Electronic Transactions on Numerical Analysis.
Volume 26, pp. 103-120, 2007.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

A FETI-DP PRECONDITIONER FOR MORTAR METHODS IN THREE DIMENSIONS*

HYEA HYUN KIM†

Abstract. A FETI-DP method is developed for three dimensional elliptic problems with mortar discretization. Mortar matching conditions are considered as the continuity constraints in the FETI-DP formulation. Among them, face average constraints are selected as primal constraints in our FETI-DP formulation to achieve an algorithm as scalable as two dimensional problems. A Neumann-Dirichlet preconditioner is used in the FETI-DP formulation and it gives the condition number bound

$$C \max_{i=1,\cdots,N} \left\{ (1+\log(H_i/h_i))^2 \right\},$$

where H_i and h_i are sizes of domain and mesh for each subdomain, respectively. When the subdomain with the smaller coefficient is chosen as the nonmortar side across the interface, the constant C is independent of H_i, h_i, and the coefficients of the elliptic problem. The proposed algorithm can be applied to two dimensional elliptic problems with edge average constraints only as primal constraints and it can be generalized to geometrically nonconforming subdomain partitions. Numerical results present the performance of the algorithm for elliptic problems with discontinuous coefficients.

Key words. FETI-DP, non-matching grids, mortar methods, preconditioner

AMS subject classifications. 65N30, 65N55

1. Introduction. FETI-DP methods were introduced by Farhat *et al.* [7] and applied to solving elliptic problems with conforming discretizations both in two and three dimensions [8]. In three dimensions, subdomains intersect with neighboring subdomains on faces, edges, or at corners, while they intersect on edges or at corners in two dimensions; the continuity of solution is imposed on faces and edges with dual variables and at corners with primal variables in the dual-primal FETI (FETI-DP) methods. However, numerical results in [7, 8] show that we need additional primal constraints for three dimensional problems to attain the same efficiency as two dimensional problems. For these primal constraints, additional Lagrange multipliers are introduced and they are treated as primal variables in the FETI-DP formulation. FETI-DP methods with various redundant constraints have been studied and their condition number bound was analyzed by Klawonn *et al.* [16, 17] for elliptic problems with heterogeneous coefficients. Numerical results were further provided in [14].

FETI-DP methods have been also applied to mortar finite elements methods [5, 6, 11, 20, 4]. In [5, 6], the condition number bound of FETI-DP operator was analyzed for various types of preconditioners but it depends on ratios of mesh sizes between neighboring subdomains. In [11], a Neumann-Dirichlet preconditioner was proposed and analyzed for elliptic problems with discontinuous coefficients. In this case, the condition number bound does not depend on the mesh sizes and the coefficients when the subdomain with the smaller coefficient is chosen as the nonmortar side. Moreover, numerical results show that the Neumann-Dirichlet preconditioner works much more efficiently than other FETI-DP preconditioners for elliptic problems with highly discontinuous coefficients. Recently, a preconditioner, with its weight factor depending on mesh parameters and the problem coefficients, was introduced and analyzed to be independent of coefficients and mesh parameters for two dimensional elliptic problems, see Dokeva *et. al.* [4]. For three dimensional problems, FETI methods with mortar discretizations were developed and their numerical results were provided in [19].

*Received November 11, 2005. Accepted for publication November 22, 2006. Recommended by Y. Kuznetsov. This work was supported by BK21 Project.

†National Institute for Mathematical Sciences, 385-16, Doryong-dong, Yuseong-gu, Daejeon 305-340, South Korea (hhk@nims.re.kr).

ETNA
Kent State University
etna@mcs.kent.edu

The primary contribution of our work is the extension of the FETI-DP method in [11] to three dimensional problems and to the second generation of mortar methods. In [11], vertex continuity constraints are introduced as primal constraints. However, for the three dimensional case we need primal constraints other than the vertex constraints to obtain a method as scalable as the two dimensional case in [11]. We select as constraints that averages of the solution across subdomain interfaces are the same, which are the so-called face constraints in [17]. Similarly to the previous work in [11], we propose a Neumann-Dirichlet preconditioner for the FETI-DP formulation and show that the condition number bound

$$C \max_{i=1,\cdots,N} \left\{ \left(1 + \log\left(H_i/h_i\right)\right)^2 \right\}$$

holds for elliptic problems with discontinuous constant coefficients. Here, H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and the constant C is independent of H_i, h_i, and the coefficients of elliptic problems. In our FETI-DP formulation, we follow a change of basis formulation introduced in [15, 18]. The change of basis makes the analysis of FETI-DP algorithms easier when primal constraints other than the vertex continuity constraints are used. Moreover it gives an efficient and robust implementation of FETI-DP algorithms [12, 13].

We note that edge average constraints can be considered as primal constraints for two dimensional problems. The continuity constraints at vertices can not be selected as primal constraints for the second generation of mortar methods [1]. We are able to extend the result in [11] to the second generation of mortar methods by introducing edge average constraints and using the change of basis formulation. Furthermore, the condition number bound estimate of this case can be carried out similarly to three dimensional case presented in this paper.

This paper is organized as follows. In Section 2, we introduce finite element spaces and norms and in Section 3, we derive the FETI-DP formulation with the Neumann-Dirichlet preconditioner. Section 4 is devoted to analyzing the condition number bound of the FETI-DP algorithm. Numerical tests are presented in Section 5.

Throughout the paper, C or c $(\leq C)$ denotes a generic positive constant that does not depend on any mesh parameters or on the coefficients of the elliptic problems.

