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OPEN PROBLEMS IN CONSTRUCTIVE FUNCTION THEORY*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. A number of open problems on constructive function theory are presented. These were submitted by participants of Constructive Function Theory Tech-04.

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AMS subject classifications. 41A17, 41A21, 41A55, 41A99, 42C15

1. Introduction. A number of open problems posed at Constructive Functions Tech-04 are presented below together with recent progress in solving them. Many involve potential theory and its applications to weighted approximation, number theory, and numerical integration on the sphere. Equally many are related to the theory of orthogonal polynomials and their relatives: Bergman and Sobolev polynomials, entropy of orthogonal polynomials, and Heine-Stieltjes polynomials. The list of problems grew out of a problem session held at Constructive Functions Tech-04. At the end of each section, the contributing author is mentioned in parenthesis.

2. Asymptotic Discretization of a Potential. Let μ be compactly supported probability measure in the open unit ball B_n of $\mathbb{R}^n$, and P_μ its Newtonian potential (logarithmic if $n = 2$). Assume that the support of μ is a smooth $n-1$ submanifold with boundary on which μ has smooth nonvanishing density with respect to Lebesgue measure. Let μ_N be a discrete probability measure whose support consists of at most N points $x_1^N, \ldots, x_m^N$, $m \le N$, $x_j^N \in B_n$, and whose potential P_{μ_N} minimizes

$$\sup_{\|x\|=1} |p_\mu(x) - p_{\mu_N}(x)|$$

among all such measures. Is it true that the counting measure of the x_j^N, namely the discrete probability measure with equal mass at each x_j^N, converges weak-*, as N goes to infinity, to the Green equilibrium distribution of supp μ with respect to the unit ball of $\mathbb{R}^n$?

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3. Heine-Stieltjes Polynomials. Let $\mathbb{P}_n$ stand for the class of all algebraic polynomials of degree at most $n \in \mathbb{N}$. The *generalized Lamé differential* equation (in algebraic form) is

$$A(x)E''(x) + B(x)E'(x) + C(x)E(x) = 0\,, \tag{3.1}$$

where A, B are polynomials of degree $p+1$, p, respectively, and $C \in \mathbb{P}_{p-1}$. The case $p = 1$ corresponds to the hypergeometric differential equation, and $p = 2$, to the Heun's equation; see [31]. Heine [14] proved that for every $N \in \mathbb{N}$ there exist *at most*

$$\sigma(N) = \binom{N+p-1}{N}$$

different polynomials C in (3.1) such that this equation admits a polynomial solution $y_N \in \mathbb{P}_N$. These coefficients C are called Van Vleck polynomials, and the corresponding polynomial solutions E are known as Heine-Stieltjes polynomials. Moreover, Szegő [36, §6.8] cites this result saying that “Heine asserts that, in general, there are exactly $\sigma(N)$ determinations of C of this kind”.

QUESTION 3.1. *If you are lucky enough to be able to read XIX century style German, try* [14]. *Heine uses some complicated arguments about the number of solutions of a system of nonlinear equations. As far as I know, there is no modern proof of Heine's result. Also, is it possible to characterize the exceptional cases when the number of different Van Vleck polynomials is strictly less than $\sigma(N)$?*

Assume that the coefficients of (3.1) depend on the parameter N, and we are interested in the asymptotic zero distribution of the Heine-Stieltjes polynomials $\{y_N\}$ as $N \to \infty$; see [19] for some results in this sense. One way we may proceed is to reduce (3.1) to the Riccati form, rewriting it in terms of $h_N = y_N'/y_N$:

$$A_N(z)\left(h_N^2(z) + h_N'(z)\right) + B_N(z)h_N(z) + C_N(z) = 0\,.$$

Observing that h_N/N is the Cauchy transform of the unit zero counting measure of y_N's, and assuming that A_N, B_N/N and C_N/N^2 are uniformly bounded, we may take limits along appropriate subsequences in order to find an expression for the Cauchy transform

$$\widehat{\mu}(z) = \int \frac{1}{z-t}\, d\mu(t)$$

of the limit distribution μ. In general it will have the form

$$\widehat{\mu}(z)^2 = r(z)\,, \tag{3.2}$$

where r is an analytic function. This expression is valid in *every* connected component of $\mathbb{C} \setminus \operatorname{supp}(\mu)$, where r must be holomorphic.

