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ETNA
Kent State University
etna@mcs.kent.edu

MOMENT MATRIX OF SELF-SIMILAR MEASURES*

C. ESCRIBANO†, M. A. SASTRE‡, AND E. TORRANO‡

Abstract. We give in this paper an expression for the moment matrix associated to a self-similar measure given by an Iterated Function Systems (IFS). This expression translates the self-similarity property of a measure to its moment matrix.

This matrix relation shows that the properties of a measure are reflected, not only in the equation of its Jacobi matrix, as stated in Krein theorem, but also in the moment matrix.

Key words. self-similar measures, orthogonal polynomials, moment matrix

AMS subject classifications. 42C05, 28A80

1. Introduction.

In this paper we study properties of self-similar measures in the complex plane related to the theory of orthogonal polynomials. In this section we recall the concepts of self-similar measure and iterated function system (IFS), and some results about moment matrix that we will need later. In the second section we prove that the moment matrix of a self similar measure satisfies a matrix equation which allows us to obtain a recurrence relation for the moments. This recurrence generalizes the expression obtained for G. Mantica in [15]. We give some examples in the third section. Finally, given an IFS, we define an operator which takes a moment matrix of a probability measure to that of another probability measure. The n-th composition of this operator converges, element to element, to the moment of the self-similar measure.

Let Π be the space of polynomials and μ a measure with an infinite number of increasing points. Then, there exists a unique sequence of normalized orthogonal polynomials $\{\widehat{P}_n(z)\}_{n=1}^{\infty}$ associated to it (see [2] or [8],[20]). Given two polynomials $P(z), Q(z) \in \Pi$, the expression

$$\langle P(z), Q(z)\rangle = \int_{\mathrm{supp}(\mu)} P(z)\overline{Q}(z)d\mu(z)$$

defines an inner product. If we take the canonical basis $\{z^n\}_{n=0}^{\infty}$ in Π, then the product $\langle z^i, z^j\rangle = \displaystyle\int z^i\overline{z}^j d\mu(z)$, gives the (i,j)-moment c_{ij} of the infinite Gram matrix

$$M = \begin{pmatrix} c_{00} & c_{10} & c_{20} & \cdots \\ c_{01} & c_{11} & c_{21} & \cdots \\ c_{02} & c_{12} & c_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We denote by M_n the main section of order $n \times n$ of M (from now on n-section). We know that if the support is bounded then the moments determine the measure. This will be the case for self-similar measures studied here.

1.1. Self-similar Measures.

Given a family $\{\varphi_j\}_{j=1}^{n}$ of contractive maps defined on a complete metric space, there exists a unique compactum K satisfying $K = \bigcup_{j=1}^{n} \varphi_j(K)$.

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†Departamento de Matemática Aplicada, Facultad de Informática Universidad Politécnica de Madrid, Spain (cescribano@fi.upm.es).
‡Facultad de Informática Universidad Politécnica de Madrid, Spain (masastre, emilio@fi.upm.es).

This compactum is obtained as a limit in the metric space of compacta with the Hausdorff metric, iterating the maps, taking as initial set any compactum of the space. We call this family $\{\varphi_j\}_{j=1}^n$ an Iterated Functions System (IFS) [1]. If we assign a probability $p_j \geq 0$ to every φ_j, with $\sum_{j=1}^n p_j = 1$, there exists a unique probability measure μ invariant for the Markov operator T, defined over the Borel regular probability measures as $T\nu = \sum_{j=1}^{n} p_j \nu \varphi_j^{-1}$.

We say that K and μ are self-similar when the maps φ_j $(j = 1, \dots, n)$ are contractive similarities (recall φ is a contractive similarity when $|\varphi(x) - \varphi(y)| = r|x - y|, 0 \leq r \leq 1$, for all x, y). Moreover, if the $\varphi_j(K)$ are disjoints sets, then the measure μ restricted to each subset $\varphi_j(K)$ is essentially the same measure [12, 14]. This measure is called the self-similar measure μ associated to the IFS with probabilities $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_n; p_1, p_2, \dots p_n\}$, and satisfies

$$\mu = \sum_{j=1}^{n} p_j \mu \varphi_j^{-1}, \quad \int_{\mathrm{supp}(\mu)} f d\mu = \sum_{i=1}^{n} p_i \int_{\mathrm{supp}(\mu)} f \circ \varphi_i d\mu.$$

For more information about this subject see the books of K. J. Falconer and P. Mattila [5, 16].

