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ON THE EIGENSTRUCTURE OF THE BERNSTEIN KERNEL*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. In approximation theory a common technique is to assume duality behavior between the Bernstein operator and the transformation matrix between the standard basis and the Bernstein basis. In this paper we shall produce an example that this assumption is not always correct. In particular, the eigenstructures of the operator and the transformation Matrix are distinguished.

Key words. eigenstructure, Bernstein operator, Bernstein polynomial basis

AMS subject classifications. 42C15, 42C30

1. Introduction. Let $P_{i,n}(x)$, $0 \leq i \leq n$ be a family of linear independent polynomials. Let

$$O_n(f(t);x) = \sum_{i=0}^{n} P_{i,n}(x) f\left(\frac{i}{k}\right),$$

where $f(t)$ is a continuous function of $[0,1]$.

The connection between the transformation matrix from the standard basis to a given basis and the induced operator is widely used. The most significant example is when the matrix is totally positive (strictly total positive), i.e., all its minors are positive (strictly positive), then the operator is totally positive (strictly totally positive), totally positive and strictly total positive, and vice versa. Therefore, a duality is often assumed between the matrix and the operator. In many cases this duality is valid and natural.

In this paper we shall produce an example that shows that this assumption is not always correct. In particular, the eigenstructures of the operator and the transformation matrix are distinct, in focusing on the Bernstein polynomials.

The Bernstein polynomial is [6]

$$b_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad 0 \leq i \leq n.$$

From these polynomials an approximation operator is reduced

$$B_n(f(t);x) = \sum_{i=0}^{n} b_{i,n}(x) f\left(\frac{i}{k}\right).$$

This operator is totally positive [3] and has many uses in approximation theory, probability, computer graphics, CAGD etc.

These polynomials are linearly independent. Hence, they form a polynomial basis for Π_n (the polynomials of order n). The transformation matrix from the monomials basis to the Bernstein basis is as follows [7]:

$$(\mathbf{B}_n)_{j,i} = \begin{cases} \binom{i}{j}\binom{n}{j}^{-1} & \text{for } i \geq j \\ 0 & \text{for } i > j. \end{cases} \tag{1.1}$$

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Furthermore, [7]:

$$(\mathbf{B}_n^{-1})_{j,i} = \begin{cases} \binom{i}{j}\binom{n}{i}(-1)^{i-j} & \text{for } i \geq j \\ 0 & \text{for } i > j. \end{cases} \tag{1.2}$$

REMARK 1.1. In this paper we consider the polynomials as a vector space, where the first coefficient of the vector is the leading coefficient of the polynomial.

REMARK 1.2. In this paper all the matrices have real eigenvalues arranged in decreasing order.

2. The Bernstein Operator Eigenstructure. In this section we discuss the eigenstructure of the Bernstein operator. The description will be brief, for more details see [1] or [4]. However, the results here are in contrast of those to the next section. The last corollary is new.

Since

$$B_n(f(t) = 1; x) = 1$$
$$B_n(f(t) = t; x) = x,$$

one can conclude that $f(t) = 1$ and $f(t) = t$ are eigenpolynomials with associated eigenvalue $\lambda = 1$. The remaining eigenvalues are $\lambda_n = \frac{n!}{n^k}\frac{1}{(n-k)!}$ with associated eigenfunction (polynomial) $p_n = x^k - \frac{k}{2}x^{k-1} +$ lower order terms. Hence, the Bernstein operator B_n is diagonalizable.

THEOREM 2.1 ([4]). *For each fixed* n *and for each function* f,

$$\lim_{m\to\infty} B_n^m(f;x) = [f(1) - f(0)]\,x + f(0), \tag{2.1}$$

where $B_n^m = B_n \circ \ldots \circ B_n$.

