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ETNA
Kent State University
etna@mcs.kent.edu

SYSTEMS OF ORTHOGONAL POLYNOMIALS DEFINED BY HYPERGEOMETRIC TYPE EQUATIONS*

NICOLAE COTFAS†

Abstract. A hypergeometric type equation satisfying certain conditions defines either a finite or an infinite system of orthogonal polynomials. We present in a unified and explicit way all these systems of orthogonal polynomials, the associated special functions and the corresponding raising/lowering operators. This general formalism allows us to extend some known results to a larger class of functions.

Key words. orthogonal polynomials, associated special functions, raising operator, lowering operator, special functions

AMS subject classifications. 33C45, 81R05, 81R30

1. Introduction.

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{1.1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [14], and each can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0$$

by choosing a function ϱ such that $[\sigma(s)\varrho(s)]' = \tau(s)\varrho(s)$.

The equation (1.1) is usually considered on an interval (a, b), chosen such that

$$\begin{array}{rl} \sigma(s) > 0 & \text{for all} \quad s \in (a,b) \\ \varrho(s) > 0 & \text{for all} \quad s \in (a,b) \\ \lim_{s\to a} \sigma(s)\varrho(s) = \lim_{s\to b} \sigma(s)\varrho(s) = 0. & \end{array}$$

Since the form of the equation (1.1) is invariant under a change of variable $s \mapsto cs + d$, it is sufficient to analyse the cases presented in table 1.1. Some restrictions must be imposed on α, β in order for the interval (a, b) to exist.

The equation (1.1) defines either a finite or an infinite system of orthogonal polynomials depending on the set $\{\gamma \in \mathbb{R} \mid \lim_{s\to a} \sigma(s)\varrho(s)s^\gamma = \lim_{s\to b} \sigma(s)\varrho(s)s^\gamma = 0\}$. A unified view on all the systems of orthogonal polynomials defined by (1.1) was presented in [8]. We think that certain results known in particular cases can be extended to a larger class of functions by using this general formalism, and our aim is to present some attempts in this direction.

The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. Since the expression of the raising/lowering operators depends directly on the normalizing condition we use, a unified approach is not possible without a unified definition for the associated special functions. Our results are based on a definition presented in section 2. The table 1.1 allows one to pass in each case from our parameters α, β to the parameters used in different approach. For classical polynomials we use the definitions from [14].

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†Faculty of Physics, University of Bucharest, PO Box 76-54, Post Office 76, Bucharest, Romania (ncotfas@yahoo.com).

ETNA
Kent State University
etna@mcs.kent.edu

TABLE 1.1
The main particular cases

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	α,β	(a,b)
1	$\alpha s+\beta$	$\mathrm{e}^{\alpha s^2/2+\beta s}$	$\alpha<0$	$(-\infty,\infty)$
s	$\alpha s+\beta$	$s^{\beta-1}\mathrm{e}^{\alpha s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
$1-s^2$	$\alpha s+\beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$\alpha<\beta<-\alpha$	$(-1,1)$
s^2-1	$\alpha s+\beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$-\beta<\alpha<0$	$(1,\infty)$
s^2	$\alpha s+\beta$	$s^{\alpha-2}\mathrm{e}^{-\beta/s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
s^2+1	$\alpha s+\beta$	$(1+s^2)^{\alpha/2-1}\mathrm{e}^{\beta\arctan s}$	$\alpha<0$	$(-\infty,\infty)$

2. Orthogonal polynomials and associated special functions. In this section we review certain results concerning the systems of orthogonal polynomials defined by equation (1.1) and the corresponding associated special functions. It is well-known [14] that for $\lambda=\lambda_l$, where

$$\lambda_l=-\frac{\sigma''(s)}{2}l(l-1)-\tau'(s)l \qquad l\in\mathbb{N}$$

the equation (1.1) admits a polynomial solution $\Psi_l=\Psi_l^{(\alpha,\beta)}$ of at most l degree

$$\sigma(s)\Psi_l''+\tau(s)\Psi_l'+\lambda_l\Psi_l=0. \tag{2.1}$$

If the degree of the polynomial Ψ_l is l then it satisfies the Rodrigues formula

$$\Psi_l(s)=\frac{B_l}{\varrho(s)}\frac{\mathrm{d}^l}{\mathrm{d}s^l}[\sigma^l(s)\varrho(s)]$$

where B_l is a constant. We do not impose any normalizing condition. Each polynomial Ψ_l is defined only up to a multiplicative constant. One can remark that

