)} \text{ for } 0 \le j \le k \right\}.$$

Sketch of the proof. Given a function $f \in H_4$, we can split it as a sum of three functions by using an appropriate partition of unity. The function in $[a_1,a_4]$ can be approximated with similar arguments than the measures of type 1. In the approximation of the functions in $[a,a_2]$ and $[a_3,b]$ we use shift arguments that allow us to construct a convolution with an approximation of the identity.

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The next theorem makes the results of this paper more valuable since it allows to deal with weights which can be obtained by "gluing" simpler ones.

THEOREM 2.7. *Let us consider strictly increasing sequences of real numbers $\{a_n\}$, $\{b_n\}$ (n belonging to a finite set, to $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$) with $b_{n-1} < a_{n+1} < b_n$ for every n. Let $w = (w_0, \dots, w_k)$ be a vectorial weight in the interval $I := \cup_n [a_n, b_n]$. Assume that for each n there exists an interval $I_n \subset [a_{n+1}, b_n]$ with $w_1, \dots, w_k \in L^\infty(I_n)$. Let us assume also that for each n we have either w is of type 1 in $[a_n, b_n]$, or $1/w_k \in L^\infty([a_n, b_n])$. Then the closure of $C^k(I) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ is*

$$H_3 := \left\{ f \in W^{k,\infty}(I,w) :\ f^{(k)} \in \overline{C(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\} .$$

We can deduce the following consequence.

THEOREM 2.8. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ in the interval I, with $w \in L^\infty_{loc}(I)$ and $1/w_k \in L^1_{loc}(I)$. Then the closure of $C^k(I) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ is*

$$H_3 := \left\{ f \in W^{k,\infty}(I,w) :\ f^{(k)} \in \overline{C(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\} .$$

Proof. This theorem is a direct consequence of Theorems 2.3 and 2.7. It is enough to split I as a union of compact intervals $[a_n, b_n]$ (n belonging to a finite set, to $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$), with $b_{n-1} < a_{n+1} < b_n$ for every n. We have that w is of type 1 in each $[a_n, b_n]$, since $w \in L^\infty([a_n, b_n])$ and $1/w_k \in L^1([a_n, b_n])$ for every n. If we choose $I_n := [a_{n+1}, b_n]$, then we can apply Theorems 2.3 and 2.7. $\square$

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