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AN INTEGRAL REPRESENTATION OF SOME HYPERGEOMETRIC FUNCTIONS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. The Euler integral representation of the ${}_2F_1$ Gauss hypergeometric function is well known and plays a prominent role in the derivation of transformation identities and in the evaluation of ${}_2F_1(a, b; c; 1)$, among other applications. The general ${}_{p+k}F_{q+k}$ hypergeometric function has an integral representation where the integrand involves ${}_pF_q$. We give a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$. The values of certain ${}_3F_2$ and ${}_4F_3$ functions at $x = 1$, some of which can be derived using other methods, are deduced from our integral formula.

Key words. 3F2 hypergeometric functions, general hypergeometric functions, integral representation

AMS subject classification. 15A15

1. Introduction. The general hypergeometric function is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!}, \quad |x| < 1 $$

where

$$ (\alpha)_k = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & , \quad k \geq 1, \\ 1 & , \quad k = 0,\ \alpha \neq 0 \end{cases} $$

is Pochhammer's symbol. The Euler integral representation of the Gauss hypergeometric function, or ${}_2F_1$ function, is well known in the literature (cf. [1], [3]) and is formulated as follows (cf. [1], p. 65, Theorem 2.2.1):

THEOREM 1.1. *If $Re(c) > Re(b) > 0$, then*

$$ {}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt $$

in the x plane cut along the real axis from 1 to ∞. Here, it is understood that arg t = arg $(1-t) = 0$ and $(1-xt)^{-a}$ has its principle value.

This integral may be viewed as the analytic continuation of the ${}_2F_1$ series for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and it yields the famous Gauss summation formula of 1812, namely

THEOREM 1.2 (cf. [1], p. 66, Theorem 2.2.2). *For $Re(c-a-b) > 0$, we have*

$$ {}_2F_1(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{(c)_n\ n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

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The case where one of the numerator parameters is a negative integer, thereby making the ${}_2F_1$ a finite sum, is known as the Chu-Vandermonde identity (cf. [1], p. 67, Corollary 2.2.3)

$$ {}_2F_1(-n, a; c; 1) = \frac{(c-a)_n}{(c)_n}. $$

The values of ${}_{q+1}F_q$ functions at 1 with $q \geq 2$ encompass the beautiful identities due to Pfaff-Saalschutz, Dougall, Dixon, Rogers, Ramanujan, Whipple and other authors. For further discussion, see [1], Chapters 2 and 3.

2. Results. In [3], Theorem 38, Rainville proves a general integral representation for ${}_{p+k}F_{q+k}$ where k is a positive integer and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, namely

$$ \begin{aligned} &{}_{p+k}F_{q+k}\left(\begin{matrix} a_1, & \dots & , a_p, & \frac{\alpha}{k}, & \frac{\alpha+1}{k}, & \dots, & \frac{\alpha+k-1}{k} \\ b_1, & \dots & , b_q, & \frac{\alpha+\beta}{k}, & \frac{\alpha+\beta+1}{k}, & \dots & , \frac{\alpha+\beta+k-1}{k} \end{matrix} ; ct^k\right) \\ &= \frac{t^{1-\alpha-\beta}}{B(\alpha,\beta)} \int_0^t x^{\alpha-1}\,(t-x)^{\beta-1}\,{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}; cx^k\right) dx. \end{aligned} $$

We begin with a direct proof of an Euler integral representation for the special class of hypergeometric functions ${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right)$.

THEOREM 2.1. *For Re*$(c) >$ *Re*$(b) > 0$,

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a}dt. \tag{2.1} $$

Proof. Suppose that $|x| < 1$. Then the left hand side of (2.1) becomes

$$ \sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{b}{2}\right)_k \left(\frac{b+1}{2}\right)_k}{\left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k\, k!}\, x^k = \sum_{k=0}^{\infty} \frac{(a)_k (b)_{2k}}{(c)_{2k}\, k!}\, x^k \tag{2.2} $$

since (cf. [3], p. 22)

$$ (\alpha)_{2k} = 2^{2k} \left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k. $$

Now

$$ \frac{(b)_{2k}}{(c)_{2k}} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\, \frac{\Gamma(b+2k)\Gamma(c-b)}{\Gamma(c+2k)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\, B(b+2k, c-b). $$

