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q-ORTHOGONAL POLYNOMIALS RELATED TO THE QUANTUM GROUP $U_q(so(5))$*

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Abstract. Orthogonal polynomials in two discrete variables related to finite-dimensional irreducible representations of the quantum algebra $U_q(\mathfrak{so}(5))$ are studied. The polynomials we consider here can be treated as two-dimensional q-analogs of Krawtchouk polynomials. Some properties of these polynomials are investigated: the difference equation of the Sturm-Liouville type, the weight function, the corresponding eigenvalues including the explicit description of their multiplicities.

Key words. quantum group, discrete orthogonal polynomials, eigenvalues

AMS subject classifications. 33D80, 33C45

1. Introduction. This work is devoted to a description of a type of orthogonal polynomials in two discrete variables related to representations of the quantum group $U_q(\mathfrak{so}(5))$. This group (more exactly, algebra) is a q-deformation of the universal enveloping algebra $U(\mathfrak{so}(5))$ which corresponds to the classical group $SO(5)$ of the rotations in 5-dimensional Euclidian space. The obtained results generalize some of the results on orthogonal polynomials related to representations of the classical group $SO(5)$ [1].

The orthogonal polynomials that we study here are some modifications of eigenfunctions of infinitesimal operators (generators) of representations of the quantum group $U_q(\mathfrak{so}(5))$. Formulas for these generators are generalizations of classical Gelfand-Tsetlin formulas for generators of representations of the group $SO(5)$. Such formulas for the algebra $U_q(\mathfrak{so}(n))$ were constructed in [2]. In the case of the group $SO(5)$, as shown in [1], investigation of the eigenfunctions of generators of representation lead to a class of orthogonal polynomials that may be considered as two-dimensional analogs of classical Krawtchouk polynomials [3]. Accordingly, orthogonal polynomials related to representations of the quantum group $U_q(\mathfrak{so}(5))$ may be treated as two-dimensional q-analogs of the Krawtchouk polynomials.

2. q-analogs of Gel'fand-Tsetlin formulas. These formulas for the representations of the classical type of the algebra $U_q(\mathfrak{so}(n))$ were obtained in [2]. Let's rewrite these formulas for the case $n = 5$. Finite-dimensional irreducible representations of the classical type of the algebra $U_q(\mathfrak{so}(5))$ are given by two integral or half-integral numbers n_1 and n_2, such that $n_1 \geq n_2 \geq 0$. We will consider the case of integers. The q-analog of Gelfand-Tsetlin basis in the representation space corresponds to successive reduction of the representation of $U_q(\mathfrak{so}(5))$ to subalgebras $U_q(\mathfrak{so}(4))$, $U_q(\mathfrak{so}(3))$, and $U_q(\mathfrak{so}(2))$. The basis vectors ξ_α can be enumerated by tableaux $\alpha = (m_1, m_2; l; k)$, where the components of α satisfy the conditions

$$\begin{cases} n_1 \geq m_1 \geq n_2 \geq m_2 \geq -n_2, \\ m_1 \geq l \geq |m_2|,\ l \geq |k|. \end{cases}$$

Let's introduce the following notations

$$\begin{cases} a_1 = n_1 + 2,\ a_2 = n_2 + 1,\ a_3 = l + 1, \\ x = m_1 + 1,\ y = m_2. \end{cases}$$

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Then the new parameters satisfy the conditions

$$\left\{ \begin{array}{l} a_2 \le x \le a_1 - 1 \\ a_3 \le x \end{array} \right. , \left\{ \begin{array}{l} |y| \le a_2 - 1 \\ |y| \le a_3 - 1 \end{array} \right. . \tag{2.1}$$