$$\lim_{s\to a}\sigma(s)\varrho(s)s^\gamma=\lim_{s\to b}\sigma(s)\varrho(s)s^\gamma=0 \qquad \text{for} \quad \gamma\in[0,\infty)$$

in the case $\sigma(s)\in\{1,\ s,\ 1-s^2\}$, and

$$\lim_{s\to a}\sigma(s)\varrho(s)s^\gamma=\lim_{s\to b}\sigma(s)\varrho(s)s^\gamma=0 \qquad \text{for} \quad \gamma\in[0,-\alpha)$$

in the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$. Let

$$\Lambda=\begin{cases}\infty & \text{for} \quad \sigma(s)\in\{1,\ s,\ 1-s^2\}\\ \frac{1-\alpha}{2} & \text{for} \quad \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}.\end{cases}$$

PROPOSITION 2.1. [14, 8] *a) $\{\Psi_l \mid l<\Lambda\}$ is a system of polynomials orthogonal with weight function $\varrho(s)$ in (a,b).*

b) Ψ_l is a polynomial of degree l for any $l<\Lambda$.

c) The function $\Psi_l(s)\sqrt{\varrho(s)}$ is square integrable on (a,b) for any $l<\Lambda$.

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Kent State University
etna@mcs.kent.edu

d) A three term recurrence relation

$$s\Psi_l(s) = \alpha_l \Psi_{l+1}(s) + \beta_l \Psi_l(s) + \gamma_l \Psi_{l-1}(s)$$

is satisfied for $1 < l+1 < \Lambda$.

e) The zeros of Ψ_l *are simple and lie in the interval* (a,b)*, for any* $l < \Lambda$.

The polynomials $\Psi_l^{(\alpha,\beta)}$ can be expressed in terms of the classical orthogonal polynomials but in certain cases the relation is not very simple.

PROPOSITION 2.2. [8, 9] *Up to a multiplicative constant*

$$\Psi_l^{(\alpha,\beta)}(s) = \begin{cases} H_l\left(\sqrt{\frac{-\alpha}{2}}\, s - \frac{\beta}{\sqrt{-2\alpha}}\right) & \text{if} \quad \sigma(s) = 1 \\ L_l^{\beta-1}(-\alpha s) & \text{if} \quad \sigma(s) = s \\ P_l^{(-(\alpha+\beta)/2-1,\, (-\alpha+\beta)/2-1)}(s) & \text{if} \quad \sigma(s) = 1 - s^2 \\ P_l^{((\alpha-\beta)/2-1,\, (\alpha+\beta)/2-1)}(-s) & \text{if} \quad \sigma(s) = s^2 - 1 \\ \left(\frac{s}{\beta}\right)^l L_l^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{if} \quad \sigma(s) = s^2 \\ \mathrm{i}^l P_l^{((\alpha+\mathrm{i}\beta)/2-1,\, (\alpha-\mathrm{i}\beta)/2-1)}(\mathrm{i}s) & \text{if} \quad \sigma(s) = s^2 + 1 \end{cases}$$

where H_n*,* L_n^p *and* $P_n^{(p,q)}$ *are the Hermite, Laguerre and Jacobi polynomials, respectively.*

Let $l \in \mathbb{N}$, $l < \Lambda$, and let $m \in \{0, 1, ..., l\}$. By differentiating the equation (2.1) m times we obtain the equation satisfied by the polynomials $\psi_{l,m} = \frac{\mathrm{d}^m}{\mathrm{d}s^m}\Psi_l$, namely

$$\sigma(s)\psi''_{l,m} + [\tau(s) + m\sigma'(s)]\psi'_{l,m} + (\lambda_l - \lambda_m)\psi_{l,m} = 0. \tag{2.2}$$

This is an equation of hypergeometric type, and we can write it in the self-adjoint form

$$[\sigma(s)\varrho_m(s)\psi'_{l,m}]' + (\lambda_l - \lambda_m)\varrho_m(s)\psi_{l,m} = 0$$

by using the function $\varrho_m(s) = \sigma^m(s)\varrho(s)$.

DEFINITION 2.3. *The functions*

$$\Psi_{l,m}(s) = \kappa^m(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Psi_l(s) \qquad where \quad \kappa(s) = \sqrt{\sigma(s)} \tag{2.3}$$

$l \in \mathbb{N}$*,* $l < \Lambda$ *and* $m \in \{0, 1, ..., l\}$*, are called the* associated special functions.