QUESTION 3.2. *Provided we have obtained that*

$$\widehat{\mu}(z)^2 = \frac{1}{z^2-1}$$

in every connected component of $\mathbb{C} \setminus \operatorname{supp}(\mu)$, *is it true that necessarily μ is the equilibrium measure of* $[-1, 1]$*?*

QUESTION 3.3. *More generally, if we know that μ is a positive unit Borel measure compactly supported on* $\mathbb{C}$, *and that r is a rational function (the same in every connected* *component of* $\mathbb{C} \setminus \operatorname{supp}(\mu)$*), does equation* (3.2) *determine uniquely* μ*? Is it true that in this case* $\operatorname{supp}(\mu)$ *does not have interior points in* $\mathbb{C}$*?*

If the answer to this question is "yes", the support of μ must be a union of trajectories of the quadratic differential $r(z)dz^2$. In the case when A_N and B_N do not depend on N these trajectories are characterized by their minimum capacity in the class of all cuts of the plane making $\sqrt{r}$ single-valued, i.e. by a min-max property of the logarithmic energy; see [33].

QUESTION 3.4. *Is there an electrostatic model for the zeros of Heine-Stieltjes polynomials, describing their location in terms of a min-max property of the corresponding energy?*

The answer is "yes", for instance, for the classical Jacobi polynomials, but the situation is not clear in general.

(Martínez-Finkelshtein)

4. Non-Standard Orthogonal Polynomials. Let μ_0 and μ_1 be two Borel measures on $\mathbb{C}$ such that at least one of them has an infinite number of points of increase. Monic *Sobolev orthogonal polynomials* Q_n are those minimizing the norm

$$\|q\|^2 = \|q\|^2_{L^2(\mu_0)} + \|q'\|^2_{L^2(\mu_1)} \,,$$

(where prime denotes derivative) in the class of all monic polynomials of degree n. If μ_1 is a discrete measure, polynomials Q_n are known as *discrete* Sobolev, otherwise they are *continuous*. Their asymptotic properties have been studied by several authors; see e.g. [17] for the discrete case, and [16, 20, 21] for the continuous. Nevertheless, many open questions still remain, among which my favorites are:

QUESTION 4.1. *Is there any matrix Riemann-Hilbert characterization of this kind of orthogonality?*

QUESTION 4.2. *What is the strong asymptotics of* $\{Q_n\}$ *if* μ_1*,* μ_2 *are absolutely continuous measures from the Szegő class with disjoint or at least non-coincident supports? For the* n*-th root asymptotics, see* [12]*, and for zero location,* [9].

The *balanced* monic Sobolev orthogonal polynomials minimize the norm depending on their degree,

$$\|q\|^2_n = \|q\|^2_{L^2(\mu_0)} + \left\| \frac{q'}{n} \right\|^2_{L^2(\mu_1)} \,,$$

in the class of all monic polynomials of degree n. For a narrow class of measures (absolutely continuous on $[-1,1]$ and satisfying a certain algebraic relation) it have been proved in [1] that Q_n's behave as the standard polynomials orthogonal with respect to the "combined" measure

$$d\mu^*(x) = \left(\mu_0'(x) + \frac{\mu_1'(x)}{1-x^2} \right) dx \,, \quad x \in (-1,1) \,.$$

QUESTION 4.3. *Is this result valid under in a more general situation?*

Finally, let γ be a Jordan arc or curve (say, an interval), and φ a function holomorphic in a neighborhood of γ. In some cases it is possible to prove the existence and uniqueness (up to normalization) of a sequence of polynomials P_n, $\deg P_n = n$, such that

$$\int_\gamma P_n(z) \varphi^k(z) w(z)\, dz = 0, \quad k = 0, 1, \ldots, n-1.$$

QUESTION 4.4. *Is it possible to characterize polynomials P_n in terms of a matrix Riemann-Hilbert problem?*

Observe that orthogonality with respect to powers z^k is naturally connected with the Cauchy integral, and this in its turn, has a nice Riemann-Hilbert problem associated via Sokhotski-Plemelj formulas. A related question is: is there any other kernel with a nice jump condition associated?

(Martínez-Finkelshtein)

5. Entropy of Orthogonal Polynomials. Assume that $\{p_n\}$ is a sequence of orthonormal polynomials with respect to a unit positive weight ω on $\mathbb{R}$. The Boltzmann-Shannon entropy of these polynomials is defined as

$$E_n \stackrel{\text{def}}{=} -\int p_n^2(x) \log\left(p_n^2(x)\right) \, \omega(x) dx \, .$$

There are two mathematical problems of absolutely different nature related with these quantities:

(i) Explicit formula or at least a stable numerical algorithm for computation of E_n for any reasonable $n \in \mathbb{N}$;

(ii) The asymptotic analysis of E_n as $n \to \infty$.

Regarding (i), general explicit formulas for E_n have been obtained for the Gegenbauer polynomials with index $\lambda = 0$ (Chebyshev), $\lambda = 1$ (Chebyshev of the second kind) and $\lambda = 2$ only. In particular, it has been shown in [8, 43] that for Chebyshev polynomials, $E_n = \log(2) - 1$, for all $n \in \mathbb{N}$. Moreover, as it follows from [2], for the unit weight ω on $[-1, 1]$, $\log(2) - 1$ is the asymptotically maximum value that E_n can achieve.