1.2. Moments matrix of the image measure under a similarity. In order to study the relationship of a self-similar measure μ with each image measure under the similarities of the IFS which defines μ, we will use the following result due to E. Torrano [21] on the transformation of a measure by a similarity.

PROPOSITION 1.1. *Let M_n be the n-section of the moment matrix of a measure μ on $\mathbb{C}$. Consider the similarity $f(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$. Then, the moment matrix of the measure $\mu \circ f^{-1}$ is given by $M_n^f = A_n^H M_n A_n$, where A_n is the n-section of*

$$A = \begin{pmatrix} \binom{0}{0}\alpha^0\beta^0 & \binom{1}{0}\alpha^0\beta^1 & \binom{2}{0}\alpha^0\beta^2 & \binom{3}{0}\alpha^0\beta^3 & \dots \\ 0 & \binom{1}{1}\alpha^1\beta^0 & \binom{2}{1}\alpha^1\beta^1 & \binom{3}{1}\alpha^1\beta^2 & \dots \\ 0 & 0 & \binom{2}{2}\alpha^2\beta^0 & \binom{3}{2}\alpha^2\beta^1 & \dots \\ 0 & 0 & 0 & \binom{3}{3}\alpha^3\beta^0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and A_n^H denotes the conjugated transposed matrix of A_n.

REMARK 1.2. *We can interprete the matrix relation $M^f = A^H M A$ as a change of basis, with M^f as the Gram matrix of the inner product with the new basis $\{1, f(z), [f(z)]^2, [f(z)]^3, \dots\}$.*

EXAMPLE 1.3. Consider Lebesgue measure normalized (so that $c_{00} = 1$) over the unit circle, such that

$$c_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [e^{i\theta}]^j [e^{-i\theta}]^k d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)\theta} d\theta = \delta_{jk}.$$

In this case the moment matrix is the identity matrix, which is obviously a Toeplitz matrix. If we take $\alpha = 1$ and $\beta = 1$, the unit circle is transformed by $f(z) = z+1$ into a new unit circle centered in $(1, 0)$ with Lebesgue measure. The new moment matrix is the Pascal matrix. For

ETNA
Kent State University
etna@mcs.kent.edu

example, for $n = 5$ we have

$$M_5^f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}.$$

2. Moment Matrix of Self-similar Measures. In this section we obtain a recurrence relation for a self-similar measure moments which generalizes the expression given by G. Mantica in [15] for the real case.

THEOREM 2.1. *Let $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k; p_1, p_2, \ldots p_k\}$ be an IFS with probabilities, where $\varphi_s(x) = \alpha_s x + \beta_s$, $p_s > 0$ for every $s \in \{1, 2, \ldots, k\}$ and $\sum_{s=1}^{k} p_s = 1$, and let μ be the associated invariant measure. Then, the sections M_n of the moment matrix of μ satisfy the following matrix relation*

$$M_n = \sum_{s=1}^{k} p_s M_n^{\varphi_s},$$

where $M_n^{\varphi_s}$ is the n-section of the moment matrix of the image measure under the similarity φ_s. Therefore,

$$c_{i,j} = \frac{1}{1 - \sum_{s=1}^{k} p_s \alpha_s^i \overline{\alpha_s}^j} \sum_{s=1}^{k} p_s \sum_{m=0, l=0 (m,l) \neq (i,j)}^{i,j} \binom{i}{m} \binom{j}{l} \beta_s^{i-m} \overline{\beta_s}^{j-l} \alpha_s^m \overline{\alpha_s}^l c_{m,l}. \tag{2.1}$$