COROLLARY 2.2.

$$B_n^m(f;x) - \{[f(1) - f(0)]\,x + f(0)\} = O\left(\left(1 - \frac{1}{n}\right)^m\right).$$

Proof.

$$B_n^m(f;x) - \{[f(1) - f(0)]\,x + f(0)\} = O\left(\Lambda_n^m - I'\right),$$

where I' is the direct sum of the unit matrix of order 2×2 and a null matrix i.e.,

$$I' = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Since $\lambda_{0,n} = \lambda_{1,n} = 1$ and $\lambda_{2,n} > \lambda_{j,n}\ \forall\, j \geq 3$, we get:

$$O\left(B_n^m - I'\right) = O\left(\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_{2,n}^m & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_{n,n}^m \end{pmatrix}\right) = O\left(\lambda_{2,n}^m\right).$$

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However,

$$\lambda_{2,n} = \frac{n!}{(n-2)!n^2} = 1 - \frac{1}{n}\,.$$

Which completes the proof. □

COROLLARY 2.3. *Let C_n be the $n+1 \times n+1$ matrix,*

$$C_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then

$$C = \lim_{m\to\infty} B_n^m.$$

Proof. By Theorem 2.1

$$c_{i,j} = 0 \quad \text{for } j \geq 2. \tag{2.2}$$

Choose a polynomial

$$p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n.$$

By Theorem 2.1

$$\lim_{m\to\infty} B_n^m(f;x) = [f(1) - f(0)]\,x + f(0). \tag{2.3}$$

Hence,

$$\lim_{m\to\infty} B_n^m(p;x) = [\alpha_1 + \alpha_2 + \cdots + \alpha_n]\,x + \alpha_0.$$

Write the matrix equation

$$\lim_{m\to\infty}(B_n^m p)_0 = (C)_0 = c_{0,0}\alpha_0 + c_{1,0}\alpha_1 + \alpha_2 c_{2,0} + \cdots + \alpha_n c_{n,0} = \alpha_0,$$

and

$$\lim_{m\to\infty}(B_n^m p)_1 = (C)_1 = \alpha_0 c_{1,1} + \alpha_1 c_{1,1} + \alpha_2 c_{1,1} + \cdots + \alpha_n c_{1,n} = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Since $P(t)$ is an arbitrary polynomial:

$$c_{0,n} = c_{1,n} = c_{n-1,n} = 0, \quad c_{n,n} = 1.$$

Furthermore,

$$c_{0,1} = 0 \quad c_{0,n-1} = c_{1,n-1} = \cdots = c_{n-1,n-1} = 1 \quad c_{0,1} = 0.$$

This completes the proof. □

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3. The Bernstein Polynomial Transformation Matrix Eigenvectors. The Bernstein polynomials are linearly independent. Therefore, they constitute a polynomial basis for Π_n. The transformation matrix from the standard basis to the Bernstein basis eigenpolynomials behaves quite differently from the Bernstein operator. First, unlike the Bernstein operator, almost all the Bernstein transformation matrix eigenvalues have algebraic multiplicity two but the Bernstein operator, have geometrical multiplicity one. Hence, the Bernstein transformation matrix is not diagonalizable.

3.1. The Eigenvalues. In this section we describe the eigenstructure of the transformation matrix between the standard basis and the Bernstein basis.

Since $\mathbf{B}_n$ is a triangular matrix, all of its principal diagonal elements are eigenvalues. Hence, $\binom{n}{i}^{-1}$ for each $i \leq n$ is an eigenvalue. Since,

$$\binom{n}{i} = \binom{n}{n-i} \qquad \forall\, i \leq n, \tag{3.1}$$

the algebraic multiplicity of each eigenvalue is two, except if n is odd, then $i = \frac{n+1}{2}$ is a simple eigenvalue. Finding the geometrical multiplicity is identical to finding the dimension of the solution space of the following equation:

$$\mathbf{B}_n V_i = \binom{n}{i}^{-1} V_i. \tag{3.2}$$

Let V_i be an eigenvector of $\mathbf{B}_n$. Then it is a solution of equation (3.2).

REMARK 3.1. In this chapter we look for the eigenvectors, by (3.1) we can assume that $i < \lfloor \frac{n+1}{2} \rfloor$.

THEOREM 3.2. *Let M be a positive lower triangular matrix that for all natural n, satisfies the following*

$$M_{i,j} > 0 \quad j \leq i \quad M_{i,i} = M_{n-i,n-i} \quad \forall\, i \leq n \quad M_{j,j} > M_{i,i} \quad j \leq i \leq \frac{n}{2}.$$

Let V_i be the eigenvector associated with the eigenvalue $\lambda_i = M_{i,i}$. Then for every $i < \lfloor \frac{n+1}{2} \rfloor$ we have

$$\begin{aligned} v_i^j &= 0 \qquad \forall\, j < n-i \\ v_i^{n-i} &\neq 0\,. \end{aligned}$$

Proof. For $j < i$ the theorem is trivial since the matrix is triangular. If $i = j$, we consider two cases.

