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Kent State University
etna@mcs.kent.edu

DISTRIBUTION OF PRIMES AND A WEIGHTED ENERGY PROBLEM*

IGOR E. PRITSKER†

Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We discuss a recent development connecting the asymptotic distribution of prime numbers with weighted potential theory. These ideas originated with the Gelfond-Schnirelman method (circa 1936), which used polynomials with integer coefficients and small sup norms on $[0, 1]$ to give a Chebyshev-type lower bound in prime number theory. A generalization of this method for polynomials in many variables was later studied by Nair and Chudnovsky, who produced tight bounds for the distribution of primes. Our main result is a lower bound for the integral of Chebyshev's ψ-function, expressed in terms of the weighted capacity for polynomial-type weights. We also solve the corresponding potential theoretic problem, by finding the extremal measure and its support. This new connection leads to some interesting open problems on weighted capacity.

Key words. distribution of prime numbers, polynomials, integer coefficients, weighted transfinite diameter, weighted capacity, potentials

AMS subject classifications. 11N05, 31A15, 11C08

1. Asymptotic distribution of prime numbers.

Let $\pi(x)$ be the number of primes not exceeding x. The well known Prime Number Theorem states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty. \tag{1.1}$$

It had been conjectured by Gauss and Legendre in the end of 18th century, but proved only about one hundred years later. We sketch the history, referring for details to many excellent books and surveys available on this subject (see, e.g., [18], [6], [32] and [8]). Chebyshev [4] made the first important contribution to the Prime Number Theorem in 1852, by proving the bounds

$$0.921\frac{x}{\log x} \le \pi(x) \le 1.106\frac{x}{\log x} \quad \text{as } x \to \infty. \tag{1.2}$$

The famous Riemann's paper [28], published in 1859, introduced complex analytic methods related to the zeta function. This culminated in the proofs of the Prime Number Theorem by Hadamard and de la Vallée Poussin in 1896, via establishing that $\zeta(s)$ does not have zeros on the line $\{1 + it,\ t \in \mathbb{R}\}$. But the "elementary" approaches, which do not use complex analysis and the zeta function, still remained of great interest. Selberg [30] and Erdős [9] found the first elementary proof of the Prime Number Theorem in 1949. A survey of elementary methods may be found in Diamond [8]. We consider the elementary method of Gelfond and Schnirelman (see Gelfond's comments in [4, pp. 285–288]), proposed in 1936, and its later generalizations. Define the Chebyshev function

$$\psi(x) := \sum_{p^m \le x} \log p, \tag{1.3}$$

where the summation extends over the primes p. Note that $\psi(x) = \log \operatorname{lcm}(1, \ldots, x)$ for $x \in \mathbb{N}$. It is well known that the Prime Number Theorem is equivalent to

$$\psi(x) \sim x \quad \text{as } x \to +\infty \tag{1.4}$$

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†Department of Mathematics, 401 Mathematical Sciences, Oklahoma State University, Stillwater, OK 74078-1058 (igor@math.okstate.edu).

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Kent State University
etna@mcs.kent.edu

(see [18], [6], [20, Ch. 10] and [8]). Gelfond and Schnirelman utilized polynomials with integer coefficients $p_n(x) = \sum_{k=0}^{n} a_k x^k$, and their integrals over $[0,1]$:

$$\int_0^1 p_n(x)\,dx = \sum_{k=0}^{n} \frac{a_k}{k+1}.$$

Observe that multiplying the above integral by the least common multiple $\mathrm{lcm}(1,\ldots,n+1)$ gives an integer. Clearly,

$$\mathrm{lcm}(1,\ldots,n+1)\left|\int_0^1 p_n(x)\,dx\right| \ge 1, \tag{1.5}$$

provided $\int_0^1 p_n(x)\,dx \ne 0$. Taking the log of (1.5), we have

$$\psi(n+1) \ge -\log\left|\int_0^1 p_n(x)\,dx\right| \ge -\log \max_{x\in[0,1]} |p_n(x)|.$$

Hence

$$\liminf_{n\to\infty} \frac{\psi(n+1)}{n} \ge -\log \limsup_{n\to\infty} \left(\max_{x\in[0,1]} |p_n(x)|\right)^{1/n}. \tag{1.6}$$

If one could find a sequence of polynomials p_n with sufficiently small sup norms $\|p_n\|_{[0,1]}$, so that

$$\lim_{n\to\infty} \|p_n\|_{[0,1]}^{1/n} \stackrel{?}{=} 1/e, \tag{1.7}$$

then the Prime Number Theorem followed from (1.6). A detailed analysis of the original Gelfond-Schnirelman argument is contained in Montgomery [20, Ch. 10] (also see Chudnovsky [5]). We are led by this method to the so-called integer Chebyshev problem on polynomials with integer coefficients minimizing the sup norm (see, e.g., Borwein [3]). Let $\mathbb{Z}_n[x]$ be the set of polynomials over integers, of degree at most n. In view of (1.6)-(1.7), we are interested in the integer Chebyshev constant

$$t_{\mathbb{Z}}([0,1]) := \lim_{n\to\infty} \left(\inf_{0\not\equiv p_n\in\mathbb{Z}_n[x]} \|p_n\|_{[0,1]}\right)^{1/n}. \tag{1.8}$$

It was found by Gorshkov [15] in 1956 that (1.7) can never be achieved. In fact, $0.4213 < t_{\mathbb{Z}}([0,1]) < 0.4232$ (see [24] for a survey of recent results on this problem). Thus the Gelfond-Schnirelman method failed in its original form, but one can generalize it for polynomials in many variables. Such an idea apparently had first appeared in Trigub [33], and was independently implemented by Nair [23] and Chudnovsky [5]. The basis of their argument lies in another equivalent form of the Prime Number Theorem [18]:

$$\int_1^x \psi(t)\,dt \sim \frac{x^2}{2} \quad \text{as } x \to +\infty. \tag{1.9}$$

Both Nair and Chudnovsky used the following weighted version of the Vandermonde determinant

$$\begin{aligned} V_n^w(x_1,\ldots,x_n) &:= \prod_{1\le i<j\le n} (x_i - x_j)w(x_i)w(x_j) \\ &= \prod_{i=1}^{n} w^{n-1}(x_i) \prod_{1\le i<j\le n} (x_i - x_j), \end{aligned} \tag{1.10}$$

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Kent State University
etna@mcs.kent.edu

where $x_i \in [0,1]$ and $w(x) = (x(1-x))^{\alpha_1}$, $\alpha_1 > 0$, to generate multivariate polynomials with small sup norms on the cube $[0,1]^n$. They obtained the numerical bound

$$\int_1^x \psi(t)\, dt \geq 0.99035\, \frac{x^2}{2} \quad \text{as } x \to +\infty, \tag{1.11}$$

produced by the optimal choice $\alpha_1 \approx 0.195$ (our notations differ from those of [23] and [5]). Chudnovsky [5] also indicated how this approach can be generalized for weights of the form

$$w(x) = \prod_{i=1}^{k} |Q_{m_i}(x)|^{\alpha_i}, \tag{1.12}$$

where $Q_{m_i} \in \mathbb{Z}_{m_i}[x]$ and $\alpha_i > 0$, $i = 1, \ldots, k$. We develop the ideas of [23] and [5], and establish a connection with the weighted potential theory (or potential theory with external fields) that originated in the work of Gauss [13] and Frostman [11] (see [29] for a modern account on this theory). An important part of the method is the analysis of the asymptotic behavior for the supremum norms of the weighted Vandermonde determinants (1.10), which is governed by the weighted capacity c_w of $[0,1]$ corresponding to the weight w (cf. Section 2 below and [29]). This method leads to the following lower bound for the integral of the ψ-function via c_w.