The equation (2.2) multiplied by $\kappa^m(s)$ can be written as

$$\mathbf{H}_m \Psi_{l,m} = \lambda_l \Psi_{l,m}$$

where $\mathbf{H}_m$ is the differential operator

$$\mathbf{H}_m = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + \frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)}$$

$$+\frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s).$$

PROPOSITION 2.4. [8] *a) For each* $m < \Lambda$*, the functions* $\Psi_{l,m}$ *with* $m \leq l < \Lambda$ *are orthogonal with weight function* $\varrho(s)$ *in* (a,b).

ETNA
Kent State University
etna@mcs.kent.edu

b) $\Psi_{l,m}(s)\sqrt{\varrho(s)}$ *is square integrable on* (a,b) *for* $0\le m\le l<\Lambda$.

c) The three term recurrence relation

$$\Psi_{l,m+1}(s)+\left(\frac{\tau(s)}{\kappa(s)}+2(m-1)\kappa'(s)\right)\Psi_{l,m}(s)+(\lambda_l-\lambda_{m-1})\Psi_{l,m-1}(s)=0 \tag{2.4}$$

is satisfied for any $l<\Lambda$ *and any* $m\in\{1,2,...,l-1\}$. *In addition, we have*

$$\left(\frac{\tau(s)}{\kappa(s)}+2(l-1)\kappa'(s)\right)\Psi_{l,l}(s)+(\lambda_l-\lambda_{l-1})\Psi_{l,l-1}(s)=0. \tag{2.5}$$

For any $l\in\mathbb{N}$, $l<\Lambda$ and any $m\in\{0,1,...,l-1\}$, by differentiating (2.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_{l,m}(s)=m\kappa^{m-1}(s)\kappa'(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Psi_l+\kappa^m(s)\frac{\mathrm{d}^{m+1}}{\mathrm{d}s^{m+1}}\Psi_l(s)$$

that is, the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_{l,m}(s)=m\frac{\kappa'(s)}{\kappa(s)}\Psi_{l,m}(s)+\frac{1}{\kappa(s)}\Psi_{l,m+1}(s)$$

which can be written as

$$\left(\kappa(s)\frac{d}{ds}-m\kappa'(s)\right)\Psi_{l,m}(s)=\Psi_{l,m+1}(s). \tag{2.6}$$

If $m\in\{1,2,...,l-1\}$ then by substituting (2.6) into (2.4) we get

$$\left(\kappa(s)\frac{d}{ds}+\frac{\tau(s)}{\kappa(s)}+(m-2)\kappa'(s)\right)\Psi_{l,m}(s)+(\lambda_l-\lambda_{m-1})\Psi_{l,m-1}(s)=0$$

that is,

$$\left(-\kappa(s)\frac{d}{ds}-\frac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s)\right)\Psi_{l,m+1}(s)=(\lambda_l-\lambda_m)\Psi_{l,m}(s). \tag{2.7}$$

for all $m\in\{0,1,...,l-2\}$. From (2.5) it follows that this relation is also satisfied for $m=l-1$.

The relations (2.6) and (2.7) suggest we should consider the first order differential operators [11, 7, 9]

$$A_m=\kappa(s)\frac{d}{ds}-m\kappa'(s)\qquad A_m^+=-\kappa(s)\frac{d}{ds}-\frac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s)$$

for $m+1<\Lambda$.

PROPOSITION 2.5. [10, 6, 11, 8] *We have*

$$A_m\Psi_{l,m}=\Psi_{l,m+1}\qquad A_m^+\Psi_{l,m+1}=(\lambda_l-\lambda_m)\Psi_{l,m}\qquad for\ \ 0\le m<l<\Lambda. \tag{2.8}$$

$$\Psi_{l,m}=\frac{A_m^+}{\lambda_l-\lambda_m}\frac{A_{m+1}^+}{\lambda_l-\lambda_{m+1}}...\frac{A_{l-1}^+}{\lambda_l-\lambda_{l-1}}\Psi_{l,l}\qquad for\ \ 0\le m<l<\Lambda. \tag{2.9}$$

$$||\Psi_{l,m+1}||=\sqrt{\lambda_l-\lambda_m}\,||\Psi_{l,m}||\qquad for\quad 0\le m<l<\Lambda. \tag{2.10}$$

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Kent State University
etna@mcs.kent.edu

$$\mathbf{H}_m - \lambda_m = A_m^+ A_m \qquad \mathbf{H}_{m+1} - \lambda_m = A_m A_m^+ \qquad for \quad m+1 < \Lambda$$

$$\mathbf{H}_m A_m^+ = A_m^+ \mathbf{H}_{m+1} \qquad A_m \mathbf{H}_m = \mathbf{H}_{m+1} A_m \qquad for \quad m+1 < \Lambda.$$