Case 1. $n = 2i + 1$.
Let us write the eigenequation

$$v_i^i M_{i,i+1} + v_i^{i+1} M_{i+1,i+1} = v_i^{i+1} M_{i,i}.$$

By the condition on the matrix we get $v_i^i = 0$ which completes the proof in this case.

Case 2. $n \neq 2i + 1$.
In this case we assume that the theorem is false. Without loss of generality, we can consider $v_i^i = 1$. For the rest of the proof the following lemma is needed.

LEMMA 3.3. *If $v_i^i = 1$ then $v_{i+l}^j = 1$, $i + l < n - i$.*

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Proof. By induction on l. If $l = 0$ then there is nothing to prove. Therefore, we may assume that the lemma is true for every $0 \leq k < l$. Write the eigenequation as:

$$\sum_{r=0}^{l} v_{i+r} M_{i+l,i+r} = v_i^{i+l} M_{i,i},$$

or

$$\sum_{r=0}^{l-1} v_{i+r} M_{i+l,i+r} = (v_i - v_{i+l}) M_{i+l,i+l}.$$

By the induction hypothesis the left hand side of the equation is strictly greater than zero and $(M_{i,i} - v_{i+l,i+l}) > 0$ implies that $v_{i+l} > 0$ which completes the proof of the lemma. ∎

Proof of Theorem 3.2 (continued). When $l + i = n - i$ we get

$$\sum_{r=0}^{l} v_{i+r} M_{i+l,i+r} = \lambda_i v_{n-i} = M_{n-i,n-i} v_{n-i},$$

or

$$\sum_{r=0}^{l-1} v_{i+r} M_{i+l,i+r} = (M_{i,i} - M_{n-i,n-i}) v_{n-i}.$$

But $M_{i,i} - M_{n-i,n-i} = 0$. By the assumption $v_i^i = 1$ and by Lemma 3.3 $v_i^j > 0$. The left hand side is strictly greater than zero. Hence, a contradiction, $v_i^i = 0$. Since that and the fact the the matrix is triangular we get

$$v_i^j = 0 \qquad j < n - i.$$

For the case $n - i$ we consider the theorem to be false i.e., $v_i^{n-i} = 0$ since the matrix is triangular we get

$$v_i^j = 0 \qquad j > n - i.$$

Hence, V_i is a null vector which cannot be an eigenvector. Hence, $v_i^{n-i} \neq 0$ which completes the proof. ∎

COROLLARY 3.4. *Assume that the condition Theorem 3.2 holds. Then each eigenvalue of the M has geometrical multiplicity one.*

Proof. We consider the theorem to be false, i.e., the geometrical multiplicity is greater than one. However, Theorem 3.2 ensures that the choice of the orientation $v_i^{n-i} = 0$ is not valid. Hence, the geometrical multiplicity is one. The Bernstein transformation matrix is the classical example of the theorem and the corollary. ∎

COROLLARY 3.5. *The Bernstein transformation matrix $\mathbf{B}_n$ is not diagonalizable for all $n \geq 1$.*

Proof. Since the algebraic multiplicity of each eigenvalue is two (except, if n is odd, for $(n+1)/2$ then it is one) and the geometrical multiplicity is one. Hence, $\mathbf{B}_n$ is not diagonalizable for $n \geq 1$. ∎

COROLLARY 3.6. *Let B_n be the Bernstein operator of dimension n and $\mathbf{B}_n$ be the transformation matrix from the standard basis to the Bernstein basis. Then*

$$B_n|_{\Pi_n} \neq \mathbf{B}_n, \qquad \forall\, n \geq 1. \tag{3.3}$$

Proof. The transformation matrix is not diagonalizable and the operator is diagonalizable. Hence,

$$B_n|_{\Pi_n} \neq \mathbf{B}_n, \qquad \forall\, n \geq 1.$$

This completes the proof. ◻

THEOREM 3.7. *Assume that the condition of Theorem 3.2 holds and*

$$sign\ M_{i,j} = (-1)^{i+j}.$$

Let V_i be an eigenvector of the matrix associated with the eigenvalue $\binom{n}{i}^{-1}$, and let the orientation be $v_i^{n-i} = 1$. Then,

$$sign\left(v_i^j\right) = (-1)^{j+i-n} \qquad \forall\, j \geq n-i. \tag{3.4}$$

Proof. If $j = n - i$, there is nothing to prove ($v_i^{n-i} = 1$). We use induction for the rest of the proof. We consider Theorem 3.7 to be true for all $n - i \leq j < k$, and we will check it for $j = k$.