QUESTION 5.1. *Does the property $E_n \equiv const$ characterize the family of Chebyshev polynomials? The answer is "yes" in the Bernstein-Szegő class* (*see* [2]), *but what about, say the whole Szegő class?*

QUESTION 5.2. *Is there any other family of polynomials for which E_n has a compact closed expression?*

In [4] a stable algorithm for computing E_n for a unit weight ω on $[-1, 1]$ has been found. It uses the coefficients of the three-terms recurrent relation for the p_n's as the only input, in the spirit of the well-known procedure for the computation of the gaussian quadratures.

QUESTION 5.3. *Is it possible to extend this method (or to find an alternative one) for the case of the unbounded interval of orthogonality?*

Problem (ii) is closely related to the strong (or at least, $L^{2+\epsilon}$) asymptotics of the polynomials. In [18] it was shown that under rather general conditions,

$$E_n = \int_{a_{-n}}^{a_n} \frac{\log \omega(x)}{\sqrt{(a_n - x)(x - a_{-n})}} \, dx \; (1 + o(1)) \, , \quad n \to \infty \, , \tag{5.1}$$

where $a_{\pm n}$ are the well-known Mhaskar-Rakhmanov-Saff numbers.

QUESTION 5.4. *The Pollaczek polynomials $p_n(x; \lambda, a, b)$ are orthogonal on $[-1, 1]$ with respect to the unit weight*

$$\omega(x; \lambda, a, b) = \frac{2^{2\lambda}\,(\lambda + a)}{2\pi\, \Gamma(2\lambda)} \, (1 - x^2)^{\lambda - 1/2} \, e^{(2 \arccos x - \pi) \frac{ax+b}{\sqrt{1-x^2}}} \left| \Gamma(\lambda + i \frac{ax + b}{\sqrt{1 - x^2}}) \right|^2 .$$

They also satisfy the three-term recurrence relation

$$x\, p_n(x; \lambda, a, b) = a_{n+1}\, p_{n+1}(x; \lambda, a, b) + b_n\, p_n(x; \lambda, a, b) + a_n\, p_{n-1}(x; \lambda, a, b) \, ,$$

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with

$$a_n = \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda+a)(n+\lambda+a-1)}}\,, \quad b_n = -\frac{b}{n+\lambda+a}\,.$$

As it follows from [2]*, for these polynomials E_n diverges. However, they do not satisfy the conditions of* [18]. *Is formula* (5.1) *still valid?*

If μ and ν are two Borel (generally speaking, real signed) measures on $\mathbb{C}$, we denote by

$$I[\nu,\mu] = -\iint \ln|z-t|\,d\nu(t)\,d\mu(z)$$

their mutual energy. With each polynomial

$$p_n(z) = \gamma_n \prod_{j=1}^{n}\left(z-\zeta_j^{(n)}\right)$$

we can associate naturally two probability measures:

$$\lambda_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{\zeta_j^{(n)}} \qquad \text{and} \qquad d\nu_n(x) = p_n^2(x)\,d\mu(x)\,.$$

Both measures are standard objects of study in the analytic theory of orthogonal polynomials. For instance, the normalized zero counting measure λ_n is closely connected with the n-th root asymptotics of p_n, and as was shown by Rakhmanov in his pioneering work [30], ν_n is associated with the behavior of the ratio p_{n+1}/p_n as $n\to\infty$.

A nice link between the entropy and the logarithmic potential theory is established by the following identity: if $\gamma_n > 0$ is the leading coefficient of p_n, then

$$E_n = -2\log(\gamma_n) + 2n\,I[\lambda_n,\nu_n]\,. \tag{5.2}$$

Hence, if we know the asymptotic behavior of the entropy, it gives us information about the mutual energy $I[\lambda_n,\nu_n]$, and viceversa.

Assume that ω is a weight on $[-1,1]$, strictly positive a. e. on the interval. Then both λ_n and ν_n tend (in a weak-star sense) to the equilibrium measure of the interval, μ, and it is not surprising at all to find out that $I[\lambda_n,\nu_n]\to I[\mu,\mu]=\log(2)$. But what about the next term of the asymptotics of the energy, does it depend on the weight ω? From the results of [2] it follows that for a large subset of the Szegő class (for instance, for weights satisfying that

$$\sup_n \int_{-1}^{1}\left(p_n^2(x)\right)^{1+\varepsilon}\omega(x)\,dx < \infty$$

for a $\varepsilon>0$), we have

$$I[\lambda_n,\nu_n] = \log(2) - \frac{1}{2n} + o\left(\frac{1}{n}\right)\,, \quad n\to\infty\,. \tag{5.3}$$