Proof. For each $s = 1, 2, \ldots, k$, we can obtain, from the above results, the nth-section of the moment matrix of the image measure of μ under $\varphi_s(x)$

$$M_n^{\varphi_s} = A_{s,n}^H \; M_n \; A_{s,n}, \text{ with } A_{s,n} = (a_{i,j})_{i,j=1}^n = \begin{cases} \binom{j-1}{i-1} \alpha^{i-1} \beta^{j-i} & j \geq i \\ 0 & j < i, \end{cases}$$

Now, using the "balance property", $\int_X f d\mu = \sum_{s=1}^{k} p_s \int_X f \circ \varphi_s d\mu$, for self-similar measures [6, pg. 36],[15], we get the moment matrix of μ in terms of the image measures under the similarities of the IFS Φ and also in congruence terms of itself

$$M_n = \sum_{s=1}^{k} p_i M_n^{\varphi_s} = \sum_{s=1}^{k} p_s A_{s,n}^H \; M_n \; A_{s,n}.$$

From this formula we can compute each moment $c_{i,j}$ by the recurrent relation 2.1.

ETNA
Kent State University
etna@mcs.kent.edu

In the real case the moment matrix is a Hankel matrix and consequently $c_{i,j} = c_{0,i+j} = S_{i+j}$, and then we obtain the recurrent formula

$$S_n = \frac{1}{1 - \sum_{s=1}^{k} p_s \alpha_s^n} \sum_{s=1}^{k} p_s \sum_{j=0}^{n-1} \binom{n}{j} \beta_s^{n-j} \alpha_s^j S_j .$$

$\square$

3. Examples. The IFS defined by $\Phi = \{f_1(x) = rx+1, f_2(x) = rx-1; p_1 = p_2 = \frac{1}{2}\}$ for $r \in (0,1)$. The corresponding invariant measure μ is the infinitely convolved Bernoulli measure (ICBM) (see the works of P. Erdös [3], A. Garsia [9] and B. Jessen and A. Wintner [13] for definition and properties) nonetheless have been studied since 1935 and there are still many questions to solve.

In the last years, several papers ([4, 7, 10, 11, 15, 18]) related to this problem have been published based on the fact that this measures are self-similar. It has been recently proved ([17, 19]) that for $\frac{1}{2} < r < 1$ the ICBM μ_r is absolute continuous almost everywhere. In the following, we consider an ICBM ν_r normalized in order to have $\operatorname{supp}(\nu_r) \subset [-1,1]$. For every $r \in (0, \frac{1}{2})$ the ICBM μ_r is singular and its spectrum is a Cantor set. For $r = \frac{1}{2}$ we get Lebesgue distribution and Legendre polynomials.

EXAMPLE 3.1. The ICBM μ_r is the self-similar measure given by the IFS with probabilities $\Phi_r = \{\varphi_1(x) = rx + r - 1, \varphi_2(x) = rx + 1 - r = rx + r^2; p_1 = p_2 = \frac{1}{2}\}$. The self-similar set for this IFS is included in the interval $E = [-1,1]$ and μ_r satisfies $\mu_r(A) = \frac{1}{2}\mu_r(\varphi_1^{-1}(A)) + \frac{1}{2}\mu_r(\varphi_2^{-1}(A))$.

In this case the two matrices used to obtain the moment matrix for the image measure under the similarities are very similar, because the contraction ratio are equals for the two similarities and the translation coefficient are opposite, so that $A_{1,5}(i,j) = (-1)^{i+j} A_{2,5}(i,j)$. Indeed,

$$A_{1,5} = \begin{pmatrix} \binom{0}{0} r^0 (r-1)^0 & \binom{1}{0} r^0 (r-1)^1 & \binom{2}{0} r^0 (r-1)^2 & \binom{3}{0} r^0 (r-1)^3 & \binom{4}{0} r^0 (r-1)^4 \\ 0 & \binom{1}{1} r^1 (r-1)^0 & \binom{2}{1} r^1 (r-1)^1 & \binom{3}{1} r^1 (r-1)^2 & \binom{4}{1} r^1 (r-1)^3 \\ 0 & 0 & \binom{2}{2} r^2 (r-1)^0 & \binom{3}{2} r^2 (r-1)^1 & \binom{4}{2} r^2 (r-1)^2 \\ 0 & 0 & 0 & \binom{3}{3} r^3 (r-1)^0 & \binom{4}{3} r^3 (r-1)^1 \\ 0 & 0 & 0 & 0 & \binom{4}{4} r^4 (r-1)^0 \end{pmatrix},$$