Using these notations, we denote the basis vectors ξ_α as $\xi_{x,y}^{a_1,a_2,a_3,k}$. Let $I = I_{5,4}$ be a generator in the representation space corresponding to the rotation in the plane (e_5, e_4) in the 5-dimensional space. Then the generator I acts on basis elements $\xi_{x,y}^{a_1,a_2,a_3,k}$ by the formula

$$\begin{aligned} I\xi_{x,y}^{a_1,a_2,a_3,k} &= \frac{1}{q^x + q^{-x}} \left\{ I_{x,y}^{a_1,a_2,a_3} \xi_{x+1,y}^{a_1,a_2,a_3,k} - I_{x-1,y}^{a_1,a_2,a_3} \xi_{x-1,y}^{a_1,a_2,a_3,k} \right\} \\ &+ \frac{1}{q^y + q^{-y}} \left\{ I_{y,x}^{a_1,a_2,a_3} \xi_{x,y+1}^{a_1,a_2,a_3,k} - I_{y-1,x}^{a_1,a_2,a_3} \xi_{x,y-1}^{a_1,a_2,a_3,k} \right\}, \end{aligned} \tag{2.2}$$

were

$$I_{x,y}^{a_1,a_2,a_3} = \left\{ \frac{[a_1+x][a_1-x-1][a_2+x][a_2-x-1][a_3+x][a_3-x-1]}{[x+y][x-y][x+y+1][x-y+1]} \right\}^{1/2}. \tag{2.3}$$

Here square brackets mean q-numbers:

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

The representation space V has the dimension

$$\dim V = \frac{(2a_1 - 1)(2a_2 - 1)(a_1 - a_2)(a_1 + a_2 - 1)}{6}.$$

3. Discrete equation related to the eigenvectors of the operator I. Let's fix the parameters a_1, a_2, a_3, and k such that $1 \le a_2 < a_1$, $1 \le a_3 < a_1$, $|k| \le a_3 - 1$, and consider the subspace $W \subset V$ spanned by the vectors $\xi_{x,y}^{a_1,a_2,a_3,k}$, where the parameters x and y satisfy the inequalities (2.1). It follows from the formulas (2.2) and (2.3) that subspace W is invariant with respect to the operator I and has the dimension

$$\dim W = (a - b + 1)(2c + 1),$$

where

$$\left\{ \begin{array}{l} a = a_1 - 1, \\ b = \max(a_2, a_3), \\ c = \min(a_2, a_3) - 1. \end{array} \right. \tag{3.1}$$

Let $L(\Omega)$ be the space of all complex-valued functions of two discrete variables defined on the lattice

$$\Omega = \{(x, y)| - c \le y \le c < b \le x \le a\}. \tag{3.2}$$

It follows from (2.1) – (3.1) that space W is isomorphic to the space $L(\Omega)$, and operator I acts in the space $L(\Omega)$ by the formula: $\forall f \in L(\Omega)$,

$$
\begin{aligned}
(If)(x,y) &= \tfrac{1}{q^x+q^{-x}}\left\{-A^{a,b,c}_{x,y} f(x+1,y) + A^{a,b,c}_{x-1,y} f(x-1,y)\right\}\\
&+ \tfrac{1}{q^y+q^{-y}}\left\{-A^{a,b,c}_{y,x} f(x,y+1) + A^{a,b,c}_{y-1,x} f(x,y-1)\right\}.
\end{aligned}
$$

where

$$A^{a,b,c}_{x,y} = \left\{\frac{[a+x+1][a-x][b+x][b-x-1][c+x+1][c-x]}{[x+y][x-y][x+y+1][x-y+1]}\right\}^{1/2}.$$

We will consider the problem of diagonalization of the operator I in the space $L(\Omega)$. Let $Q \in L(\Omega)$ be an eigenfunction of the operator I with the eigenvalue σ:

$$(IQ)(x,y) = \sigma Q(x,y), \tag{3.3}$$

Let's make the following substitution in the equation (3.3)

$$Q(x,y) = i^{x+y}\sqrt{\rho(x,y)}P(x,y),$$

where

$$\rho(x,y) = \frac{[x-y][x+y][x+b-1]![b-y-1]![b+y-1]![x-c-1]!}{[a-x]![a+x]![x-b]![a-y]![a+y]![x+c]![y+c]![c-y]!}. \tag{3.4}$$

Here q-factorials are defined as $[k]! = [k][k-1]\ldots[2][1]$.
Then the function $P(x,y)$ satisfies the equation

$$
\begin{aligned}
&\tfrac{[x]}{[2x]}\left\{[a-x][x+b][x-c]P(x+1,y) + [x+a][x-b][x+c]P(x-1,y)\right\}\\
+\;&\tfrac{[y]}{[2y]}\left\{[a-y][y+b][c-y]P(x,y+1) + [y+a][b-y][y+c]P(x,y-1)\right\}\\
=\;&\lambda\left\{[x-y][x+y]\right\}P(x,y), \quad (x,y)\in\Omega, \quad \lambda = \sigma i.
\end{aligned} \tag{3.5}
$$