QUESTION 5.5. *What is the explanation of this "universal" behavior of the second term? Is the constant $1/2$ appearing there the logarithmic capacity of the support of the measure of orthogonality? Is there any direct proof of* (5.3) *which does not use* (5.2)*? Is* (5.3) *still valid in a larger class of weights ω?*

Finally, if p_n is now a sequence of polynomials orthonormal with respect to a varying weight,

$$\int_\Delta p_n(x)p_k(x)\omega^n(x)\,dx = \delta_{nk}\,, \quad k = 0, 1, \ldots, n\,,$$

the asymptotic behavior or E_n is described in terms of the equilibrium measure in the external field $Q(x) = -\log(\omega(x))$: if K is its support, and μ_K is the Robin distribution on K, then

$$E_n = -2n \int_\Delta Q\, d\mu_K\ + o(n)\,, \quad n \to \infty\,. \tag{5.4}$$

This formula was established in [3] only for the case when ω is the Jacobi or Laguerre weight.

QUESTION 5.6. *Is this result valid in a more general setting, say when K is an interval?*

We note that after the conference, there has been some progress in the study of the asymptotic behavior of the entropy in the case when the weight does not satisfy the Szegő condition (Question 5.4). According to [22], formula (5.1) is valid also for Pollaczek polynomials $p_n(x; \lambda, a, 0)$ with $a \geq 0$ and $\lambda \geq 1$; see notation in the statement of Question 5.4. Moreover, for these polynomials the following asymptotic behavior for the mutual energy $I[\lambda_n, \nu_n]$ holds:

$$I[\lambda_n, \nu_n] = \log(2) - \frac{1-2a}{2n} + o\left(\frac{1}{n}\right)\,, \quad n \to \infty\,.$$

Obviously, for $a = 0$ we recover (5.3). Since $p_n(x; \lambda, 0, 0)$ are just the well-known Gegenbauer polynomials, a natural addendum to Question 5.5 is: *does the asymptotic behavior* (5.3) *characterize the class of weights on* $[-1, 1]$ *satisfying the Szegő condition?*

(Martínez-Finkelshtein)

6. Minimizing Multiplier Polynomials. In some investigations of Padé approximation, the following problem arises: let Q be a given polynomial of degree n, and $|z| < 1$. Find a polynomial S of degree n that minimizes

$$\Lambda_n\left(Q; z\right) = \frac{\max_{|t|=1} |QS|\left(t\right)}{|QS|\left(z\right)}$$

over all polynomials S of degree $\leq n$. That is,

$$\Lambda_n\left(Q; z\right) = \inf_{\deg(S) \leq n} \frac{\max_{|t|=1} |QS|\left(t\right)}{|QS|\left(z\right)}.$$

If

$$Q\left(z\right) = \prod_{j=1}^{n} \left(z - z_j\right),$$

then it is easy to see that

$$\Lambda_n\left(Q; z\right) \leq \prod_{j=1}^{n} \Lambda_1\left(\ell_j; z\right),$$

where for each j,

$$\ell_j\left(t\right) = t - z_j.$$

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QUESTION 6.1. *(I) Compute* $\Lambda_1(\ell; z)$*, where* $\ell(t) = t - a$ *is a linear function.*
(II) Ultimately, one would like to estimate quantities such as

$$meas\left\{z : |z| \leq 1 \text{ and } \Lambda_n(Q; z) \geq r\right\},$$

where $meas$ *denotes planar Lebesgue measure and* $r > 0$.

Even part (I) of this problem is less trivial than it seems.

The following estimate for $\Lambda_1(\ell; z)$ was proved by David Jimenez, a graduate student in the School of Mathematics, Georgia Institute of Technology.

THEOREM 6.1. *Let* $\ell(t) = t - a$*. Then for* $|z| \leq 1$*,*
(a)

$$\Lambda_1(\ell; z) \leq \frac{1 + |a|}{|z - a|}.$$

(b)

$$\frac{1 + |a|}{2|z - a|} \leq \Lambda_1(\ell; z) \leq \frac{1 + |a|}{|z - a|}$$

Proof. (a) Firstly,

$$\Lambda_1(\ell; z) \leq \frac{\max_{|t|=1}|t - a|}{|z - a|} \leq \frac{1 + |a|}{|z - a|}.$$

(b) Note that for each $b \in \mathbb{C}$,

$$|t - b| \geq \sqrt{1 + |b|^2}$$

for at least half of all t on the unit circle. (If $b = 0$, then it hold for all t. If $b \neq 0$, it holds as long as the triangle with vertices b, t and 0 is obtuse.) Therefore, for at least two points t on the unit circle,

$$|t - b|\,|t - a| \geq \sqrt{1 + |b|^2}\sqrt{1 + |a|^2}.$$

So

$$\Lambda_1(\ell; z) \geq \inf_{b \in \mathbb{C}} \frac{\sqrt{1 + |b|^2}\sqrt{1 + |a|^2}}{|z - b|\,|z - a|}.$$