$$M^{-1} V_i = \binom{n}{i} V_i. \tag{3.5}$$

Therefore,

$$\sum_{l=n-i}^{k} \left(M_n^{-1}\right)_{l,k} v_i^l = \binom{n}{i} v_i^k\,, \tag{3.6}$$

or

$$\sum_{l=n-i}^{k-1} \left(M_n^{-1}\right)_{l,k} v_i^l + \left(M_n^{-1}\right)_{k,k} v_i^k = \left(M_n^{-1}\right)_{i,i} v_i^k. \tag{3.7}$$

By the properties of the matrix M^{-1} then

$$\sum_{l=n-i}^{k-1} \left(M_n^{-1}\right)_{lk} v_i^l = \left(M_n^{-1}\right)_{i,i} - \left(M_n^{-1}\right)_{k,k} v_i^k \tag{3.8}$$

$k > j \geq n - i$ yields

$$\binom{n}{i} - \binom{n}{k} > 0.$$

Hence,

$$\text{sign}\left(v_i^k\right) = \text{sign}\left(\sum_{l=n-i}^{k-1} \left(M_n^{-1}\right)_{l,k} v_i^l\right).$$

From

$$\text{sign}\ M_{i,j} = (-1)^{i+j},$$

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and the definition of M we get

$$\operatorname{sign}\left(v_i^j\right) = (-1)^{j+i-n} \qquad \forall\, j \geq n-i, \tag{3.9}$$

which completes the proof. □

The choice of the orientation was only for the convenience of the proof and the proof goes through with any real orientation. The classical example of this theorem is the Bernstein transformation matrix.

3.2. The Asymptotic Behavior of the Transformation Matrix. Unlike the operator's asymptotic behavior which converge and the limit matrix is described in Corollary 2.3. The transformation matrix between the standard (power) basis and the Bernstein basis asymptotic behavior, however, is very different: it diverges. From that we can deduce that the transformation matrix norm is greater than one, for every subordinate (operator) norm.

LEMMA 3.8. *The transformation matrix from the standard basis to the Bernstein basis asymptotic behavior diverges.*

Proof. Consider the vector e_0. Then,

$$(\mathbf{B}_n e_0)^0 = 1 \qquad (\mathbf{B}_n e_0)^1 = \frac{1}{n}\,.$$

Hence,

$$(\mathbf{B}_n^m e_0)^0 = 1 \qquad (\mathbf{B}_n^m e_0)^1 = \frac{m}{n}\,.$$

Therefore,

$$\lim_{m\to\infty} (\mathbf{B}_n^m e_0)^1 \geq \sum_{m=1}^{\infty} \frac{1}{n} + \lim_{m\to\infty} \left(\frac{1}{n}\right)^m = \infty.$$

Using the same computation, we can determine that:

$$\lim_{m\to\infty} (\mathbf{B}_n^m e_0)^j = \infty \qquad \forall\, j \geq 1.$$

Since $\mathbf{B}_n$ is a finite dimensional matrix, there exist entries of $\lim_{m\to\infty} \mathbf{B}_n^m$ that diverge [2]. This completes the proof. □

COROLLARY 3.9. *Let* $|||\cdot|||$ *be a matrix norm. Then,*

$$|||\mathbf{B}_n||| > 1 \qquad \textit{for every } n \geq 1. \tag{3.10}$$

Proof. Since the greatest eigenvalue is 1, $|||\mathbf{B}_n||| \geq 1$ ([5] p. 150). By Lemma 3.8 the matrix $\lim_{m\to\infty} \mathbf{B}_n^m$ diverges. Hence,

$$\lim_{m\to\infty} |||\mathbf{B}_n^m||| = \infty.$$

By the subordinate of a matrix norm,

$$\lim_{m\to\infty} |||\mathbf{B}_n|||^m \geq \lim_{m\to\infty} |||\mathbf{B}_n^m||| = \infty.$$

Hence,

$$|||\mathbf{B}_n||| > 1.$$

This completes the proof. □

REMARK 3.10. The infimum of all norms is 1 since it is the spectral radius.

REMARK 3.11. All the results of this section are valid for the transformation matrix from the standard polynomial basis to the reduced Bernstein polynomial basis matrix.

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