From (2.8), (2.9) and (2.10) it follows that the *normalized associated special functions* $\tilde{\Psi}_{l,m} = \Psi_{l,m}/||\Psi_{l,m}||$ satisfy the relations

$$(2.11) \qquad \begin{array}{c} A_m \tilde{\Psi}_{l,m} = \sqrt{\lambda_l - \lambda_m}\,\tilde{\Psi}_{l,m+1} \qquad A_m^+ \tilde{\Psi}_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\,\tilde{\Psi}_{l,m} \\ \tilde{\Psi}_{l,m} = \frac{A_m^+}{\sqrt{\lambda_l - \lambda_m}} \frac{A_{m+1}^+}{\sqrt{\lambda_l - \lambda_{m+1}}} \cdots \frac{A_{l-1}^+}{\sqrt{\lambda_l - \lambda_{l-1}}} \tilde{\Psi}_{l,l}. \end{array}$$

3. A group theoretical approach based on projection method. The system of functions $\tilde{\Psi}_{l,m}$ is the projection of the system of functions [1, 2, 13]

$$(3.1) \qquad |l,m) : (a,b) \times [-\pi,\pi] \longrightarrow \mathbb{C} \qquad |l,m) = \mathrm{e}^{\mathrm{i}m\varphi} \tilde{\Phi}_{l,m}$$

orthogonal with respect to the scalar product

$$\langle F, G\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \int_a^b \overline{F(s,\varphi)}\, G(s,\varphi)\, \varrho(s) \mathrm{d}s \mathrm{d}\varphi.$$

More exactly, we can identify each function $\tilde{\Psi}_{l,m}$ with the restriction of $|l,m)$ to the subset $(a,b) \times \{0\}$. By using the relation $\frac{\partial}{\partial\varphi}|l,m) = \mathrm{i}m\,|l,m)$ obtained directly from definition (3.1), and (2.11) we get

$$\begin{array}{l} \mathrm{e}^{\mathrm{i}\varphi}\left(\kappa \frac{\partial}{\partial s} + \mathrm{i}\kappa' \frac{\partial}{\partial\varphi}\right) |l,m) = \sqrt{\lambda_l - \lambda_m}\, |l,m+1) \\ \mathrm{e}^{-\mathrm{i}\varphi}\left(-\kappa \frac{\partial}{\partial s} + \mathrm{i}\kappa' \frac{\partial}{\partial\varphi} - \frac{\tau}{\kappa} + 2\kappa'\right) |l,m+1) = \sqrt{\lambda_l - \lambda_m}\, |l,m). \end{array}$$

These relations suggest we should consider the first order differential operators

$$(3.2) \qquad \begin{array}{l} L_+ = \mathrm{e}^{\mathrm{i}\varphi}\left(\kappa \frac{\partial}{\partial s} + \mathrm{i}\kappa' \frac{\partial}{\partial\varphi}\right) \\ L_- = \mathrm{e}^{-\mathrm{i}\varphi}\left(-\kappa \frac{\partial}{\partial s} + \mathrm{i}\kappa' \frac{\partial}{\partial\varphi} - \frac{\tau}{\kappa} + 2\kappa'\right) \\ L_0 = -\mathrm{i}\frac{\partial}{\partial\varphi} \end{array}$$

satisfying the relations

$$\begin{array}{l} L_+|l,m) = \sqrt{\lambda_l - \lambda_m}\, |l,m+1) \\ L_-|l,m) = \sqrt{\lambda_l - \lambda_{m-1}}\, |l,m-1) \\ L_0|l,m) = m|l,m). \end{array}$$

One can remark that $L_+|l,l\rangle = 0$ and

$$|l,m) = \frac{1}{\sqrt{\lambda_l - \lambda_m}} \frac{1}{\sqrt{\lambda_l - \lambda_{m+1}}} \cdots \frac{1}{\sqrt{\lambda_l - \lambda_{l-1}}} (L_-)^{l-m} |l,l)$$