Here

$$\sqrt{1 + |b|^2} \geq \frac{1}{\sqrt{2}}(1 + |b|) \geq \frac{1}{\sqrt{2}}|z - b|.$$

So

$$\Lambda_1(\ell; z) \geq \frac{1}{\sqrt{2}}\frac{\sqrt{1 + |a|^2}}{|z - a|} \geq \frac{1 + |a|}{2|z - a|}. \qquad \square$$

(Lubinsky)

7. Quadrature Formulae and Weighted Approximation. It is well known [24, 32] that under suitable conditions on $W(x) = \exp(-Q(x))$, there exists, for every integer $n \geq 1$, a unique probability measure $\mu_{W,n}$, supported on $[-1, 1]$ that maximizes

$$\int\int \log|W(a_n x)W(a_n t)(x-t)|\, d\nu(x) d\nu(t)$$

among all compactly supported probability measures ν supported on $\mathbb{R}$, where a_u is defined by

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\frac{dt}{\sqrt{1-t^2}}, \qquad u > 0.$$

It will be interesting to prove the following analogue of a theorem of Erdős, Kroó, and Szabados [10].

QUESTION 7.1. *Let $x_{k,n}$ be distinct points on $\mathbb{R}$, W be a weight function such that the measures $\mu_{W,n}$ are supported on $[-1, 1]$. The following are equivalent.*
(a) *To every f with $Wf \in C_0(\mathbb{R})$ and $\epsilon > 0$, there exists a sequence of polynomials $r_n \in \Pi_{n(1+\epsilon)}$ such that $r_n(x_{k,n}) = f(x_{k,n})$ for $k = 1, \cdots, n$, and $\|(f - r_n)W\|_{\infty,\mathbb{R}} \to 0$ as $n \to \infty$.*
(b) *We have*

$$\limsup_{n\to\infty} \frac{\#\{k \,:\, x_{k,n}/a_n \in I_n\}}{n\mu_{W,n}(I_n)} \leq 1 \tag{7.1}$$

for every sequence of intervals $I_n \subseteq [-1, 1]$ for which $\lim_{n\to\infty} n\mu_{W,n}(I_n) \to \infty$, and

$$\liminf_{n\to\infty} n\mu_{W,n}([x_{k+1,n}/a_n, x_{k,n}/a_n]) > 0, \qquad 1 \leq k \leq n. \tag{7.2}$$

It is worth mentioning here that the location and distribution of node systems $\{x_{k,n}\}$ that provide a "good" interpolation process has been studied by Szabados [35], Damelin [7], and Vertesi [39, 37].

QUESTION 7.2. *Let $q \geq 1$ be an integer, $\mathbb{S}^q$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$, and μ be the volume element of $\mathbb{S}^q$. We are interested in quadrature formulas of the form*

$$\sum_{\xi\in\mathcal{C}} w_\xi f(\xi) \approx \int_{\mathbb{S}^q} f(\mathbf{x})d\mu(\mathbf{x}),$$

where $\mathcal{C}$ is a finite set of points on $\mathbb{S}^q$, w_ξ are positive numbers, and the formula is required to be exact for spherical polynomials of degree as high as possible. The highest degree of polynomials for which the formula is exact will be called the order of the formula. The formula will be called interpolatory if P is any spherical polynomial of degree at most $|\mathcal{C}|$, and $P(\xi) = 0$ for each $\xi \in \mathcal{C}$, then $P \equiv 0$. Numerical experiments in [23] suggest that if $\mathcal{C}$ is the Saff-Kuijlaars system then one obtains quadrature formulas of nearly the highest order. The questions are: (1) If $\mathcal{C}$ is an extremal system of points with respect to some energy problem, does there exists an interpolatory quadrature formula based at these points? (2) If an interpolatory quadrature formula exists, then is it necessarily based on the extremal points for some discretized energy problem? In this connection, it is noteworthy that Prestin and Roşca [27] have recently obtained interpolatory quadrature formulas, some of which can be thought of as based on a set of tensor product Fekete points with respect to a suitable energy functional.