$$A_{2,5} = \begin{pmatrix} \binom{0}{0} r^0 (1-r)^0 & \binom{1}{0} r^0 (1-r)^1 & \binom{2}{0} r^0 (1-r)^2 & \binom{3}{0} r^0 (1-r)^3 & \binom{4}{0} r^0 (1-r)^4 \\ 0 & \binom{1}{1} r^1 (1-r)^0 & \binom{2}{1} r^1 (1-r)^1 & \binom{3}{1} r^1 (1-r)^2 & \binom{4}{1} r^1 (1-r)^3 \\ 0 & 0 & \binom{2}{2} r^2 (1-r)^0 & \binom{3}{2} r^2 (1-r)^1 & \binom{4}{2} r^2 (1-r)^2 \\ 0 & 0 & 0 & \binom{3}{3} r^3 (r-1)^0 & \binom{4}{3} r^3 (1-r)^1 \\ 0 & 0 & 0 & 0 & \binom{4}{4} r^4 (1-r)^0 \end{pmatrix}.$$

The ICBM for $r = \frac{1}{3}$ is the Cantor measure in the interval $[-1,1]$. In this case we have

$$M_5^{\varphi 2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 & 0 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} & 0 \\ \frac{16}{81} & \frac{32}{81} & \frac{8}{27} & \frac{8}{81} & \frac{1}{81} \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{7}{20} \\ 0 & \frac{1}{2} & 0 & \frac{7}{20} & 0 \\ \frac{1}{2} & 0 & \frac{7}{20} & 0 & \frac{205}{728} \\ 0 & \frac{7}{20} & 0 & \frac{205}{728} & 0 \\ \frac{7}{20} & 0 & \frac{205}{728} & 0 & \frac{10241}{42640} \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} \\ 0 & \frac{1}{3} & \frac{4}{9} & \frac{4}{9} & \frac{32}{81} \\ 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{8}{27} \\ 0 & 0 & 0 & \frac{1}{27} & \frac{8}{81} \\ 0 & 0 & 0 & 0 & \frac{1}{81} \end{pmatrix}.$$

ETNA
Kent State University
etna@mcs.kent.edu

In this case, we obtain the same relation between $M_n^{\varphi_1}$ and $M_n^{\varphi_2}$ as for the matrices $A_{1,n}$ and $A_{2,n}$, i.e. $M_n^{\varphi_1}(i,j) = (-1)^{i+j} M_n^{\varphi_2}(i,j)$. Finally, $M_n = \dfrac{1}{2} M_n^{\varphi_1} + \dfrac{1}{2} M_n^{\varphi_2}$.

If we assign different probabilities to each similarity in the ICBM we obtain an infinitely convolved weighted binomial measure (ICWBM). For $r = \dfrac{1}{3}$ and probabilities $\dfrac{2}{3}, \dfrac{1}{3}$ we have a non-symmetric measure in a Cantor set contained in the unit interval. Since the similarities are the same as that as the previous example, the matrices $A_{1,n}$ and $A_{2,n}$ are the same as before. Then, we have