This equation can also be written in terms of finite differences using the operators

$$(\Delta_x f)(x,y) = f(x+1,y) - f(x,y), \;\; (\nabla_x f)(x,y) = f(x,y) - f(x-1,y),$$

and similarly Δ_y and ∇_y:

$$
\begin{aligned}
&\frac{[x]}{[2x]}\frac{[a+x]![a-x]![x-b]![x+c]!}{[x+b-1]![x-c-1]!}\\
&\times\Delta_x\left\{\frac{[x+b-1]![x-c-1]!}{[a+x-1]![a-x]![x-b-1]![x+c-1]!}\nabla_x P(x,y)\right\}\\
&+\frac{[y]}{[2y]}\frac{[a+y]![a-y]![c+y]![c-y]!}{[b+y-1]![b-y-1]!}\\
&\times\Delta_y\left\{\frac{[b+y-1]![b-y]!}{[a+y-1]![a-y]![c+y-1]![c-y]!}\nabla_y P(x,y)\right\}\\
&= \mu[x-y][x+y]P(x,y), \quad \mu = \lambda - [a-b+c], \;\; (x,y)\in\Omega.
\end{aligned} \tag{3.6}
$$

If we let $q \to 1$ (case of the classical group $SO(5)$), the equation (3.6) becomes

$$
\begin{aligned}
&\frac{(a+x)!(a-x)!(x-b)!(x+c)!}{(x+b-1)!(x-c-1)!}\\
&\times\Delta_x\left\{\frac{(x+b-1)!(x-c-1)!}{(a+x-1)!(a-x)!(x-b-1)!(x+c-1)!}\nabla_x P(x,y)\right\}\\
&+\frac{(a+y)!(a-y)!(c+y)!(c-y)!}{(b+y-1)!(b-y-1)!}\\
&\times\Delta_y\left\{\frac{(b+y-1)!(b-y)!}{(a+y-1)!(a-y)!(c+y-1)!(c-y)!}\nabla_y P(x,y)\right\}\\
&= 2\mu(x^2-y^2)P(x,y).
\end{aligned} \tag{3.7}
$$

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This equation describes eigenvectors of infinitesimal operator of irreducible representations of the group $SO(5)$. As it is shown in [1], equation (3.7) can be considered as two-dimensional analog of the equation for Krawtchouk polynomials. Equation (3.6) can be treated as two-dimensional q-analog of the equation for Krawtchouk polynomials.

Let's make the following substitution in the equation (3.7): $x = h^{-1}x_1$, $y = h^{-1}x_2$, where x_1 and x_2 are new variables, $h > 0$. If we let $h \to 0$, and a, b, $c \to \infty$ such that $ah^2 \to 1$, $bh \to \alpha$, $b - c \to s$, then the discrete equation (3.7) is transformed to the following differential equation

$$
\begin{aligned}
&\frac{1}{x_2^2 - x_1^2} e^{x_1^2}(x_1^2 - \alpha^2)^{-s} \frac{\partial}{\partial x_1}\left\{ e^{-x_1^2}(x_1^2 - \alpha^2)^{s+1} \frac{\partial P(x_1, x_2)}{\partial x_1} \right\} \\
&+ \frac{1}{x_1^2 - x_2^2} e^{x_2^2}(x_2^2 - \alpha^2)^{-s} \frac{\partial}{\partial x_2}\left\{ e^{-x_2^2}(x_2^2 - \alpha^2)^{s+1} \frac{\partial P(x_1, x_2)}{\partial x_2} \right\} = \sigma P(x_1, x_2).
\end{aligned} \tag{3.8}
$$

$$
-\alpha \le x_2 \le \alpha \le x_1 < \infty.
$$

The equation (3.8) can be considered as two-dimensional analog of the equation for Hermite polynomials. The explicit formulas for the complete set of polynomial solutions of the equation (3.8) are given in [1].