THEOREM 1.1. *Let $w(x)$ be as in (1.12) and let $\alpha := \sum_{i=1}^k \alpha_i m_i$. Then*

$$\int_1^x \psi(t)\, dt \geq \frac{-2\log c_w}{4\alpha + 3}\, \frac{x^2}{2} + O(x \log^2 x) \quad \textit{as } x \to +\infty. \tag{1.13}$$

We recover the results of Nair and Chudnovsky as a special case of Theorem 1.1.

COROLLARY 1.2. *If $w(x) = x^{\alpha_1}(1-x)^{\alpha_2}$, $x \in [0,1]$, $\alpha_1 = \alpha_2 = 0.195$, then $c_w \approx 0.1045575588$ and (1.11) holds true.*

It is natural to try improving the bound (1.11) by choosing a weight with a proper combination of factors $Q_{m_i}(x)$ and exponents α_i. The most interesting question is, of course, whether one can find a weight $w(x)$ of the form (1.12) such that

$$\frac{-2\log c_w}{4\alpha + 3} = 1?$$

It turns out this is impossible to achieve for any *fixed* weight of the type (1.12). The reason for such a conclusion transpires from the error term in (1.13), which is "too good." Indeed, it is known from Littlewood's theorem that the difference $\int_1^x \psi(t)\, dt - x^2/2$ takes both positive and negative values of the amplitude $cx^{3/2}$, $c > 0$, infinitely often as $x \to +\infty$. This is conveniently written in the notation

$$\int_1^x \psi(t)\, dt - \frac{x^2}{2} = \Omega_{\pm}(x^{3/2}) \quad \text{as } x \to +\infty;$$

cf. [18, pp. 91-92]. Hence the correct error term should be of the order $O(x^{3/2})$. Relating this to (1.9) and (1.13), we obtain in such an indirect way the following.

PROPOSITION 1.3. *Given a weight $w(x)$ of the form (1.12), we have*

$$B(w) := \frac{-2\log c_w}{4\alpha + 3} < 1, \tag{1.14}$$

where $\alpha = \sum_{i=1}^k \alpha_i m_i$.

We should also note that if the Riemann hypothesis is true, then

$$\int_1^x \psi(t)\,dt - \frac{x^2}{2} = O(x^{3/2}) \quad \text{as } x \to +\infty;$$

see Theorem 30 in [18, p. 83]. It would be very interesting to find a direct potential theoretic argument explaining (1.14). Although (1.13) cannot provide a proof of the PNT for a fixed weight w, this does not preclude the possibility that such a proof can be obtained by finding a *sequence* of weights w_n with $B(w_n) \to 1$, as $n \to \infty$. On the other hand, we did not observe a numerical improvement of the estimate (1.11) when using further factors of the one-dimensional integer Chebyshev polynomials for the weight w, beyond the factors x and $1-x$ (see [20], [5] and [24]). Thus one needs a better insight into the arithmetic nature of such factors, to address the problem stated below.

PROBLEM 1.4. *For $w(x)$ as in (1.12) and $\alpha = \sum_{i=1}^k \alpha_i m_i$, find*

$$B := \sup_w \frac{-2\log c_w}{4\alpha+3}. \tag{1.15}$$

If $B = 1$ then find a sequence of weights that gives this value. If $B < 1$ then investigate whether B is attained for a weight of the form (1.12).

The solution of this problem also requires a detailed knowledge of potential theory with external fields generated by the weights (1.12), which is considered below. Our results discussed in this section were originally obtained in [25].

2. Potential theory with external fields. We consider a special case of the weighted energy problem on a segment of the real line $[a,b]$, which is associated with the "polynomial-type" weights (1.12). A comprehensive treatment of potential theory with external fields, or weighted potential theory, is contained in the book of Saff and Totik [29], together with historical remarks and numerous references. It is convenient to rewrite the weight function in the following more general form:

$$w(x) = A\prod_{i=1}^K |x-z_i|^{p_i}, \qquad x \in [a,b], \tag{2.1}$$

where $A > 0,\ p_i > 0$ and $z_i \in \mathbb{C}$. Let $\mathcal{M}([a,b])$ be the set of positive unit Borel measures supported on $[a,b]$. For any measure $\mu \in \mathcal{M}([a,b])$ and weight w of the form (2.1), we define the energy functional

$$\begin{aligned} I_w(\mu) &:= \iint \log\frac{1}{|z-t|w(z)w(t)}\,d\mu(z)d\mu(t) \\ &= \iint \log\frac{1}{|z-t|}\,d\mu(z)d\mu(t) - 2\int \log w(t)\,d\mu(t), \end{aligned} \tag{2.2}$$

and consider the minimum energy problem

$$V_w := \inf_{\mu\in\mathcal{M}([a,b])} I_w(\mu). \tag{2.3}$$

It follows from Theorem I.1.3 of [29] that V_w is finite, and there exists a unique equilibrium measure $\mu_w \in \mathcal{M}([a,b])$ such that $I_w(\mu_w) = V_w$. Thus μ_w minimizes the energy functional (2.2) in presence of the external field generated by the weight w. Furthermore, we have for the potential of μ_w that

$$U^{\mu_w}(x) - \log w(x) \ge F_w, \qquad x \in [a,b], \tag{2.4}$$

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Kent State University
etna@mcs.kent.edu

and

$$U^{\mu_w}(x) - \log w(x) = F_w, \qquad x \in S_w, \tag{2.5}$$

where $U^{\mu_w}(x) := -\int \log|x-t|\,d\mu_w(t)$, $F_w := V_w + \int \log w(t) d\mu_w(t)$ and $S_w :=$ supp μ_w; see Theorems I.1.3 and I.5.1 in [29]. The weighted capacity of $[a,b]$ is defined by

$$\text{cap}([a,b],w) := e^{-V_w}. \tag{2.6}$$

In agreement with the notation of Section 1, we set

$$c_w := \text{cap}([0,1],w).$$

If $w \equiv 1$ on $[a,b]$, then we obtain the classical logarithmic capacity $\text{cap}([a,b],1) = (b-a)/4$; cf. [27].

The support S_w plays a crucial role in determining the equilibrium measure μ_w itself, as well as other components of this weighted energy problem. Indeed, if S_w is known then μ_w can be found as a solution of the singular integral equation

$$\int \log \frac{1}{|x-t|}\,d\mu(t) - \log w(x) = F, \qquad x \in S_w,$$

where F is a constant; cf. (2.5) and [29, Ch. IV]. For w given by (2.1) or (1.12), this equation can be solved by potential theoretic methods, using balayage techniques, so that μ_w is expressed as a linear combination of harmonic measures; see [24] and [25]. We follow another path here, via the methods of singular integral equations, which gives a more explicit solution. This approach for polynomial-type weights with real zeros was suggested by Chudnovsky [5] and developed further by Amoroso [1]. For more general weights, one should consult Chapter IV of [29] and the paper of Deift, Kreicherbauer and McLaughlin [7]. The first complete solution of the weighted energy problem for the weight (2.1) with *real* zeros z_i was found by Martínez-Finkelshtein and Saff [19]. In fact, they considered a related problem for vector potentials, which implies the solution of our problem. The solution was also obtained in [25] by a different method, but still for the case of real zeros. We present a general result for the weight (2.1) with *complex* zeros in the following theorem. Furthermore, we give a self contained proof of this result, which is considerably simpler than previously known arguments; see, e.g., [7]. It is based on the ideas of [26].