$$
\begin{aligned}
M_5^{\varphi_1} &= A_{1,5}^H M_5 A_{1,5} \\
&= \begin{pmatrix}
1 & -\frac{7}{9} & \frac{53}{81} & -\frac{1829}{3159} & \frac{75139}{142155} \\
-\frac{7}{9} & \frac{53}{81} & -\frac{1829}{3159} & \frac{75139}{142155} & -\frac{15233305}{30961359} \\
\frac{53}{81} & -\frac{1829}{3159} & \frac{75139}{142155} & -\frac{15233305}{30961359} & \frac{559966709}{1207493001} \\
-\frac{1829}{3159} & \frac{75139}{142155} & -\frac{15233305}{30961359} & \frac{559966709}{1207493001} & \\
\frac{75139}{142155} & -\frac{15233305}{30961359} & \frac{559966709}{1207493001} & -\frac{5236263395783}{11878108650837} & \frac{9240140043317737}{21915110460794265}
\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
M_5^{\varphi_2} &= A_{2,5}^H M_5 A_{2,5} \\
&= \begin{pmatrix}
1 & \frac{5}{9} & \frac{29}{81} & \frac{823}{3159} & \frac{29299}{142155} \\
\frac{5}{9} & \frac{29}{81} & \frac{823}{3159} & \frac{29299}{142155} & \frac{5319755}{30961359} \\
\frac{29}{81} & \frac{823}{3159} & \frac{29299}{142155} & \frac{5319755}{30961359} & \frac{178630637}{1207493001} \\
\frac{823}{3159} & \frac{29299}{142155} & \frac{5319755}{30961359} & \frac{178630637}{1207493001} & \frac{1544687605501}{11878108650837} \\
\frac{29299}{142155} & \frac{5319755}{30961359} & \frac{178630637}{1207493001} & \frac{1544687605501}{11878108650837} & \frac{2541691310236297}{21915110460794265}
\end{pmatrix},
\end{aligned}
$$

$$
M_5 = \begin{pmatrix}
1 & -\frac{1}{3} & \frac{5}{9} & -\frac{35}{117} & \frac{739}{1755} \\
-\frac{1}{3} & \frac{5}{9} & -\frac{35}{117} & \frac{739}{1755} & -\frac{34495}{127413} \\
\frac{5}{9} & -\frac{35}{117} & \frac{739}{1755} & -\frac{34495}{127413} & \frac{593765}{1656369} \\
-\frac{35}{117} & \frac{739}{1755} & -\frac{34495}{127413} & \frac{593765}{1656369} & -\frac{1360743665}{5431233951} \\
\frac{739}{1755} & -\frac{34495}{127413} & \frac{593765}{1656369} & -\frac{1360743665}{5431233951} & \frac{1068026794537}{3340208879865}
\end{pmatrix}.
$$

Therefore, $M_5 = \dfrac{2}{3} M_5^{\varphi_1} + \dfrac{1}{3} M_5^{\varphi_2}$.

4. Computation of the Moment Matrix of a Self-similar Measure. Let $\mathcal{P}_c$ be the space of probability measures with compact support. In this section we consider the set $\mathcal{M}_m$ of $\mathcal{P}_c$ moment matrix. We denote by $M_\nu \in \mathcal{M}_m$ the moment matrix of the probability measure $\nu \in \mathcal{P}_c$.

DEFINITION 4.1. *Given an IFS with probabilities*

$$\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k; p_1, p_2, \ldots p_k\},$$

where $p_s > 0$ *for every* $s \in \{1, 2, \dots, k\}$ *and* $\sum_{s=1}^{k} p_s = 1$, *we define the transformation in the space* $\mathcal{M}_m$, $\mathcal{T} : \mathcal{M}_m \to \mathcal{M}_m$ *by*

$$\mathcal{T}(M_\nu) = \sum_{s=1}^{k} p_s A_s^H M_\nu A_s,$$

where A_s *is the matrix given in the first section, for the similarity* φ_s.

REMARK 4.2. *Note that the transformation* $\mathcal{T}$ *is well defined: For every* s, *the matrix* $M_\nu^{\varphi_s} = A_s^H M_\nu A_s$ *is the moment matrix of the induced probability measure* $\nu \circ \varphi_s^{-1}$ *and the convex linear combination of probabilities with compact support is a measure in* $\mathcal{P}_c$.