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Kent State University
etna@mcs.kent.edu

for all $m\in\{0,1,2,...,l-1\}$, but, generally, $L_-|l,0)\neq 0$. For example, in the case of Legendre polynomials $\kappa(s)=\sqrt{1-s^2}$, $\tau(s)=-2s$ and

$$L_+=\mathrm{e}^{\mathrm{i}\varphi}\left(\kappa\frac{\partial}{\partial s}+\mathrm{i}\kappa'\frac{\partial}{\partial\varphi}\right)$$
$$L_-=\mathrm{e}^{-\mathrm{i}\varphi}\left(-\kappa\frac{\partial}{\partial s}+\mathrm{i}\kappa'\frac{\partial}{\partial\varphi}\right)=-\overline{\mathrm{e}^{\mathrm{i}\varphi}\left(\kappa\frac{\partial}{\partial s}+\mathrm{i}\kappa'\frac{\partial}{\partial\varphi}\right)}=-\overline{L_+}$$

whence

$$(L_-)^m|l,0)=(-1)^m\mathrm{e}^{-\mathrm{i}m\varphi}\Psi_{l,m}=(-1)^m\overline{|l,m)}\qquad\text{for all }\ m\in\{1,2,...,l\}$$

and $(L_-)^{l+1}|l,0)=0$. The $(2l+1)$-dimensional vector space spannied by the set $\{\ (L_-)^q|l,l)\ |\ q\in\{0,1,2,...,2l\}\ \}$ is invariant under the action of L_+, L_- and L_0.

The operators defined by (3.2) satisfy the relations $[L_0,L_\pm]=\pm L_\pm$ and

$$\begin{aligned}[L_+,L_-]&=(-\tau'+2\kappa\kappa''+2\kappa'^2)\mathbb{I}+\mathrm{i}(2\kappa\kappa''+2\kappa'^2)\frac{\partial}{\partial\varphi}\\&=\begin{cases}-\alpha\mathbb{I} & \text{for}\quad \sigma(s)\in\{1,\ s\}\\ 2\left(L_0-\frac{\alpha+2}{2}\mathbb{I}\right) & \text{for}\quad \sigma(s)=1-s^2\\ -2\left(L_0+\frac{\alpha-2}{2}\mathbb{I}\right) & \text{for}\quad \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{cases}\end{aligned}$$

where $\mathbb{I}$ is the identity operator. The Lie algebra $\mathcal{L}$ generated by L_+ and L_- is finite dimensional.

THEOREM 3.1.

$$\mathcal{L}\ \ is\ \ isomorphic\ \ to\begin{cases}Heisenberg\,algebra\ h(2) & if\quad \sigma(s)\in\{1,\ s\}\\ su(2) & if\quad \sigma(s)=1-s^2\\ su(1,1) & if\quad \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{cases}$$

Proof. If $\sigma(s)\in\{1,\ s\}$ then the operators $K_+=\sqrt{-1/\alpha}\,L_+$ and $K_-=-\sqrt{-1/\alpha}\,L_-$ satisfy the relations $[K_+,K_-]=-\mathbb{I}$ and $[\mathbb{I},K_\pm]=0$.
In the case $\sigma(s)=1-s^2$ the operators $K_+=L_+$, $K_-=L_-$ and $K_0=L_0-\frac{\alpha+2}{2}\mathbb{I}$ satisfy the relations $[K_+,K_-]=2K_0$ and $[K_0,K_\pm]=\pm K_\pm$.
If $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$ the operators $K_+=L_+$, $K_-=L_-$ and $K_0=L_0+\frac{\alpha-2}{2}\mathbb{I}$ satisfy the relations $[K_+,K_-]=-2K_0$ and $[K_0,K_\pm]=\pm K_\pm$. $\square$

In the case $\sigma(s)=1-s^2$, the functions $|l,m)$ satisfy the relations

$$\begin{aligned}&K_0|l,m)=(\Phi+m-l)\,|l,m) && \text{for}\quad m\in\{0,1,...,l\}\\&K_+|l,m)=\sqrt{(l-m)(l+m-\alpha-1)}\,|l,m+1) && \text{for}\quad m\in\{0,1,...,l-1\}\\&K_-|l,m)=\sqrt{(l-m+1)(l+m-\alpha-2)}\,|l,m-1) && \text{for}\quad m\in\{1,2,...,l\}\\&C|l,m)=\Phi(\Phi+1)\,|l,m) && \text{for}\quad m\in\{0,1,...,l\}\end{aligned}$$

where $C=K_-K_++K_0(K_0+\mathbb{I})$ is the Casimir operator of $su(2)$ and $\Phi=l-\frac{\alpha}{2}-1$.
In the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$, the functions $|l,m)$ satisfy the relations