THEOREM 2.1. *Let $Z := \bigcup_{i=1}^{K}\{z_i\}$ be the set of zeros for w of (2.1), where $z_1 = a$ and $z_K = b$. There exist an integer L, $1 \le L \le K-1$, and L intervals $[a_l, b_l] \subset [a,b] \setminus Z$, with $a < a_1 < b_1 < a_2 < b_2 < \ldots < a_L < b_L < b$, such that*

$$S_w = \bigcup_{l=1}^{L} [a_l, b_l].$$

Set $R(z) := \prod_{l=1}^{L}(z-a_l)(z-b_l)$, and consider the branch of $\sqrt{R(z)}$ defined in the domain $\overline{\mathbb{C}} \setminus S_w$ by $\lim_{z\to\infty} \sqrt{R(z)}/z^L = 1$. By the values of $\sqrt{R(z)}$ on S_w, we understand the limiting values from the upper half plane. Let

$$F(z) := \frac{\sqrt{R(z)}}{2\pi i} \sum_{k=1}^{K} p_k \left(\frac{1}{(z_k - z)\sqrt{R(z_k)}} + \frac{1}{(\bar{z}_k - z)\sqrt{R(\bar{z}_k)}} \right). \tag{2.7}$$

ETNA
Kent State University
etna@mcs.kent.edu

Then

$$d\mu_w(x) = F(x)\,dx, \qquad x \in S_w. \tag{2.8}$$

Furthermore, the endpoints of S_w satisfy the equations

$$\sum_{k=1}^{K} p_k \left(\frac{z_k^l}{\sqrt{R(z_k)}} + \frac{\bar{z}_k^l}{\sqrt{R(\bar{z}_k)}} \right) = 0, \qquad l = 0, \ldots, L-1, \tag{2.9}$$

$$\sum_{k=1}^{K} \frac{p_k}{2} \left(\frac{z_k^L}{\sqrt{R(z_k)}} + \frac{\bar{z}_k^L}{\sqrt{R(\bar{z}_k)}} \right) = 1 + \sum_{k=1}^{K} p_k, \tag{2.10}$$

and the equations

$$\int_{b_l}^{a_{l+1}} F(x)\,dx = 0, \qquad l = 1, \ldots, L-1, \tag{2.11}$$

where the integrals are defined as Cauchy principal values for $z_k \in [b_l, a_{l+1}]$.

Recall that we need the quantity $-\log \operatorname{cap}([a,b], w) = V_w$ for Theorem 1.1. This can be found from (2.5) as

$$V_w = U^{\mu_w}(x) - \log w(x) - \int \log w \, d\mu_w,$$

for any $x \in S_w$.

The assumption that the weight w vanishes at the endpoints of $[a, b]$ seems appropriate in this case, due to the role of the factors x and $1-x$, for $w(x)$ on $[0,1]$, in the work of Nair and Chudnovsky. The general case of weights (2.1) can be handled similarly, along the lines of this paper. One can predict the possible forms of the result, if the conditions $z_1 = a$ and $z_K = b$ are dropped, from Theorem 1.38 of [7]. The density of μ_w may behave like $(x-a)^{-1/2}$ (or $(b-x)^{-1/2}$) near the endpoints of $[a, b]$, which is similar to the classical (not weighted) case.

Note that equations (2.9)-(2.11) may be used to find the endpoints of S_w. For example, if $K = 2$, i.e., we have the so-called Jacobi-type weight w on $[a, b]$, then $L = 1$ and (2.9) gives just two equations for the endpoints of $S_w = [a_1, b_1]$. It is easy to solve them explicitly, and find the well known representation for S_w and μ_w; see, e.g., Examples IV.1.17 and IV.5.2 of [29].

COROLLARY 2.2. *Suppose $w(x) = x^{p_1}(1-x)^{p_2}$, $x \in [0, 1]$, where $p_1, p_2 > 0$. Then*

$$d\mu_w(x) = \frac{(1 + p_1 + p_2)\sqrt{(x-a_1)(b_1-x)}}{\pi x(1-x)}\,dx, \qquad x \in [a_1, b_1]. \tag{2.12}$$

The endpoints of the support are

$$a_1 = \frac{1 + r_1^2 - r_2^2 - \sqrt{(1 + r_1^2 - r_2^2)^2 - 4r_1^2}}{2} \tag{2.13}$$

and

$$b_1 = \frac{1 + r_1^2 - r_2^2 + \sqrt{(1 + r_1^2 - r_2^2)^2 - 4r_1^2}}{2}, \tag{2.14}$$

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Kent State University
etna@mcs.kent.edu

where we set

$$r_1 := \frac{p_1}{1+p_1+p_2} \quad \textit{and} \quad r_2 := \frac{p_2}{1+p_1+p_2}.$$

The connection between potential theory with external fields and this version of the Gelfond-Schnirelman method arises in the need for asymptotics of the weighted Vandermonde determinant (1.10). It is known that

$$\lim_{n\to\infty}\left(\max_{x_1,\ldots,x_n\in[a,b]}|V_n^w(x_1,\ldots,x_n)|\right)^{\frac{2}{n(n-1)}} = \mathrm{cap}([a,b],w); \tag{2.15}$$

see Theorem III.1.3 of [29]. The quantity on the left-hand side of (2.15) is called the weighted transfinite diameter of $[a,b]$. In the case $w\equiv 1$, it was introduced by Fekete [10] for arbitrary compact sets in the plane. Szegő [31] showed that the transfinite diameter coincides with the logarithmic capacity, so that (2.15) is a generalization of his result. We quantify the rate of convergence in (2.15).

LEMMA 2.3. *Let w be as in* (2.1). *There exist constants $d = d(w) > 1$ and $D = D(w) > 0$ such that*

$$(\mathrm{cap}([a,b],w))^{n(n-1)} \le \max_{[a,b]^n}(V_n^w)^2 \le D\, d^{n\log^2 n}\,(\mathrm{cap}([a,b],w))^{n(n-1)}. \tag{2.16}$$

Equation (2.16) is the only fact from potential theory needed in the proof of Theorem 1.1. It is very likely that $\log^2 n$ can be replaced by $\log n$, matching the classical case; see Theorem 1.3.3 in [2]. This would give a corresponding improvement in the error term of (1.13), but we do not pursue this direction.

3. Proofs. *Proof of Theorem* 1.1. The proof is based on an argument similar to the original Gelfond-Schnirelman idea (cf. [23] and [5]). We consider the integrals of small polynomials with integer coefficients over the cube $[0,1]^n$, $n\in\mathbb{N}$. It is important that the integrals are non-zero, so that we work with the square of the weighted Vandermonde determinant (1.10), instead:

$$\Delta_n^w(x_1,\ldots,x_n) := (V_n^w)^2 = \prod_{i=1}^n w^{2(n-1)}(x_i)\prod_{1\le i<j\le n}(x_i-x_j)^2. \tag{3.1}$$

If $w(x)\equiv 1$ then Δ_n^w is the classical discriminant. In general, Δ_n^w is not a polynomial in x_i's because of the real exponents α_i's in the weight (1.12). Hence we modify it further into

$$\tilde\Delta_n^w(x_1,\ldots,x_n) := \prod_{j=1}^n\prod_{i=1}^k (Q_{m_i}(x_j))^{2\lceil\alpha_i(n-1)\rceil}\prod_{1\le i<j\le n}(x_i-x_j)^2, \tag{3.2}$$

where $\lceil a\rceil$ denotes the ceiling function: the smallest integer at least a. It is now clear that $\tilde\Delta_n^w(x_1,\ldots,x_n)$ is a positive polynomial with integer coefficients that has the following form

$$\tilde\Delta_n^w(x_1,\ldots,x_n) = \sum a_{i_1\ldots i_n}x_1^{i_1}\ldots x_n^{i_n}. \tag{3.3}$$

Recall the definition of the classical Vandermonde determinant

$$V_n := \begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1}\\ 1 & x_2 & \ldots & x_2^{n-1}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & x_n & \ldots & x_n^{n-1}\end{vmatrix} = \prod_{1\le i<j\le n}(x_i-x_j).$$