Therefore, for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $|x| < 1$, the right hand side of (2.2) becomes

$$ \begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k\, x^k}{k!} \int_0^1 t^{b+2k-1}(1-t)^{c-b-1}dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \frac{(a)_k\, (xt^2)^k}{k!} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a}dt. \end{aligned} $$

This proves the result for $|x| < 1$. Since the integral is analytic in the cut plane, the result holds for x in this region as well. $\square$

COROLLARY 2.2. *For Re*$(c) >$ *Re*$(b) > 0$*,*

$$ {}_3F_2\left(-n, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^n dt. $$

THEOREM 2.3. *If Re*$(c) >$ *Re*$(b) > 0$ *and Re*$(c-a-b) > 0$*, then*

$$ {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}\, {}_2F_1(a, b; c-a; -1). \tag{2.3} $$

Proof. Putting $x = 1$ in equation (2.1), the left hand side of (2.3) becomes

$$ \begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}(1+t)^{-a} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1} \sum_{k=0}^{\infty} \binom{-a}{k} t^k dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} \int_0^1 t^{b+k-1}(1-t)^{c-b-a-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} \frac{\Gamma(b+k)\Gamma(c-b-a)}{\Gamma(c-a+k)}. \end{aligned} $$

Now $\binom{-a}{k} = \frac{(-1)^k (a)_k}{k!}$ and $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$. Therefore we have

$$ \begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} \frac{\Gamma(b+k)\Gamma(c-b-a)}{\Gamma(c-a+k)} \\ &= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k}{k!} \frac{(b)_k}{(c-a)_k} \\ &= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a, b; c-a; -1). \qquad \square \end{aligned} $$

REMARK 1. *An alternative proof of Theorem 2.3 can be found in [4], formula (3.7).*

COROLLARY 2.4. *If Re*$(c) >$ *Re*$(b) > 0$*, then*

$$ {}_3F_2\left(-n, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{(c-b)_n}{(c)_n}\, {}_2F_1(-n, b; c+n; -1). $$

Generalising Theorem 2.1 in the obvious way yields the following result.

THEOREM 2.5. *If Re*$(c) >$ *Re*$(b) > 0$*, then*

$$ {}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \dots \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \dots, \frac{c+q-1}{q}; x\right) $$

$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a} dt. $$

Proof. We use (cf. [3], p. 22, lemma 6)

$$(\alpha)_{qk} = q^{qk}\left(\frac{\alpha}{q}\right)_k\left(\frac{\alpha+1}{q}\right)_k\cdots\left(\frac{\alpha+q-1}{q}\right)_k$$

and the proof proceeds as in Theorem 2.1. $\square$

It follows from Theorem 2.5 with $q=3$ that for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$\begin{aligned}&{}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; x\right)\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^3)^{-a}dt\end{aligned}$$

and this leads to the following result.

THEOREM 2.6. *For* $Re(c-a-b) > 0$,

$$\begin{aligned}&{}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right)\\ &= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k(b)_k}{k!\,(c-a)_k}\,{}_2F_1(-k, b+k; c-a+k; -1).\end{aligned}$$

Proof. Letting $q=3$ and $x=1$ in Theorem 2.5, we have

$$\begin{aligned}&{}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t^3)^{-a}dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}(1+t+t^2)^{-a}dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k}{k!}\,t^k(1+t)^k dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k}{k!}\int_0^1 t^{k+b-1}(1-t)^{c-b-a-1}(1+t)^k dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k}{k!}\int_0^1 t^{k+b-1}(1-t)^{c-b-a-1}\sum_{r=0}^{k}\binom{k}{r}t^r dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(a)_k(-1)^k}{k!}\binom{k}{r}\int_0^1 t^{k+r+b-1}(1-t)^{c-b-a-1}dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(a)_k(-1)^k}{k!}\binom{k}{r}\frac{\Gamma(b+r+k)\,\Gamma(c-b-a)}{\Gamma(c-a+k+r)}\\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(a)_k(-1)^k}{k!}\binom{k}{r}\frac{(b)_{r+k}}{(c-a)_{r+k}}\\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(a)_k(-1)^k}{k!}\binom{k}{r}\frac{(b)_k(b+k)_r}{(c-a)_k(c-a+k)_r} \qquad \text{from [3], p. 65}\end{aligned}$$