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8. Distribution of Primes and a Weighted Energy Problem. Let $\pi(x)$ be the number of primes not exceeding x. The celebrated Prime Number Theorem (PNT), suggested by Legendre and Gauss, states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

Chebyshev [5] made the first important step towards the PNT in 1852, by proving the bounds

$$0.921\frac{x}{\log x} \le \pi(x) \le 1.106\frac{x}{\log x} \quad \text{as } x \to \infty.$$

Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem in 1896, via establishing that $\zeta(s)$ does not have zeros on the line $\{1+it,\, t \in \mathbb{R}\}$. From a different perspective, Gelfond and Schnirelman (see [5, pp. 285–288]) proposed an interesting "elementary" method aimed at producing the PNT with a good error term, in 1936. It used polynomials with integer coefficients and the Chebyshev function $\psi(x) = \log \operatorname{lcm}(1,\dots,x)$, $x \in \mathbb{N}$, together with the well known fact that the PNT is equivalent to $\psi(x) \sim x$ as $x \to +\infty$; see [15]. Later work showed that the original Gelfond-Schnirelman method cannot give a proof of the PNT [25, Ch. 10]. However, Nair [26] and Chudnovsky [6] found a generalization based on an equivalent form of the Prime Number Theorem [15]

$$\int_1^x \psi(t)\, dt \sim \frac{x^2}{2} \quad \text{as } x \to +\infty. \tag{8.1}$$

Using the following weighted version of Vandermonde determinant

$$V_n^w(x_1,\dots,x_n) := \prod_{i=1}^n w^{n-1}(x_i) \prod_{1\le i<j\le n} (x_i - x_j),$$

where $x_i \in [0,1]$ and $w(x) = (x(1-x))^{\alpha_1}$, $\alpha_1 > 0$, they obtained the bound

$$\int_1^x \psi(t)\, dt \ge 0.99035\,\frac{x^2}{2} \quad \text{as } x \to +\infty, \tag{8.2}$$

produced by the optimal choice $\alpha_1 \approx 0.195$. We developed the ideas of [26] and [6], and established a connection with the weighted capacity c_w (cf. [32]) of $[0,1]$ corresponding to the weight w of the form

$$w(x) = \prod_{i=1}^k |Q_{m_i}(x)|^{\alpha_i}, \quad \alpha_i > 0,\ i = 1,\dots,k, \tag{8.3}$$

where Q_{m_i} is a polynomial with integer coefficients of degree at most m_i. For $\alpha := \sum_{i=1}^k \alpha_i m_i$, we proved [28] that

$$\int_1^x \psi(t)\, dt \ge \frac{-2\log c_w}{4\alpha+3}\,\frac{x^2}{2} + O(x\log^2 x) \quad \text{as } x \to +\infty. \tag{8.4}$$

In particular, if $w(x) = x^{\alpha_1}(1-x)^{\alpha_2}$, $x \in [0,1]$, $\alpha_1 = \alpha_2 = 0.195$, then $c_w \approx 0.1045575588$ and (8.2) holds true.

It is natural to try improving the bound (8.2) by choosing a weight with a proper combination of factors $Q_{m_i}(x)$ and exponents α_i. The most interesting question is, of course, whether one can find a weight $w(x)$ of the form (8.3) such that

$$\frac{-2\log c_w}{4\alpha+3} = 1?$$

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It turns out this is impossible to achieve for any *fixed* weight of the type (8.3). The reason for such a conclusion transpires from the error term in (8.4), which is "too good." Indeed, it is known from Littlewood's theorem that the difference $\int_1^x \psi(t)\,dt - x^2/2$ takes both positive and negative values of the amplitude $cx^{3/2}$, $c > 0$, infinitely often as $x \to +\infty$. This is conveniently written in the notation

$$\int_1^x \psi(t)\,dt - \frac{x^2}{2} = \Omega_{\pm}(x^{3/2}) \quad \text{as } x \to +\infty;$$

cf. [15, pp. 91–92]. Hence the correct error term should be of the order $O(x^{3/2})$. Relating this to (8.1) and (8.4), we obtain in such an indirect way that

$$B(w) := \frac{-2\log c_w}{4\alpha + 3} < 1. \tag{8.5}$$

We should also note that if the Riemann hypothesis is true, then

$$\int_1^x \psi(t)\,dt - \frac{x^2}{2} = O(x^{3/2}) \quad \text{as } x \to +\infty;$$

see Theorem 30 in [15, p. 83]. It would be very interesting to find a direct potential theoretic argument explaining (8.5). Although (8.4) cannot provide a proof of the PNT for a fixed weight w, this does not preclude the possibility that such a proof can be obtained by finding a *sequence* of weights w_n with $B(w_n) \to 1$, as $n \to \infty$. Thus one needs an insight into the nature of such factors, to address the following problem.

QUESTION 8.1. *For $w(x)$ as in (8.3) and $\alpha = \sum_{i=1}^k \alpha_i m_i$, find*

$$B := \sup_w \frac{-2\log c_w}{4\alpha + 3}. \tag{8.6}$$

If $B = 1$ then find a sequence of weights that gives this value. If $B < 1$ then investigate whether B is attained for a weight of the form (8.3).

A solution of the minimum weighted energy problem for weights (8.3) with real zeros is given in our paper [28]. Arbitrary weights (8.3) with complex zeros are considered in [29]. These papers also contain more background material.