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Kent State University
etna@mcs.kent.edu

Using the standard expansion of V_n, we observe that each term of this expansion is a (signed) product of all powers of the x_i's from 0 to $n-1$. The expression in (3.2) is equal to V_n^2 times the weight part. Thus if we arrange the powers $i_1, \ldots, i_n$ in every term of (3.3) in an increasing order, we have that

$$i_j \le n + j - 2 + N, \quad j = 1, \ldots, n, \tag{3.4}$$

where $N := 2\sum_{i=1}^k m_i \lceil \alpha_i(n-1) \rceil$ is the contribution of the weight part in (3.2). Hence

$$\int_0^1 \ldots \int_0^1 \tilde{\Delta}_n^w(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \sum \frac{a_{i_1 \ldots i_n}}{(i_1+1)\ldots(i_n+1)} \ne 0$$

is a rational number whose denominator divides $\prod_{l=n+N}^{2n-1+N} \operatorname{lcm}(1, \ldots, l)$ by (3.4). It follows as in (1.5) that

$$\prod_{l=n+N}^{2n-1+N} \operatorname{lcm}(1, \ldots, l) \int_{[0,1]^n} \tilde{\Delta}_n^w \ge 1.$$

On taking the logarithm, we obtain that

$$\sum_{l=n+N}^{2n-1+N} \psi(l) \ge -\log \int_{[0,1]^n} \tilde{\Delta}_n^w \ge -\log \max_{[0,1]^n} \tilde{\Delta}_n^w.$$

It is clear from (3.1) and (3.2) that

$$\tilde{\Delta}_n^w = \Delta_n^w \prod_{j=1}^n \prod_{i=1}^k |Q_{m_i}(x_j)|^{2(\lceil \alpha_i(n-1) \rceil - \alpha_i(n-1))},$$

which gives

$$\max_{[0,1]^n} \tilde{\Delta}_n^w \le \max_{[0,1]^n} \Delta_n^w \prod_{i=1}^k \left(\max(1, \|Q_{m_i}\|_{[0,1]}) \right)^{2n}.$$

Since $\psi(x)$ is constant between integers, we arrive at the estimate

$$\int_{n+N}^{2n+N} \psi(y)\, dy \ge -\log \max_{[0,1]^n} \Delta_n^w + O(n) \quad \text{as } n \to \infty. \tag{3.5}$$

We now need the following consequence of Lemma 2.3:

$$\log \max_{[0,1]^n} \Delta_n^w = n^2 \log c_w + O(n \log^2 n) \quad \text{as } n \to \infty.$$

Applying this in (3.5), we have

$$\int_{n+N}^{2n+N} \psi(y)\, dy \ge -n^2 \log c_w + O(n \log^2 n) \quad \text{as } n \to \infty.$$

Note that $2\alpha(n-1) \le N \le 2\alpha(n-1) + 2\sum_{i=1}^k m_i$. If we set $2n(\alpha+1) = x$, then

$$\int_{\frac{2\alpha+1}{2\alpha+2}x}^{x} \psi(y)\, dy \ge -\frac{\log c_w}{4(\alpha+1)^2}\, x^2 + O(x \log^2 x) \quad \text{as } x \to \infty. \tag{3.6}$$

Let $q := \frac{2\alpha+1}{2\alpha+2}$. Recall that $\psi(y)$ is a positive increasing function satisfying $\psi(y) = O(y)$ as $y \to \infty$. Therefore,

$$\int_{qx+a}^{x+b} \psi(y)\,dy = \int_{qx}^{x} \psi(y)\,dy + O(x) \quad \text{as } x \to \infty$$

for any pair of constants $a, b \in \mathbb{R}$. It follows that (3.6) holds for any $x \in \mathbb{R},\ x \to \infty$, because it holds along the arithmetic sequence $x = 2n(\alpha+1),\ n \in \mathbb{N}$.

Given $x > 1$, we choose $l \in \mathbb{N} \cup \{0\}$ so that $q^{l+1}x < \sqrt{x} \le q^l x$. Using the substitution $x \to qx$ iteratively in (3.6) and summing up the results, we obtain

$$\begin{aligned}
\int_{\sqrt{x}}^{x} \psi(y)\,dy &\ge \frac{-x^2 \log c_w}{4(\alpha+1)^2}\,\frac{1-q^{2l}}{1-q^2} + O\left(\sum_{k=0}^{l-1} q^k x \log^2 q^k x\right) \\
&\ge \frac{-\log c_w}{4\alpha+3}\, x^2 + O(x) + O\left(\sum_{k=0}^{l-1} q^k x \log^2 x\right) \\
&= \frac{-\log c_w}{4\alpha+3}\, x^2 + O\left(x \log^2 x\right) \quad \text{as } x \to \infty.
\end{aligned}$$

Hence

$$\int_{1}^{x} \psi(y)\,dy \ge \frac{-\log c_w}{4\alpha+3}\, x^2 + O(x\log^2 x) \quad \text{as } x \to \infty. \qquad \square$$

Proof of Corollary 1.2. Corollary 2.2 gives the weighted equilibrium measure μ_w for such weights in (2.12)-(2.14). Hence we have by (2.5) that the numerical value of c_w can be computed from

$$-\log c_w = U^{\mu_w}(a_1) - \log w(a_1) - \int \log w\, d\mu_w,$$

where a_1 is defined in (2.13). The same equation yields (1.11), as a consequence of Theorem 1.1. $\square$

LEMMA 3.1. *Let w be as in Theorem 2.1. The weighted equilibrium measure μ_w is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$:*

$$d\mu_w(x) = f(x)\,dx, \quad x \in S_w \subset [a,b],$$

where $f \in L_\infty([a,b])$.

Proof. Our first goal is to show that U^{μ_w} is Lipschitz continuous in $\mathbb{C}$, which implies that the directional derivatives of U^{μ_w} exist a.e. on $\mathbb{R}$, and

$$d\mu_w(t) = -\frac{1}{2\pi}\left(\frac{\partial U^{\mu_w}}{\partial \mathbf{n}_-}(t) + \frac{\partial U^{\mu_w}}{\partial \mathbf{n}_+}(t)\right)\, dt$$

by Theorem II.1.5 of [29], where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the upper and lower normals to $\mathbb{R}$. Since the normal derivatives of U^{μ_w} are bounded by the Lipschitz constant, we obtain that

$$f(t) = -\frac{1}{2\pi}\left(\frac{\partial U^{\mu_w}}{\partial \mathbf{n}_-}(t) + \frac{\partial U^{\mu_w}}{\partial \mathbf{n}_+}(t)\right),$$

with $f \in L_\infty([a,b])$.