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$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!\,(c-a)_k} \sum_{r=0}^{k} \binom{k}{r} \frac{(b+k)_r}{(c-a+k)_r}.$$

Now,

$$\begin{aligned}
&\sum_{r=0}^{k} \binom{k}{r} \frac{(b+k)_r}{(c-a+k)_r} \\
&= \sum_{r=0}^{k} \frac{(-1)^r(-k)_r}{r!} \; \frac{(b+k)_r}{(c-a+k)_r} \qquad \text{since} \qquad \frac{k!}{(k-r)!} = (-1)^r(-k)_r \\
&= \sum_{r=0}^{k} \frac{(-k)_r(b+k)_r}{(c-a+k)_r} \; \frac{(-1)^r}{r!} \\
&= {}_2F_1(-k, b+k; c-a+k; -1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&{}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right) \\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!\,(c-a)_k} \; {}_2F_1(-k, b+k; c-a+k; -1)
\end{aligned}$$

as stated. □

As another application of Theorem 2.1, we consider the value of ${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right)$ at $x = \frac{1}{2}$ and obtain the following result.

THEOREM 2.7. *If* $Re(c) > Re(b) > 0$, *then*

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} \; {}_2F_1(-k, b; c+k; -1).$$

Proof. From Theorem 2.1 with $x = \frac{1}{2}$,

$$\begin{aligned}
&{}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1 - \frac{1}{2}\,t^2\right)^{-a} dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(2-t^2)^{-a}\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1+(1-t^2)\right)^{-a} dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \binom{-a}{k} (1-t^2)^k\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k}(1+t)^k\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k} \sum_{r=0}^{k} \binom{k}{r} t^r\,dt
\end{aligned}$$

$$= \frac{2^a\ \Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \binom{-a}{k} \binom{k}{r} \int_0^1 t^{b+r-1}(1-t)^{c-b-1+k}\ dt$$
$$= \frac{2^a\ \Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \binom{-a}{k} \binom{k}{r} \frac{\Gamma(b+r)\Gamma(c-b+k)}{\Gamma(c+r+k)}$$
$$= 2^a \sum_{k=0}^{\infty} \sum_{r=0}^{k} \binom{-a}{k} \binom{k}{r} \frac{(b)_r\ (c-b)_k}{(c)_{r+k}}.$$

But since $(c)_{r+k} = (c)_k\ (c+k)_r$ (cf. [3], p. 65), we have that

$$
\begin{aligned}
{}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) &= 2^a \sum_{k=0}^{\infty} \sum_{r=0}^{k} \binom{-a}{k} \binom{k}{r} \frac{(b)_r\ (c-b)_k}{(c)_k\ (c+k)_r} \\
&= 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} \sum_{r=0}^{k} \binom{k}{r} \frac{(b)_r}{(c+k)_r}.
\end{aligned}
$$

We know that

$$\sum_{r=0}^{k} \binom{k}{r} \frac{(b)_r}{(c+k)_r} = {}_2F_1(-k, b; c+k; -1)$$

and hence

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} {}_2F_1(-k, b; c+k; -1). \qquad \square$$

COROLLARY 2.8. *If* $Re(c) > Re(b) > 0$, *then*

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k}\ {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right).$$

Proof. From Corollary 2.4, we can see that

$${}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{(c-b)_k}{(c)_k}\ {}_2F_1(-k, b; c+k; -1)$$

which along with Theorem 2.7 gives the result. $\square$

REMARK 2. *This corollary is a special case of a transformation given by Chaundy [2].*

REFERENCES

[1] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
[2] T. W. CHAUNDY, *An extension of hypergeometric functions, I.*, Quart. J. Maths. Oxford Ser., 14 (1943), pp. 55–78.
[3] E. D. RAINVILLE, *Special Functions*, The Macmillan Co., New York, 1960.
[4] F. J. W. WHIPPLE, *On series allied to the hypergeometric series with argument -1*, Proc. London Math. Soc., 30 (1930), pp. 81–94.