THEOREM 4.3. *For every* $M_\nu \in \mathcal{M}_m$ *the sequence* $\mathcal{T}^n(M_\nu)$, *where* $\mathcal{T}^n$ *denotes the nth-composition of* $\mathcal{T}$ *with itself, converges element by element to the moment matrix* M_μ *where* μ *is the invariant measure associated with the IFS.*

Proof. The invariant measure μ associated with the IFS is the invariant measure of the Markov operator defined in the space $\mathcal{P}_c$ by

$$T(\nu) = \sum_{s=1}^{k} p_s \nu \circ \varphi_s^{-1}.$$

This operator is a contraction in the complete metric space $\mathcal{P}_c$ with the Hutchinson metric [12], which agrees to weakly convergence of measures, so that, the invariant measure μ is the weakly limit of $T^n(\nu)$ for every $\nu \in \mathcal{P}_c$. The weakly convergence ensures the convergence of the moments of the measures $T^n(\mu)$, $c_{i,j}(T^n(\mu))$, to $c_{i,j}(\mu)$. As $\mathcal{T}(M_\nu) = M_{\nu_1}$ for $T(\nu) = \nu_1$, we have the result. ☐

Next examples show the efficient of the algorithm for calculate the moments.

EXAMPLE 4.4. Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1, 1]$. This is a self-similar measure for the IFS given by $\Phi = \{\varphi_1 = \frac{1}{2}x - \frac{1}{2}, \varphi_2 = \frac{1}{2}x + \frac{1}{2}; p_1 = p_2 = \frac{1}{2}\}$. The section of order six of the corresponding transformation $\mathcal{T}$ is given by

$$\mathcal{T}(M_\nu) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{5}{32} & \frac{5}{16} & \frac{5}{16} & \frac{5}{32} & \frac{1}{32} \end{pmatrix} M_\nu \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} & \frac{1}{4} & \frac{5}{32} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & \frac{5}{16} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{5}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{pmatrix} +$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{8} & \frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{16} & -\frac{1}{4} & \frac{3}{8} & -\frac{1}{4} & \frac{1}{16} & 0 \\ -\frac{1}{32} & \frac{5}{32} & -\frac{5}{16} & \frac{5}{16} & -\frac{5}{32} & \frac{1}{32} \end{pmatrix} M_\nu \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{16} & -\frac{1}{32} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{3}{8} & -\frac{1}{4} & \frac{5}{32} \\ 0 & 0 & \frac{1}{4} & -\frac{3}{8} & \frac{3}{8} & -\frac{5}{16} \\ 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{5}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{pmatrix}.$$

ETNA
Kent State University
etna@mcs.kent.edu

Let's iterate the transformation $\mathcal{T}$ twenty times starting with to the identity matrix, which is the moment matrix for the Lebesgue measure in the unit circle:

$$\mathcal{T}^{20}\left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right) =$$

$$\begin{pmatrix} 1.0 & 0.0 & 0.3333333333 & 0.0 & 0.2000000000 & 0.0 \\ 0.0 & 0.3333333333 & 0.0 & 0.2000000000 & 0.0 & 0.1428571429 \\ 0.3333333333 & 0.0 & 0.2000000000 & 0.0 & 0.1428571429 & 0.0 \\ 0.0 & 0.2000000000 & 0.0 & 0.1428571429 & 0.0 & 0.1111111111 \\ 0.2000000000 & 0.0 & 0.1428571429 & 0.0 & 0.1111111111 & 0.0 \\ 0.0 & 0.1428571429 & 0.0 & 0.1111111111 & 0.0 & 0.09090909091 \end{pmatrix}.$$

This matrix agrees (with ten digits precision) to the well known moment matrix (of six order) of the measure $\mathcal{L}$.