ETNA
Kent State University
etna@mcs.kent.edu

Recall that S_w cannot contain zeros of w, and $\log w(t)$ is a C^2 function in an open neighborhood of S_w in $\mathbb{R}$. We can modify $w(t)$ outside $[a,b]$ and in small neighborhoods of zeros on $[a,b]$, so that for the resulting function $v(t)$ we still have

$$U^{\mu_w}(t) - \log v(t) \geq F_w, \qquad t \in \mathbb{R},$$

and

$$U^{\mu_w}(t) - \log v(t) = F_w, \qquad t \in S_w,$$

because $w|_{S_w} = v|_{S_w}$ (cf. (2.4) and (2.5)), and we also have $\log v(t) \in C^2(\mathbb{R}) \cap L_1(\mathbb{R})$. This may be achieved via replacing $w(t)$ by a sufficiently small positive C^2 function in a neighborhood of every zero $z_i \in [a,b]$, and via continuing the resulting function smoothly with ce^{-x^2} outside $[a,b]$, where $c > 0$ is small. Theorem I.3.3 of [29] implies that $\mu_v = \mu_w$ and $F_v = F_w$. Thus we can work with v instead. Let u be a solution of the Dirichlet problem in the upper half plane H_+ for the boundary data $\log v(t) + F_w$. Then $u \in C^1(H_+ \cup \mathbb{R})$ by an application of Privalov's theorem (see §5 of Chap. IX in [14]) and a conformal mapping of H_+ onto the unit disk. Since $u|_{S_w} = U^{\mu_w}|_{S_w}$ and $u|_{\mathbb{R}} \leq U^{\mu_w}|_{\mathbb{R}}$ by our construction, we obtain that

$$u(z) \leq U^{\mu_w}(z), \qquad z \in H_+ \cup \mathbb{R},$$

as U^{μ_w} is superharmonic. If we continue u by setting $u(z) := u(\bar{z})$, $\Im z < 0$, then $u \in C^1(\mathbb{C})$. Furthermore,

$$u(z) \leq U^{\mu_w}(z), \qquad z \in \mathbb{C},$$

because $U^{\mu_w}(z) = U^{\mu_w}(\bar{z})$. In the proof of Lipschitz continuity of U^{μ_w}, we first consider $z \in S_w$ and $\zeta \in \mathbb{C}$, and follow an idea of Götz [16]. It is clear from the previous inequality that

$$U^{\mu_w}(z) - U^{\mu_w}(\zeta) \leq u(z) - u(\zeta) \leq C|z - \zeta|, \quad z \in S_w,\ \zeta \in \mathbb{C}, \tag{3.7}$$

where C is the Lipschitz constant for u on $H_+ \cup \mathbb{R}$. In order to prove a matching estimate from below, we consider a nearest point $\zeta^* \in S_w$ for ζ, i.e., $\text{dist}(\zeta, S_w) = |\zeta - \zeta^*| =: r$. Then

$$\begin{aligned} U^{\mu_w}(z) - U^{\mu_w}(\zeta) &= u(z) - u(\zeta^*) + U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta) \\ &\geq -C|z - \zeta^*| + U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta), \quad z \in S_w,\ \zeta \in \mathbb{C}. \end{aligned} \tag{3.8}$$

Using the area mean-value inequality, we obtain that

$$\begin{aligned} U^{\mu_w}(\zeta^*) &\geq \frac{1}{\pi(2r)^2} \int_{D_{2r}(\zeta^*)} U^{\mu_w}(x+iy)\,dxdy \\ &= \frac{1}{4\pi r^2} \int_{D_{2r}(\zeta^*)\setminus D_r(\zeta)} U^{\mu_w}(x+iy)\,dxdy + \frac{1}{4} U^{\mu_w}(\zeta), \end{aligned} \tag{3.9}$$

where the second term comes from the mean-value property for the harmonic function U^{μ_w} in $D_r(\zeta)$. Note that $U^{\mu_w}(\xi) \geq U^{\mu_w}(\zeta^*) - C|\zeta^* - \xi|$, $\xi \in \mathbb{C}$, by (3.7). Hence (3.9) implies that

$$U^{\mu_w}(\zeta^*) \geq \frac{4\pi r^2 - \pi r^2}{4\pi r^2} \left(U^{\mu_w}(\zeta^*) - C2r\right) + \frac{1}{4} U^{\mu_w}(\zeta),$$

ETNA
Kent State University
etna@mcs.kent.edu

and that

$$U^{\mu_w}(\zeta^*) - U^{\mu_w}(\zeta) \geq -6Cr.$$

Applying this in (3.8), we have

$$\begin{aligned} U^{\mu_w}(z) - U^{\mu_w}(\zeta) &\geq -C|z-\zeta^*| - 6C|\zeta-\zeta^*| \\ &\geq -C|z-\zeta| - 7C|\zeta-\zeta^*| \geq -8C|z-\zeta|. \end{aligned}$$

Consequently,

$$|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| \leq 8C|z-\zeta|, \quad z \in S_w,\ \zeta \in \mathbb{C}. \tag{3.10}$$

We now show that (3.10) is true for any $z, \zeta \in \mathbb{C}$. Observe that

$$\sup\{|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| : |z-\zeta| \leq \delta,\ z,\zeta \in \mathbb{C}\} = |U^{\mu_w}(z_0) - U^{\mu_w}(\zeta_0)|$$

for some $z_0, \zeta_0 \in \mathbb{C},\ |z_0 - \zeta_0| \leq \delta$, because

$$\lim_{\substack{z,\zeta\to\infty \\ |z-\zeta|\leq\delta}} (U^{\mu_w}(z) - U^{\mu_w}(\zeta)) = 0.$$

Consider $h(\xi) := U^{\mu_w}(\xi) - U^{\mu_w}(\xi - z_0 + \zeta_0)$, which is continuous on $\overline{\mathbb{C}}$ and harmonic in $\overline{\mathbb{C}} \setminus S_w$. By the maximum-minimum principle, we have

$$|U^{\mu_w}(z_0) - U^{\mu_w}(\zeta_0)| = |h(z_0)| \leq \max_{\xi \in S_w} |h(\xi)| = |h(\xi_0)| \leq 8C|z_0 - \zeta_0|,$$

where $\xi_0 \in S_w$, and where the last inequality follows from (3.10). Thus

$$\sup\{|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| : |z-\zeta| \leq \delta,\ z,\zeta \in \mathbb{C}\} \leq 8C\delta,$$

i.e., U^{μ_w} is Lipschitz continuous in $\mathbb{C}$. ∎

LEMMA 3.2. *For the weight w defined in Theorem 2.1, the density of μ_w satisfies*

$$f^2(x) = \frac{P(x)}{(x-a)^2(x-b)^2\prod_{k=2}^{K-1}|x-z_k|^4}, \qquad x \in S_w,$$

where $P(x)$ is a polynomial of degree at most $4K-5$. Furthermore, S_w consists of at most $2K-3$ intervals on $\mathbb{R}$, and f vanishes at the endpoints of those intervals.

Proof. The equilibrium equation (2.5) and Lemma 3.1 give that

$$-\int_a^b f(t)\log|x-t|\,dt = F_w + \log w(x), \qquad x \in S_w.$$

Differentiating this equation with respect to x, we obtain that

$$-\int_a^b \frac{f(t)\,dt}{x-t} = \frac{w'(x)}{w(x)}$$

for almost every $x \in S_w$ (cf. Lemma 2.45 of [7] for the justification of differentiation under the integral). Hence

$$\tilde{f}(x) = -\frac{1}{\pi}\frac{w'(x)}{w(x)} \qquad \text{a.e. on } S_w, \tag{3.11}$$

ETNA
Kent State University
etna@mcs.kent.edu

where

$$\tilde{f}(x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)\,dt}{x-t} = \frac{1}{\pi}\int_a^b \frac{f(t)\,dt}{x-t}$$

is the harmonic conjugate (or the Hilbert transform) of f, see [12, Chap. 3]. Since $f \in L_\infty([a,b])$, we have that $\tilde{f} \in L_p(\mathbb{R})$ for any $p < \infty$, by M. Riesz's theorem. Consider the function

$$G(z) := \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(t)\,dt}{t-z}, \qquad z \in H_+,$$

which is analytic in H_+, and recall that $G(z) = u(z) + i\tilde{u}(z)$, where u and $\tilde{u}$ have the boundary values (cf. [12])

$$u|_{\mathbb{R}} = f \quad \text{and} \quad \tilde{u}|_{\mathbb{R}} = \tilde{f}.$$