(Pritsker)

9. Positivity of the Defining Equation of a Disjoint Union of Disks.

QUESTION 9.1. *Let $D(a_i, r_i), 1 \le i \le n$, be a finite collection of disjoint open disks in the complex plane. Let $Q(z, w) = \prod_{i=1}^n [(z - a_i)(\overline{w} - \overline{a_i}) - r_i^2]$ be the polarized form of the defining function of the union. Prove by elementary means that the matrix*

$$(-Q(a_i, a_j))_{i,j=1}^n$$

is positive semi-definite.

For a more elaborated proof and the relevance of this question, see [13].

(Putinar)

10. A Conjecture on the Strong Asymptotics of Bergman Orthogonal Polynomials. Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $L := \partial G$ is a Jordan curve and let $\{P_n\}_{n=0}^\infty$ denote the sequence of *Bergman polynomials* of G. This is defined as the sequence

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots,$$

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of polynomials that are orthonormal with respect to the inner product

$$(f,g) := \int_G f(z)\overline{g(z)}dm(z),$$

where dm stands for the 2-dimensional Lebesgue measure. Also, let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the exterior (in $\overline{\mathbb{C}}$) of $\overline{G}$. Then, the *exterior* conformal map Φ associated with G is the conformal map $\Phi : \Omega \to \Delta := \{w : |w| > 1\}$, normalised so that

$$\Phi(z) = cz + \mathcal{O}(1), \quad z \to \infty, \quad c > 0.$$

The constant

$$\operatorname{cap} L = 1/c,$$

is called the (logarithmic) capacity of L.

With respect to the strong asymptotics of the leading coefficient γ_n and of $P_n(z)$, for $z \in \Omega$, we consider the following two formulas:

$$\gamma_n = \sqrt{\frac{n+1}{\pi}} \frac{1}{\operatorname{cap} L^{n+1}} \{1 + \alpha_n\},$$

and

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z)\Phi^n(z) \ \{1 + \beta_n\}, \quad z \in \overline{\Omega}.$$

If the boundary L of G is an *analytic* Jordan curve, then a result due to T. Carleman gives respectively,

$$\alpha_n = \mathcal{O}(\rho^{2n}) \text{ and } \beta = \mathcal{O}(\rho^n), \quad n \to \infty,$$

for some $\rho < 1$; see e.g. [11, pp. 12–13]. In the case where L is *smooth*, typically $L \in C(p+1, s)$, where $p + 1 \in \mathbb{N}$ and $p + s > \frac{1}{2}$, then a result of P.K. Suetin (see [34, Thms 1.1 and 1.2]) gives,

$$\alpha_n = \mathcal{O}\left(\frac{1}{n^{2(p+s)}}\right) \text{ and } \beta_n = \mathcal{O}\left(\frac{\log n}{n^{p+s}}\right), \quad n \to \infty.$$

Apart from the above very important results, we haven't been able to find, in the relevant literature, any similar result concerning the behaviour of the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, associated with more general Jordan curves.

Accordingly, our conjecture is concerned with boundary curves that encountered very frequently in the applications, namely with piecewise analytic Jordan curves. It is based on certain theoretical results and strong numerical evidence and can be stated as follows.

CONJECTURE 10.1. *Assume that the boundary L of G is a piecewise analytic Jordan curve without cusps. Then,*

$$\gamma_n = \sqrt{\frac{n+1}{\pi}} \frac{1}{\operatorname{cap} L^{n+1}} \{1 + \mathcal{O}\left(\frac{1}{n^2}\right)\}, \quad n \to \infty,$$

and

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z)\Phi^n(z) \ \{1 + \mathcal{O}\left(\frac{1}{n}\right)\}, \quad z \in \Omega, \quad n \to \infty.$$

(Stylianopoulos)

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11. Weighted Polynomial Approximation and Equilibrium Measures. Let $w = e^{-Q}$ be a weight on $[-1,1]$, μ_w the associated equilibrium measure and v its density (wherever exists) with respect to linear measure. A. B. J. Kuijlaars showed that if $v(t) \sim |t|^{-a}$ with some $a > 0$ and if $w^n P_n \to f$ uniformly on $[-1,1]$ with some sequence $\{P_n\}$ of polynomials of degree $n = 1, 2, \ldots$, then f must vanish at 0. Show this if only $v(t) \geq c|t|^{-a}$ is true.