4. Spectrum and structure of solutions of the equation (3.5). The following theorem gives the explicit description of the spectrum of the equation (3.5).

THEOREM 4.1. *Equation (3.5) has* $N = 2(a - b + c) + 1$ *distinct eigenvalues of the form*

$$
\lambda = \lambda_n = [a - b + c - n], \quad n = 0, 1, \ldots, N - 1.
$$

The multiplicity $\dim \lambda_n$ *of the values* λ_n *can be described as follows.*

If $c \le a - b$ *then*

$$
\dim \lambda_n = \begin{cases} \langle \frac{n+2}{2} \rangle, & \text{if } 0 \le n \le 2c \\ c + \frac{(-1)^n + 1}{2}, & \text{if } 2c \le n \le 2(a - b) \\ a - b + c - \langle \frac{n-1}{2} \rangle, & \text{if } 2(a - b) \le n \le 2(a - b + c) \end{cases}
$$

Here $\langle k \rangle$ *means the integer part of the number* k. *The multiplicity of eigenvalues* λ_n *can be represented by the Fig. 4.1.*

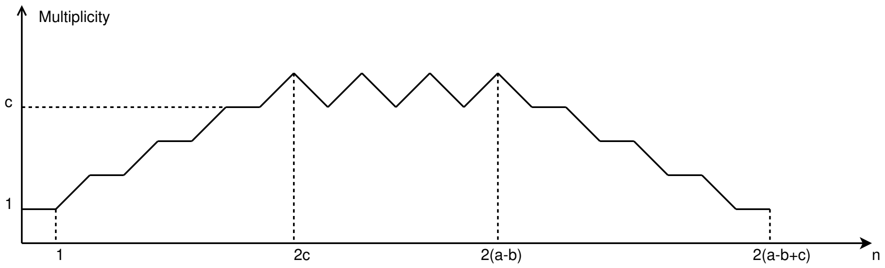

FIG. 4.1.

If $c \ge a - b$ *then*

$$
\dim \lambda_n = \begin{cases} \langle \frac{n+2}{2} \rangle, & \text{if } 0 \le n \le 2(a - b) \\ a - b + 1, & \text{if } 2(a - b) \le n \le 2c \\ a - b + c - \langle \frac{n-1}{2} \rangle, & \text{if } 2c \le n \le 2(a - b + c) \end{cases}
$$

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The multiplicity of eigenvalues λ_n can be represented by the Fig. 4.2.

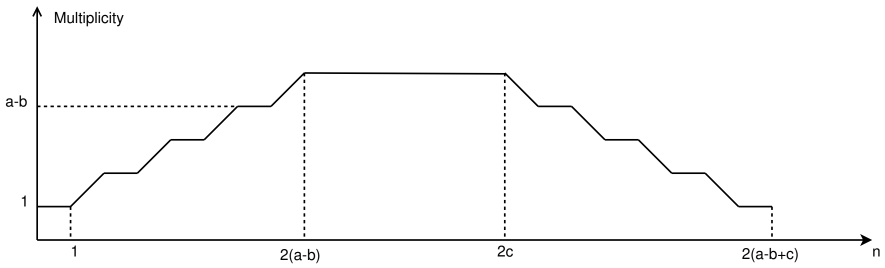

FIG. 4.2.

The structure of the solutions of the equation (3.5) is described by the following theorem.

THEOREM 4.2. *Equation (3.5) has $(a - b + 1)(2c + 1)$ linearly independent solutions. The solutions are symmetric polynomials in $[x]$ and $[y]$, which are orthogonal on the lattice (3.2) with the weight (3.4). A basis of solutions of the equation (3.5) can be obtained by the orthogonalization of the sequence*

$$\underbrace{1}_{\lambda_0}, \underbrace{[x][y]}_{\lambda_1}, \underbrace{[x]^2[y]^2, [x]^2 + [y]^2}_{\lambda_2}, \underbrace{[x]^3[y]^3, [x]^3[y] + [x][y]^3}_{\lambda_3}, \ldots$$

Theorems 4.1 and 4.2 can be proved analogously to those in [1] by constructing a special family of symmetric polynomials that form a basis in space $L(\Omega)$.

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