Clearly, the function

$$-iG^2(z) = 2u(z)\tilde{u}(z) + i(\tilde{u}^2(z) - u^2(z)), \qquad z \in H_+,$$

is analytic in H_+. It follows that the harmonic conjugate of $2u\tilde{u}$ is $\tilde{u}^2 - u^2$:

$$(2u\tilde{u})\tilde{} = \tilde{u}^2 - u^2.$$

Passing to the boundary values, we obtain that

$$2(f\tilde{f})\tilde{} = \tilde{f}^2 - f^2 \qquad \text{a.e. on } \mathbb{R},$$

and that

$$-\frac{2\,w'}{\pi w}\tilde{f} - \left(\frac{w'}{\pi w}\right)^2 - 2(f\tilde{f})\tilde{} = f^2 - \left(\tilde{f} + \frac{w'}{\pi w}\right)^2 \qquad \text{a.e. on } \mathbb{R}. \tag{3.12}$$

Denote the left hand side of (3.12) by g. Observe that the right hand side gives a decomposition for g into the positive part f^2 and the negative part $-\left(\tilde{f} + w'/(\pi w)\right)^2$, because of (3.11) and $f(t) = 0,\ t \notin S_w$. Hence $S_w = \overline{\{x : g(x) > 0\}}$ and, using (3.11) again, we obtain that

$$\begin{aligned} f^2(x) &= \left(\frac{w'(x)}{\pi w(x)}\right)^2 + 2\left(f\frac{w'}{\pi w}\right)\tilde{}(x) = \left(\frac{w'(x)}{\pi w(x)}\right)^2 + \frac{2}{\pi^2}\int_a^b \frac{w'(t)f(t)\,dt}{w(t)(x-t)} \\ &= \frac{2}{\pi^2}\int_a^b \left(\frac{w'(t)}{w(t)} - \frac{w'(x)}{w(x)}\right)\frac{f(t)\,dt}{x-t} - \left(\frac{w'(x)}{\pi w(x)}\right)^2 \end{aligned} \tag{3.13}$$

for a.e. $x \in S_w$. It is immediate to see that

$$\frac{w'(x)}{w(x)} = \sum_{k=1}^K p_k \frac{x-x_k}{(x-x_k)^2 + y_k^2} = \sum_{k=1}^K p_k \frac{x-x_k}{|x-z_k|^2}, \tag{3.14}$$

where $z_k = x_k + iy_k,\ k = 1,\ldots,K$. If we recall that $z_1 = x_1 = a$ and $z_K = x_K = b$, and insert the above representation in (3.13), it becomes clear that

$$f^2(x) = \frac{P(x)}{(x-a)^2(x-b)^2\prod_{k=2}^{K-1}|x-z_k|^4}, \qquad x \in S_w,$$

ETNA
Kent State University
etna@mcs.kent.edu

where $P(x)$ is a polynomial of degree at most $4K-5$. Since $P(x)$ has at most $4K-5$ zeros, and $S_w = \overline{\{x : g(x) > 0\}} \subset \overline{\{x : P(x) > 0\}}$, we conclude that S_w consists of at most $2K-3$ subintervals of $[a,b]$. Naturally, P and f vanish at the endpoints of those intervals. $\square$

Proof of Theorem 2.1. It is clear from (2.5) that $S_w \subset [a,b] \setminus Z$. Lemma 3.2 gives that $S_w = \cup_{l=1}^{L}[a_l, b_l]$, where $L \le 2K-1$. Furthermore, we have that $d\mu_w(x) = f(x)\,dx$, where $f \in C^\infty\left(\cup_{l=1}^{L}(a_l,b_l)\right)$, and f is Hölder continuous on S_w with exponent 1/2. Note that, by (3.11), f satisfies the singular integral equation with Cauchy kernel

$$\frac{1}{\pi i}\int_{S_w}\frac{f(t)\,dt}{t-x} = \frac{1}{\pi i}\frac{w'(x)}{w(x)} \tag{3.15}$$

for a.e. $x \in S_w$. The Hölder continuity of f implies that of $\tilde{f}$ (and of the Cauchy or the Hilbert transforms), see [12, Chap. 3] or [22, §22]. Hence both sides of (3.15) are continuous, and this equation holds for all $x \in S_w$. Denote the right-hand side of (3.15) by $h(x)$. Since, by Lemma 3.2, f vanishes at the endpoints of S_w, we obtain from the results of §88 in [22, Chap. 11] (see also [21]) that f must be the unique solution of (3.15) given by

$$f(x) = \frac{\sqrt{R(x)}}{\pi i}\int_{S_w}\frac{h(t)\,dt}{\sqrt{R(t)}(t-x)}, \qquad x \in S_w, \tag{3.16}$$

where $\sqrt{R(x)}$ is defined in the statement of Theorem 2.1, and the integral is defined as the Cauchy principal value. For further reference, we describe the values of $\sqrt{R(z)}$ on the real line:

$$\sqrt{R(x)} = \begin{cases} \sqrt{|R(x)|}, & x \ge b_L, \\ (-1)^{L+l}\, i\, \sqrt{|R(x)|}, & a_l \le x \le b_l,\ l = 1,\dots,L, \\ (-1)^{L+l}\sqrt{|R(x)|}, & b_l \le x \le a_{l+1},\ l = 1,\dots,L-1, \\ (-1)^{L}\sqrt{|R(x)|}, & x \le a_1. \end{cases} \tag{3.17}$$

Here and throughout, the values of $\sqrt{R(x)}$ for $x \in \cup_{l=1}^{L}[a_l,b_l]$ are understood as the upper limiting values of $\sqrt{R(z)}$, when $\Im z \to 0^+$.

Using (3.14), we obtain that

$$h(t) = \frac{1}{\pi i}\sum_{k=1}^{K} p_k \frac{t-x_k}{|t-z_k|^2} = \frac{1}{2\pi i}\sum_{k=1}^{K} p_k \frac{(t-z_k)+(t-\bar{z}_k)}{(t-z_k)(t-\bar{z}_k)},$$

where $t \in \mathbb{R}$. Replacing t by $z \in \mathbb{C}$ gives an extension of h into the complex plane of the form

$$h(z) = \frac{1}{2\pi i}\sum_{k=1}^{K} p_k \left(\frac{1}{z-z_k} + \frac{1}{z-\bar{z}_k}\right), \tag{3.18}$$

which is useful for the evaluation of the integral in (3.16). Note that the limiting boundary values of $\sqrt{R(z)}$ on the intervals of S_w, from above and below, are opposite in sign. Hence the same is true for the function $h(z)/((z-x)\sqrt{R(z)})$, which is analytic in $\overline{\mathbb{C}} \setminus S_w$ except for the simple poles at z_k and $\bar{z}_k$, $k = 1,\dots,K$. Passing to the contour integral over both

ETNA
Kent State University
etna@mcs.kent.edu

sides of the cut S_w, and computing the residues at z_k and $\bar{z}_k$, we obtain from (3.16) that

$$\begin{aligned} f(x) &= \frac{\sqrt{R(x)}}{2\pi i} \oint_{S_w} \frac{h(z)\,dz}{\sqrt{R(z)}(z-x)} \\ &= \frac{\sqrt{R(x)}}{2\pi i} \sum_{k=1}^{K} p_k \left(\frac{1}{(z_k - x)\sqrt{R(z_k)}} + \frac{1}{(\bar{z}_k - x)\sqrt{R(\bar{z}_k)}} \right), \quad x \in S_w. \end{aligned}$$

Thus (2.8) is proved. One can easily produce the following alternative forms for f:

$$\begin{aligned} f(x) &= \frac{\sqrt{R(x)}}{\pi i} \sum_{k=1}^{K} p_k \frac{\Re\left((z_k - x)\sqrt{R(z_k)}\right)}{|x - z_k|^2\,|R(z_k)|} \\ &= \frac{\sqrt{|R(x)|}}{\pi} \left| \sum_{k=1}^{K} p_k \frac{\Re\left((x - z_k)\sqrt{R(z_k)}\right)}{|x - z_k|^2\,|R(z_k)|} \right|, \quad x \in S_w. \end{aligned} \tag{3.19}$$