$$\begin{aligned}&K_0|l,m)=(\Phi+m-l)\,|l,m) && \text{for}\quad m\in\{0,1,...,l\}\\&K_+|l,m)=\sqrt{(m-l)(m+l+\alpha-1)}\,|l,m+1) && \text{for}\quad m\in\{0,1,...,l-1\}\\&K_-|l,m)=\sqrt{(m-l-1)(m+l+\alpha-2)}\,|l,m-1) && \text{for}\quad m\in\{1,2,...,l\}\\&C|l,m)=-\Phi(\Phi+1)\,|l,m) && \text{for}\quad m\in\{0,1,...,l\}\end{aligned}$$

where $C = K_-K_+ - K_0(K_0 + \mathbb{I})$ is the Casimir operator of $su(1,1)$ and $\Phi = l + \frac{\alpha}{2} - 1$. The Casimir operator of $h(2)$ is the identity operator $\mathbb{I}$ belonging to the algebra [5].

Our approach is different from the one presented in [12] based on $gl(2,c)$ and Miller algebra h_4. Generally, our algebra $\mathcal{L}$ does not contain L_0.

4. Some systems of coherent states. In this section we restrict us to the case $\sigma(s) \in \{1, s, 1-s^2\}$. For each $m \in \mathbb{N}$, the sequence $\tilde{\Psi}_{m,m}$, $\tilde{\Psi}_{m+1,m}$, $\tilde{\Psi}_{m+2,m}$, ... is an orthonormal basis in the Hilbert space

$$\mathcal{H} = \left\{ \psi : (a,b) \longrightarrow \mathbb{C} \;\middle|\; \int_a^b |\psi(s)|^2 \varrho(s)ds < \infty \right\} \qquad \langle \psi_1 | \psi_2 \rangle = \int_a^b \overline{\psi_1(s)}\, \psi_2(s)\, \varrho(s)ds\,.$$

The linear operator defined by (see figure 4.1)

$$U_m : \mathcal{H} \longrightarrow \mathcal{H}, \quad U_m \tilde{\Psi}_{l,m} = \tilde{\Psi}_{l+1,m+1}$$

is a unitary operator, the operators $a_m = U_m^+ A_m$, $a_m^+ = A_m^+ U_m$ are mutually adjoint, and

$$\begin{aligned} a_m \tilde{\Psi}_{l,m} &= \sqrt{\lambda_l - \lambda_m}\, \tilde{\Psi}_{l-1,m} && \text{for} \quad l \geq m+1 \\ a_m^+ \tilde{\Psi}_{l,m} &= \sqrt{\lambda_{l+1} - \lambda_m}\, \tilde{\Psi}_{l+1,m} && \text{for} \quad l \geq m \\ \tilde{\Psi}_{l,m} &= \frac{(a_m^+)^{l-m}}{\sqrt{(\lambda_l - \lambda_m)(\lambda_{l-1} - \lambda_m)\ldots(\lambda_{m+1} - \lambda_m)}} \tilde{\Psi}_{m,m} && \text{for} \quad l > m. \end{aligned} \tag{4.1}$$

Since

$$a_m a_m^+ \tilde{\Psi}_{l,m} = (\lambda_{l+1} - \lambda_m)\tilde{\Psi}_{l,m} \qquad a_m^+ a_m \tilde{\Psi}_{l,m} = (\lambda_l - \lambda_m)\tilde{\Psi}_{l,m}$$

we get the factorization $\mathbf{H}_m - \lambda_m = a_m^+ a_m$ and the relation

$$[a_m, a_m^+]\tilde{\Psi}_{l,m} = (\lambda_{l+1} - \lambda_l)\tilde{\Psi}_{l,m}. \tag{4.2}$$

By using the operator

$$R_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m \qquad R_m \tilde{\Psi}_{l,m} = \frac{-\sigma'' l - \alpha}{2} \tilde{\Psi}_{l,m}$$

the relation (4.2) can be written as $[a_m^+, a_m] = -2R_m$. Since

$$[R_m, a_m^+] = -\frac{\sigma''}{2} a_m^+ \qquad [R_m, a_m] = \frac{\sigma''}{2} a_m$$

it follows that the Lie algebra $\mathcal{L}_m$ generated by $\{a_m^+, a_m\}$ is finite dimensional.