Since the conference there has been some progress: My student Péter Varjú has proved a theorem that includes the solution as a special case [38]. Call a measure μ smooth on the interval (a,b) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any two adjacent subintervals $I, J \subseteq (a,b)$ of equal length smaller than δ we have

$$\frac{1}{1+\varepsilon} \leq \frac{\mu(I)}{\mu(J)} \leq 1 + \varepsilon.$$

THEOREM 11.1. *If $w^n P_n \to f$ uniformly on $[-1,1]$ with some sequence $\{P_n\}$ of polynomials of degree $n = 1, 2, \ldots$ and $f(0) \neq 0$, then μ_w is smooth on some interval around 0.*

This rules out a $v(t) \geq c|t|^{-a}$ behavior for the density (as well as a $v(t) \leq c|t|^{a}$ behavior) if approximation is possible for a function that is not zero at 0. Based on these, it is easy to construct a w such that v is positive at all $x \in [-1,1]$, but no nonzero function can be uniformly approximated by $w^n P_n$.

The converse is also true under some mild additional conditions (without any additional condition the converse is not true). To this end call a measure μ doubling on the interval $[a,b]$, if there is a constant M such that for any two adjacent subintervals I, J of $[a,b]$ of equal length we have

$$\frac{1}{M} \leq \frac{\mu(I)}{\mu(J)} \leq M.$$

We say that μ has a positive lower bound on the interval (a,b) if there is a $c > 0$ such that $d\mu(t)/dt \geq c$ on (a,b).

THEOREM 11.2. *Suppose that μ_w is smooth on some interval $(-\delta,\delta)$, and either of the following two conditions is true:*

a) *$\mathrm{supp}(\mu_w)$ can be written as the union of finitely many intervals J_k, and the restriction of μ_w to each J_k is a doubling measure on J_k,*

b) *μ_w has a positive lower bound in $(-\delta,\delta)$.*

Then any continuous f that vanishes outside $(-\delta,\delta)$ is the uniform limit of some sequence of weighted polynomials $w^n P_n$.

For the proofs of these theorems, see [38].

(Totik)

12. Generalized Translation Operator on the Simplex. The generalized translation operator, T_θ^κ, for the weight function

$$W_\kappa(x) = x_1^{\kappa_1} \cdots x_d^{\kappa_d} (1 - |x|_1)^{\kappa_d}, \qquad |x|_1 = (1 - x_1 - \cdots - x_d),$$

on the simplex $T^d = \{x : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |x|_1 \geq 0\}$ is defined in [41]. For $f \in L^2(W_\kappa, T^d)$, the operator T_θ^κ has the orthogonal expansion

$$T_\theta^\kappa f(x) \sim \sum_{k=0}^{\infty} \frac{P_k^{(\lambda-1/2,-1/2)}(\cos 2\theta)}{P_k^{(\lambda-1/2,-1/2)}(1)} \operatorname{proj}_n f(x),$$

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where $\lambda = \sum_{i=1}^{d+1} \kappa_i + (d-1)/2$ and proj_n denotes the projection operator from $L^2(W_\kappa, T^d)$ onto Π_n^d, the space of polynomials of degree at most n in d variables. This operator is used to define a modulus of smoothness in [41], which leads to a characterization of the weighted best approximation by polynomials (the direct and the inverse theorems) on T^d [42].

The operator T_θ^κ is associated with a corresponding one for the weight function

$$U_\kappa(x) := |x_1|^{2\kappa_1} \dots |x_d|^{2\kappa_d}(1 - \|x\|^2)^{\kappa_{d+1}-1/2}$$

on the unit ball $B^d = \{x \in \mathbb{R}^d : \|x\|^2 \le 1\}$, which in turn is associated to the weighted spherical means for the weight function $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on the unit sphere S^d [40]. In the case of the classical weight function $U_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}$ on the unit ball, an integral formula is found for the generalized translation operator [42]. The formula takes the form

$$T_\theta^\mu f(x) = A_\mu \int_{B^d} f\left(\cos\theta x + \sin\theta\, sD(x)U^T(x)\right)(1 - \|s\|^2)^{\mu-1} ds,$$

where $D(x) = \mathrm{diag}\{\sqrt{1 - \|x\|^2}, 1, \dots, 1\}$ is a diagonal matrix, $U(x)$ is a unitary matrix whose first column is $x/\|x\|$ and A_μ is a constant ($s \in \mathbb{R}^d$ is taken as a row vector). For $d = 1$, this becomes the classical generalized translation operator

$$T_s f(t) = b_{\lambda-1/2} \int_{-1}^{1} f\left(st + u\sqrt{1-s^2}\sqrt{1-t^2}\right)(1-u^2)^{\lambda-1} du$$

for the weight function $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ on $[-1, 1]$.

The open question calls for an integral formula for the generalized translation operator with respect to the weight function W_κ on the simplex. The definition of T_θ^κ is given implicitly by an integral relation in [41], relying on the intertwining operator V_κ of Dunkl's operators for $h_\kappa^2(x)$ given above. For this h_κ, the operator V_κ is an explicit integral transform, which suggests that an integral formula for T_θ^κ should exist.

(Xu)

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