Hence

$$f(x) = \frac{T(x)\sqrt{R(x)}}{i(x-a)(x-b)\prod_{k=2}^{K-1} |x - z_k|^2}, \quad x \in S_w, \tag{3.20}$$

where $T(x)$ is a polynomial with real coefficients. It is clear from (3.17) that $\sqrt{R(x)}/i$ is real, and alternates sign on the intervals of $S_w = \cup_{l=1}^{L}[a_l, b_l]$. Therefore, $T(x)$ must alternate sign too, and must have at least one zero between the neighboring intervals $[a_l, b_l]$ and $[a_{l+1}, b_{l+1}]$. Consequently, $\deg(T) \geq L - 1$. Comparing (3.20) and the representation for f^2 from Lemma 3.2, we obtain that $\deg(T^2 R) \leq 4K - 5$ and $2\deg(T) + 2L \leq 4K - 5$. Thus $2(L-1) + 2L \leq 4K - 6$ and $L \leq K - 1$, as stated in the theorem.

In the remaining part, we prove (2.9)-(2.11). Since f is the solution of (3.15) bounded at the endpoints of S_w (and vanishing there, in fact), h must satisfy the following moment conditions

$$\int_{S_w} \frac{x^l h(x)\,dx}{\sqrt{R(x)}} = 0, \qquad l = 0, \ldots, L-1,$$

by §88 of [22, Chap. 11]. These integrals are found by passing to the contour integrals over both sides of the cut S_w, and by computing residues at z_j and $\bar{z}_j$:

$$\begin{aligned} \frac{1}{\pi i} \int_{S_w} \frac{z^l h(z)\,dz}{\sqrt{R(z)}} &= \frac{1}{2\pi i} \oint_{S_w} \frac{z^l h(z)\,dz}{\sqrt{R(z)}} \\ &= \frac{1}{2\pi i} \sum_{j=1}^{K} p_j \left(\frac{z_j^l}{\sqrt{R(z_j)}} + \frac{\bar{z}_j^l}{\sqrt{R(\bar{z}_j)}} \right). \end{aligned}$$

Hence (2.9) follows from the above moment conditions.

Consider

$$G(z) := \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{h(x)\,dx}{\sqrt{R(x)}(x-z)}, \qquad z \in \mathbb{C} \setminus S_w.$$

Passing to the contour integral and computing residues at $z \in \mathbb{C} \setminus S_w$ and z_j, $\bar{z}_j$, $j = 1, \ldots, K$, we have that

$$G(z) = \frac{\sqrt{R(z)}}{2\pi i} \oint_{S_w} \frac{h(\zeta)\,d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} = h(z) + F(z). \tag{3.21}$$

ETNA
Kent State University
etna@mcs.kent.edu

On the other hand, we have in a neighborhood of ∞ that

$$G(z) = \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{h(x)\,dx}{\sqrt{R(x)}(x-z)} = -\frac{\sqrt{R(z)}}{\pi i} \sum_{k=0}^{\infty} z^{-k-1} \int_{S_w} \frac{x^k h(x)\,dx}{\sqrt{R(x)}}$$
$$= -\frac{\sqrt{R(z)}}{\pi i} \sum_{k=L}^{\infty} z^{-k-1} \int_{S_w} \frac{x^k h(x)\,dx}{\sqrt{R(x)}},$$

which gives

$$\lim_{z\to\infty} G(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} (-z)G(z) = \frac{1}{\pi i} \int_{S_w} \frac{x^L h(x)\,dx}{\sqrt{R(x)}}. \tag{3.22}$$

Let $z \in \mathbb{C} \setminus (S_w \cup Z)$. Observe that $F(\zeta)/(\zeta - z)$ is an analytic function of ζ in $\overline{\mathbb{C}} \setminus S_w$ (see (3.21) and (3.22)), except for the poles at z and $z_j,\ \bar{z}_j,\ j = 1, \ldots, K$. Using the contour integral again, we obtain

$$\frac{1}{\pi i} \int_{S_w} \frac{F(t)\,dt}{t-z} = \frac{1}{2\pi i} \oint_{S_w} \frac{F(\zeta)\,d\zeta}{\zeta - z} = F(z) + h(z) = G(z), \tag{3.23}$$

where $z \in \mathbb{C} \setminus S_w$. Since μ_w is a unit measure, it follows by combining (3.22) and (3.23) that

$$1 = \lim_{z\to\infty} (-z) \int_{S_w} \frac{F(t)\,dt}{t-z} = \pi i \lim_{z\to\infty} (-z)G(z) = \int_{S_w} \frac{x^L h(x)\,dx}{\sqrt{R(x)}}.$$

The latter integral is calculated via residues at $z_k,\ \bar{z}_k$ and ∞:

$$\int_{S_w} \frac{x^L h(x)\,dx}{\sqrt{R(x)}} = \frac{1}{2} \oint_{S_w} \frac{z^L h(z)\,dz}{\sqrt{R(z)}}$$
$$= \frac{1}{2} \sum_{k=1}^{K} p_k \left(\frac{1}{2\pi i} \oint_{S_w} \frac{z^L\,dz}{(z-z_k)\sqrt{R(z)}} + \frac{1}{2\pi i} \oint_{S_w} \frac{z^L\,dz}{(z-\bar{z}_k)\sqrt{R(z)}} \right)$$
$$= \frac{1}{2} \sum_{k=1}^{K} p_k \left(\frac{z_k^L}{\sqrt{R(z_k)}} - 1 + \frac{\bar{z}_k^L}{\sqrt{R(\bar{z}_k)}} - 1 \right),$$

which implies (2.10).

Observe from (2.5) that

$$U^{\mu_w}(a_{l+1}) - U^{\mu_w}(b_l) = \log w(a_{l+1}) - \log w(b_l), \quad l = 1, \ldots, L-1. \tag{3.24}$$

The potential $U^{\mu_w}(x)$ is continuous in $\mathbb{C}$, and is infinitely differentiable on (b_l, a_{l+1}). Hence we obtain by the fundamental theorem of calculus and (3.23) that

$$U^{\mu_w}(a_{l+1}) - U^{\mu_w}(b_l) = \int_{b_l}^{a_{l+1}} \frac{d}{dx}\left(U^{\mu_w}(x)\right)\,dx = \int_{b_l}^{a_{l+1}} \int_{S_w} \frac{d\mu_w(t)}{t-x}\,dx$$
$$= \int_{b_l}^{a_{l+1}} \left(\pi i F(x) + \frac{1}{2} \sum_{j=1}^{K} p_j \left(\frac{1}{x-z_j} + \frac{1}{x-\bar{z}_j} \right) \right) dx$$
$$= \pi i \int_{b_l}^{a_{l+1}} F(x)\,dx + \sum_{j=1}^{K} p_j \left(\log|a_{l+1} - z_j| - \log|b_l - z_j| \right),$$

where the last equality holds in the principal value sense. Therefore, we have by (3.24) that

$$\pi i \int_{b_l}^{a_{l+1}} F(x)\,dx + \log \frac{w(a_{l+1})}{w(b_l)} = \log \frac{w(a_{l+1})}{w(b_l)},$$

which proves (2.11). $\square$

Proof of Corollary 2.2. Note that $K = 2$, $L = 1$ and $S_w = [a_1, b_1] \subset (0,1)$ by Theorem 2.1. We obtain from (2.9) and (2.10) that the endpoints of the support must satisfy the equations

$$\frac{2p_1}{\sqrt{R(0)}} + \frac{2p_2}{\sqrt{R(1)}} = 0 \quad \text{and} \quad \frac{p_2}{\sqrt{R(1)}} = 1 + p_1 + p_2.$$