THEOREM 4.1.

$$\mathcal{L}_m \textit{ is isomorphic to } \begin{cases} \textit{Heisenberg algebra } h(2) & \textit{for} \quad \sigma(s) \in \{1,\ s\} \\ su(1,1) & \textit{for} \quad \sigma(s) = 1 - s^2 \end{cases}$$

Proof. In the case $\sigma(s) \in \{1, s\}$ the operator R_m is a constant operator, namely, $R_m = -\alpha$. Since $\alpha < 0$, the operators $P_+ = \sqrt{-1/\alpha}\, a_m^+$, $P_- = \sqrt{-1/\alpha}\, a_m$ and $\mathbb{I}$ form a basis of $\mathcal{L}_m$ such that

$$[P_+, P_-] = -\mathbb{I} \qquad [\mathbb{I}, P_\pm] = 0$$

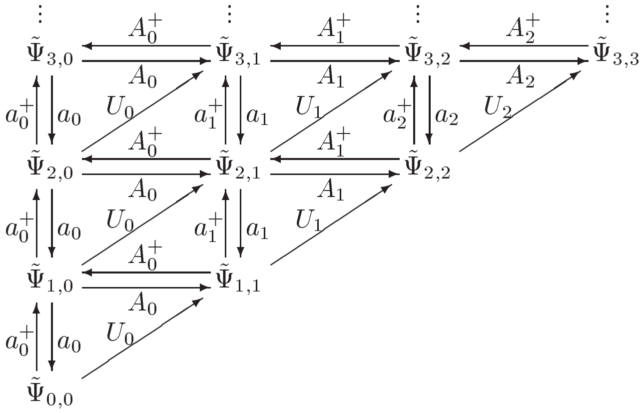

FIG. 4.1. *The operators A_m, A_m^+, a_m, a_m^+ and U_m relating the functions $\tilde\Psi_{l,m}$.*

that is, $\mathcal{L}_m$ is isomorphic to the Heisenberg-Weyl algebra $h(2)$.
If $\sigma(s)=1-s^2$ then $K_+=a_m^+$, $K_-=a_m$ and $K_0=R_m$ form a basis of $\mathcal{L}_m$ such that

$$[K_+,K_-]=-2K_0 \qquad [K_0,K_\pm]=\pm K_\pm. \qquad \square$$

In the case $\sigma(s)=1-s^2$, the functions $\tilde\Psi_{m,m}$, $\tilde\Psi_{m+1,m}$, $\tilde\Psi_{m+2,m}$, ..., satisfy the relations

$$\begin{aligned}
&K_0\tilde\Psi_{l,m}=\left(l-\tfrac{\alpha}{2}\right)\tilde\Psi_{l,m}\\
&K_+\tilde\Psi_{l,m}=\sqrt{(l-m+1)(l+m-\alpha)}\,\tilde\Psi_{l+1,m}\\
&K_-\tilde\Psi_{l,m}=\sqrt{(l-m)(l+m-1-\alpha)}\,\tilde\Psi_{l-1,m}\\
&C\tilde\Psi_{l,m}=-\left(\tfrac{\alpha}{2}-m\right)\left(\tfrac{\alpha}{2}-m+1\right)\tilde\Psi_{l,m}
\end{aligned}$$

where $C=K_-K_+-K_0(K_0+\mathbb{I})$ is the Casimir operator of $su(1,1)$. If we denote

$$E_0=m-\frac{\alpha}{2}=-\Phi \qquad |\Phi,n\rangle=\tilde\Psi_{m+n,m}$$

then the above relations can be written as

$$\begin{aligned}
&K_0|\Phi,n\rangle=(E_0+n)\,|\Phi,n\rangle\\
&K_+|\Phi,n\rangle=\sqrt{(\Phi+E_0+n+1)(E_0-\Phi+n)}\,|\Phi,n+1\rangle\\
&K_-|\Phi,n\rangle=\sqrt{(\Phi+E_0+n)(E_0-\Phi+n-1)}\,|\Phi,n-1\rangle\\
&C|\Phi,n\rangle=-\Phi(\Phi+1)\,|\Phi,n\rangle.
\end{aligned}$$

and show that [3, 15], in case $\sigma(s)=1-s^2$, the representation of $su(1,1)$ defined by (4.1) in $\mathcal{H}$ is the irreducible discrete representation $D^+\left(\frac{\alpha}{2}-m\right)$.