EXAMPLE 4.5. The Cantor measure $\mu_{\frac{1}{3}}$ in the interval $[-1,1]$ is a ICBM with $r = \dfrac{1}{3}$ (see section 3, example 2). To calculate the moments of this measure, we use the recurrent formula obtained in theorem 1 of section 2, so that, the moment matrix (of six order) is

$$M_{5,\mu_{\frac{1}{3}}} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{7}{20} \\ 0 & \frac{1}{2} & 0 & \frac{7}{20} & 0 \\ \frac{1}{2} & 0 & \frac{7}{20} & 0 & \frac{205}{728} \\ 0 & \frac{7}{20} & 0 & \frac{205}{728} & 0 \\ \frac{7}{20} & 0 & \frac{205}{728} & 0 & \frac{10241}{42640} \end{pmatrix}.$$

The Cantor measure is self-similar for the IFS $\Phi = \{\varphi_1 = \dfrac{1}{3}x - \dfrac{2}{3}, \varphi_2 = \dfrac{1}{3}x + \dfrac{2}{3}; p_1 = p_2 = \dfrac{1}{2}\}$. The section of order six of the corresponding transformation $\mathcal{T}$ is given by

$$\mathcal{T}(M_\nu) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 & 0 & 0 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} & 0 & 0 \\ \frac{16}{81} & \frac{32}{81} & \frac{8}{27} & \frac{8}{81} & \frac{1}{81} & 0 \\ \frac{32}{243} & \frac{80}{243} & \frac{80}{243} & \frac{40}{243} & \frac{10}{243} & \frac{1}{243} \end{pmatrix} M_\nu \begin{pmatrix} 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} \\ 0 & \frac{1}{3} & \frac{4}{9} & \frac{4}{9} & \frac{32}{81} & \frac{80}{243} \\ 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{8}{27} & \frac{80}{243} \\ 0 & 0 & 0 & \frac{1}{27} & \frac{8}{81} & \frac{40}{243} \\ 0 & 0 & 0 & 0 & \frac{1}{81} & \frac{10}{243} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{243} \end{pmatrix} +$$

ETNA
Kent State University
etna@mcs.kent.edu

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{4}{9} & -\frac{4}{9} & \frac{1}{9} & 0 & 0 & 0 \\ -\frac{8}{27} & \frac{4}{9} & -\frac{2}{9} & \frac{1}{27} & 0 & 0 \\ \frac{16}{81} & -\frac{32}{81} & \frac{8}{27} & -\frac{8}{81} & \frac{1}{81} & 0 \\ -\frac{32}{243} & \frac{80}{243} & -\frac{80}{243} & \frac{40}{243} & -\frac{10}{243} & \frac{1}{243} \end{pmatrix} M_\nu \begin{pmatrix} 1 & -\frac{2}{3} & \frac{4}{9} & -\frac{8}{27} & \frac{16}{81} & -\frac{32}{243} \\ 0 & \frac{1}{3} & -\frac{4}{9} & \frac{4}{9} & -\frac{32}{81} & \frac{80}{243} \\ 0 & 0 & \frac{1}{9} & -\frac{2}{9} & \frac{8}{27} & -\frac{80}{243} \\ 0 & 0 & 0 & \frac{1}{27} & -\frac{8}{81} & \frac{40}{243} \\ 0 & 0 & 0 & 0 & \frac{1}{81} & -\frac{10}{243} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{243} \end{pmatrix}.$$

Let's iterate the transformation $\mathcal{T}$ twenty times starting with to the identity matrix, which is the moment matrix for the Lebesgue measure in the unit circle:

$$\mathcal{T}^{20}\left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right) =$$

$$\begin{pmatrix} 1.0 & 0.0 & 0.5000000000 & 0.0 & 0.3500000000 & 0.0 \\ 0.0 & 0.5000000000 & 0.0 & 0.3500000000 & 0.0 & 0.2815934066 \\ 0.5000000000 & 0.0 & 0.3500000000 & 0.0 & 0.2815934066 & 0.0 \\ 0.0 & 0.3500000000 & 0.0 & 0.2815934066 & 0.0 & 0.2401735460 \\ 0.3500000000 & 0.0 & 0.2815934066 & 0.0 & 0.2401735460 & 0.0 \\ 0.0 & 0.2815934066 & 0.0 & 0.2401735460 & 0.0 & 0.2113091906 \end{pmatrix}.$$

This matrix approximates, with ten digits precision, the moment matrix (of six order) of Cantor measure.

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ETNA
Kent State University
etna@mcs.kent.edu

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