Since $\sqrt{R(0)} = -\sqrt{a_1 b_1}$ and $\sqrt{R(1)} = \sqrt{(1-a_1)(1-b_1)}$ by (3.17), it follows that

$$a_1 b_1 = r_1^2 \quad \text{and} \quad (1-a_1)(1-b_1) = r_2^2.$$

Solving those equations, we obtain (2.13) and (2.14). The representation (2.12) for μ_w follows from (2.7)-(2.8), because

$$\frac{\sqrt{R(x)}}{2\pi i}\left(-\frac{2p_1}{x\sqrt{R(0)}} + \frac{2p_2}{(1-x)\sqrt{R(1)}}\right) = \frac{\sqrt{(x-a_1)(b_1-x)}}{\pi}\left(\frac{p_1}{x r_1} + \frac{p_2}{(1-x)r_2}\right)$$

by the above argument. $\square$

Proof of Lemma 2.3. Consider the weighted Fekete points $\{\zeta_i^{(n)}\}_{i=1}^n \subset [a,b]$, $n \in \mathbb{N}$, that maximize the absolute value of the weighted Vandermonde determinant (1.10). The relation between the problems of minimizing energy (2.2)-(2.3) and maximizing (1.10) becomes transparent if we consider $-\frac{2}{n(n-1)} \log |V_n^w|$, which is essentially a discrete version of the weighted energy functional (2.2). Indeed, the normalized counting measures

$$\nu_n := \frac{1}{n}\sum_{i=1}^n \delta_{\zeta_i^{(n)}}$$

converge weakly to μ_w, the extremal measure for (2.2)-(2.3), and (2.15) holds true (cf. Section III.1 of [29]). Thus the discrete problem is a good approximation of the continuous one.

We deduce (2.16) from the results of Götz and Saff [17]. They require that $\log w(x)$ be Hölder continuous on $[a,b]$, which is not true if w of (2.1) has zeros on $[a,b]$. But we can modify w in the small neighborhoods of those zeros, outside the compact set

$$S_w^* = \{x \in [a,b] : U^{\mu_w}(x) - \log w(x) \le F_w\} \subset [a,b] \setminus Z, \tag{3.25}$$

so that the new weight $\tilde{w}$ satisfies

$$U^{\mu_w}(x) - \log \tilde{w}(x) > F_w, \qquad x \in [a,b] \setminus S_w^*, \tag{3.26}$$

and $\log \tilde{w}(x)$ is Hölder continuous on $[a,b]$. It follows from Theorem I.3.3 of [29] and (3.25)-(3.26) that the equilibrium measure $\mu_{\tilde{w}} = \mu_w$, and from Theorem III.1.2 of [29] and (3.25) that the weighted Fekete points for $\tilde{w}$ are identical to those for w. Thus all results of [17] are applicable here. Set

$$F_n(x) := \prod_{i=1}^n (x - \zeta_i^{(n)}), \qquad n \in \mathbb{N}.$$

Then $\frac{1}{n}\log|F_n(x)| = -U^{\nu_n}(x)$ and

$$U^{\mu_w}(x) + \frac{1}{n}\log|F_n(x)| \le c_0\,\frac{\log n}{n}, \qquad x\in\mathbb{C},$$

where $c_0>0$ depends only on w, by Theorem 1 of [17]. Hence

$$|F_n(x)| \le n^{c_0}\, e^{-n\,U^{\mu_w}(x)}, \qquad x\in\mathbb{C}. \tag{3.27}$$

For a small $r>0$, write

$$F_n'(x) = \frac{1}{2\pi i}\int_{|x-z|=r}\frac{F_n(z)\,dz}{(z-x)^2}, \qquad x\in S_w^*,$$

and estimate

$$w^{n-1}(x)|F_n'(x)| \le \frac{w^{n-1}(x)}{r}\max_{|x-z|=r}|F_n(z)| = O(n^{c_0})\,\frac{w^n(x)}{r}\max_{|x-z|=r} e^{-n\,U^{\mu_w}(z)},$$

as $n\to\infty$, by (3.27). Note that $U^{\mu_w}(x)$ is Hölder continuous in $\mathbb{C}$, because it is a harmonic function in $\mathbb{C}\setminus S_w$, with smooth boundary values $\log w(x)+F_w$ on S_w (see Theorem I.4.7 of [29] and Lemma 2 of [17]). If $\lambda\in(0,1]$ is the Hölder exponent for $U^{\mu_w}(x)$, then we choose $r=n^{-1/\lambda}$ and obtain

$$\max_{|x-z|=n^{-1/\lambda}} e^{-n\,U^{\mu_w}(z)} = O(1)\,e^{-n\,U^{\mu_w}(x)}, \qquad x\in S_w^*.$$

Hence

$$\begin{aligned} w^{n-1}(x)|F_n'(x)| &= O(n^{c_0+1/\lambda})\,e^{n(\log w(x)-U^{\mu_w}(x))} \\ &= O(n^{c_0+1/\lambda})\,e^{-n\,F_w}, \qquad x\in S_w^*, \end{aligned} \tag{3.28}$$

by (3.25) and (2.4). Recall that the weighted Fekete points are contained in the compact set $S_w^*\subset[a,b]\setminus Z$ (cf. Theorem III.1.2 of [29]), where $\log w(x)$ is continuous. Therefore, we have from Theorem 3 of [17] that

$$\int\log w\,d\nu_n - \int\log w\,d\mu_w = O\left(\frac{\log^2 n}{n}\right) \qquad \text{as } n\to\infty.$$

This implies

$$\prod_{i=1}^n w(\zeta_i^{(n)}) = O(e^{\log^2 n})\,e^{n\int\log w\,d\mu_w} \qquad \text{as } n\to\infty. \tag{3.29}$$

Observe that

$$(V_n^w(\zeta_1^{(n)},\ldots,\zeta_n^{(n)}))^2 = \prod_{i=1}^n w^{2(n-1)}(\zeta_i^{(n)})\prod_{i=1}^n F_n'(\zeta_i^{(n)}).$$

We now use (3.28) and (3.29) to estimate

$$\begin{aligned} \max_{[a,b]^n}(V_n^w)^2 &= \prod_{i=1}^n w^{n-1}(\zeta_i^{(n)})\prod_{i=1}^n w^{n-1}(\zeta_i^{(n)})|F_n'(\zeta_i^{(n)})| \\ &= O(d^{n\,\log^2 n})\,e^{n(n-1)(\int\log w\,d\mu_w - F_w)} \\ &= O(d^{n\,\log^2 n})\,e^{-n(n-1)V_w} = O(d^{n\,\log^2 n})\ (\mathrm{cap}([a,b],w))^{n(n-1)}, \end{aligned}$$

where $d > e$, as $n \to \infty$. Thus the upper bound in (2.16) is proved. The lower bound of (2.16) is a well known consequence of extremal properties for the weighted Fekete points and Vandermonde determinants, see Theorem III.1.1 of [29], which states that the sequence $|V_n^w(\zeta_1^{(n)}, \ldots, \zeta_n^{(n)})|^{\frac{2}{n(n-1)}}$ decreases to $\mathrm{cap}([a,b], w)$ as $n \to \infty$. $\square$

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ETNA
Kent State University
etna@mcs.kent.edu

[33] R. M. TRIGUB, *Approximation of functions with Diophantine conditions by polynomials with integral coefficients*, (in Russian), in Metric Questions of the Theory of Functions and Mappings, no. 2, G. D. Suvorov, ed., Naukova Dumka, Kiev, 1971, pp. 267–333.