Let $m\in\mathbb{N}$ be a fixed natural number. The functions $|0\rangle, |1\rangle, |2\rangle, \cdots$, where

$$|n\rangle=\tilde\Psi_{m+n,m}$$

satisfy the relations

$$\begin{aligned}
&a_m|n\rangle=\sqrt{e_n}\,|n-1\rangle\\
&a_m^+|n\rangle=\sqrt{e_{n+1}}\,|n+1\rangle\\
&(\mathbf{H}_m-\lambda_m)|n\rangle=e_n|n\rangle
\end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

where

$$e_n = \lambda_{m+n} - \lambda_m = \begin{cases} -\alpha n & \text{if} \quad \sigma(s) \in \{1, s\} \\ n(n+2m-\alpha-1) & \text{if} \quad \sigma(s) = 1 - s^2. \end{cases}$$

Some useful systems of coherent states can be defined [3] by using these relations, the confluent hypergeometric function

$${}_0F_1(c;z) = 1 + \frac{1}{c}\frac{z}{1!} + \frac{1}{c(c+1)}\frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots$$

and the modified Bessel function

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \qquad \text{where} \qquad I_\nu(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}.$$

THEOREM 4.2. *a) If $\sigma(s) \in \{1, s\}$ then $\{\,|z\rangle \mid z \in \mathbb{C}\}$, where*

$$|z\rangle = \mathrm{e}^{\frac{|z|^2}{2\alpha}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,(-\alpha)^n}} |n\rangle$$

is a system of coherent states in $\mathcal{H}$ such that

$$\langle z|z\rangle = 1 \qquad a_m|z\rangle = z|z\rangle \qquad \text{and} \qquad \frac{-1}{\pi\alpha}\int_{\mathbb{C}} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)|z\rangle\langle z| = \mathbb{I}.$$

b) If $\sigma(s) = 1 - s^2$ then $\{\,|z\rangle \mid z \in \mathbb{C}\}$, where

$$|z\rangle = \sqrt{\Gamma(2m-\alpha)} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,\Gamma(n+2m-\alpha)}} |n\rangle$$

is a system of coherent states in $\mathcal{H}$ such that

$$\langle z|z\rangle = {}_0F_1(2m-\alpha; |z|^2) \qquad a_m|z\rangle = z|z\rangle \quad \text{and} \quad \int_{\mathbb{C}} d\mu(z)\, |z\rangle\langle z| = \mathbb{I}$$

where

$$d\mu(z) = \frac{2r^{2m-\alpha}}{\pi\Gamma(2m-\alpha)} K_{\frac{\alpha+1}{2}-m}(2r)\, dr\, d\theta \qquad \text{and} \qquad z = r\mathrm{e}^{\mathrm{i}\theta}. \tag{4.3}$$

Proof. [3] By denoting $t = -\frac{r^2}{\alpha}$ and using the integration by parts we get

$$\frac{-1}{\pi\alpha}\int_{\mathbb{C}} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)|z\rangle\langle z| = \frac{-1}{\pi\alpha}\sum_{n,n'}\left(\int_0^\infty \mathrm{e}^{\frac{r^2}{\alpha}} \frac{r^{n+n'+1}}{\sqrt{n!\,n'!\,(-\alpha)^{n+n'}}} dr \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-n')\theta} d\theta\right) |n\rangle\langle n'|$$

$$= \frac{-2}{\alpha}\sum_n \left(\int_0^\infty \mathrm{e}^{\frac{r^2}{\alpha}} \frac{1}{n!}\left(\frac{r^2}{-\alpha}\right)^n r\, dr\right) |n\rangle\langle n| = \sum_n \left(\int_0^\infty \mathrm{e}^{-t}\frac{t^n}{n!}\, dt\right) |n\rangle\langle n| = \sum_n |n\rangle\langle n| = \mathbb{I}.$$

Denoting $d\mu = \mu(r)\, dr\, d\theta$ we get

$$\int_{\mathbb{C}} d\mu(z)\, |z\rangle\langle z| = \sum_{n=0}^{\infty} \frac{2\pi\Gamma(2m-\alpha)}{n!\,\Gamma(n+2m-\alpha)} \left(\int_0^\infty r^{2n}\mu(r)\, dr\right) |n\rangle\langle n|$$

and hence, we must have the relation (Mellin transformation)

$$2\pi\Gamma(2m-\alpha)\int_0^\infty r^{2n}\mu(r)\,dr = \Gamma(n+1)\,\Gamma(n+2m-\alpha). \tag{4.4}$$

The formula [4]

$$\int_0^\infty 2x^{\eta+\xi}K_{\eta-\xi}(2\sqrt{x})\,x^{n-1}dx = \Gamma(2\eta+n)\,\Gamma(2\xi+n)$$

for $x=r^2,\ \eta=\frac{1}{2},\ \xi=m-\frac{\alpha}{2}$ becomes

$$4\int_0^\infty r^{2n}K_{\frac{\alpha+1}{2}-m}(2r)\,r^{2m-\alpha}\,dr = \Gamma(n+1)\Gamma(n+2m-\alpha). \tag{4.5}$$

The relations (4.4) and (4.5) lead to (4.3). ☐

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