2. Finite element spaces and norms.

2.1. A model problem and Sobolev spaces. Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm

$$\|v\|^2_{L^2(\Omega)} := \int_\Omega v^2 \, dx.$$

The space $H^1(\Omega)$ is the set of functions, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{H^1(\Omega)} := \left(\int_\Omega \nabla v \cdot \nabla v \, dx + \frac{1}{d_\Omega^2} \int_\Omega v^2 \, dx \right)^{1/2},$$

where d_Ω denotes the diameter of Ω.

We consider the following model elliptic problem:

Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (\rho(x) \nabla u(x)) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2.1}$$

ETNA
Kent State University
etna@mcs.kent.edu

where $f(x)$ is a square integrable function and $\rho(x)$ is a positive and bounded function in Ω.

Let Ω be partitioned into non-overlapping polyhedral subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that each subdomain intersects its neighboring subdomains on a full face, a full edge or at a vertex. Each subdomain Ω_i is equipped with a quasi uniform triangulation Ω_i^h, which consists of tetrahedrons. These triangulations need not be aligned across subdomain interfaces.

Each subdomain Ω_i is equipped with a conforming linear finite element space

$$X_i := \{v \in H_D^1(\Omega_i) \ : \ v|_\tau \in P_1(\tau),\ \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) \ : \ v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ. We assume that

$$\rho(x) = \rho_i, \quad \forall x \in \Omega_i,$$

where ρ_i is a positive constant. A bilinear form $a_i(\cdot,\cdot) : X_i \times X_i \to \mathbb{R}$ is defined as

$$a_i(u_i, v_i) := \rho_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx. \tag{2.2}$$

We now introduce Sobolev spaces defined on the boundaries of subdomains. The space $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ equipped with the norm

$$\|w_i\|^2_{H^{1/2}(\partial\Omega_i)} := |w_i|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{d_{\Omega_i}}\|w_i\|^2_{L^2(\partial\Omega_i)},$$

where

$$|w_i|^2_{H^{1/2}(\partial\Omega_i)} := \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x-y|^3} \, ds(x)\, ds(y).$$

For any face $F_{ij} \in \partial\Omega_i$, $H_{00}^{1/2}(F_{ij})$ is the set of functions in $L^2(F_{ij})$ whose zero extension into $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$ and is equipped with the norm

$$\|v\|^2_{H_{00}^{1/2}(F_{ij})} := |v|^2_{H^{1/2}(F_{ij})} + \int_{F_{ij}} \frac{v^2(x)}{\text{dist}(x, \partial F_{ij})} \, ds.$$

From Section 4.1 in [25], we have the following relation for $v \in H_{00}^{1/2}(F_{ij})$:

$$c\|\widetilde{v}\|_{H^{1/2}(\partial\Omega_i)} \leq \|v\|_{H_{00}^{1/2}(F_{ij})} \leq C\|\widetilde{v}\|_{H^{1/2}(\partial\Omega_i)}, \tag{2.3}$$

where $\widetilde{v}$ is the zero extension of v to $\partial\Omega_i$, i.e., $\widetilde{v} = v$ on F_{ij} and $\widetilde{v} = 0$ on $\partial\Omega_i \setminus F_{ij}$.

2.2. Mortar matching conditions. Let us define

$$X := \left\{ v \in \prod_{i=1}^N X_i \ : \ v \text{ is continuous at subdomain vertices} \right\}$$

and

$$W := \left\{ w \in \prod_{i=1}^N W_i, \ : \ w \text{ is continuous at subdomain vertices} \right\},$$

ETNA
Kent State University
etna@mcs.kent.edu

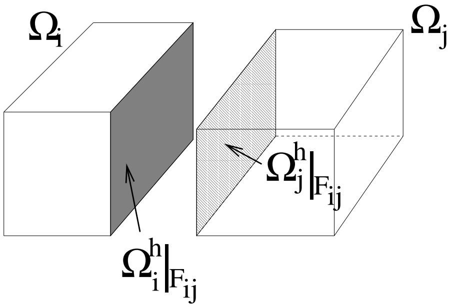

FIG. 2.1. *Mortar and nonmortar sides of F_{ij}*

where W_i is the trace space of X_i, i.e., $W_i = X_i|_{\partial\Omega_i}$. We will approximate the solution of the problem (2.1) in X. We note that the space X is not contained in $H^1(\Omega)$. In order to approximate the solution of the problem (2.1) in the nonconforming finite element space X, we impose the mortar matching condition on X, for which jumps of a function in X across a common face (interface) are orthogonal to a Lagrange multiplier space, i.e., $v = (v_1, \cdots, v_N) \in X$ satisfies

$$\int_{F_{ij}} (v_i - v_j)\lambda_{ij}\, ds = 0 \ \ \forall\ \lambda_{ij} \in M_{ij}, \forall F_{ij}, \tag{2.4}$$

where M_{ij} is a Lagrange multiplier space given on the common interface $F_{ij} := \partial\Omega_i \cap \partial\Omega_j$.

On F_{ij}, we distinguish $\Omega_i^h|_{F_{ij}}$ and $\Omega_j^h|_{F_{ij}}$ as in Figure 2.1 and choose one as a mortar side and the other as a nonmortar side. On each nonmortar side, we define a finite element space

$$W_{ij} := \{v|_{F_{ij}} \in H_0^1(F_{ij}) \ : \ v \in X_{n(ij)}\}, \tag{2.5}$$

where $n(ij)$ is the nonmortar side (nonmortar subdomain) of F_{ij}.

To get the optimal order approximation, we need the following abstract conditions on the space M_{ij};

(A.1) The basis $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ are locally supported, that is, the number of elements in $\Omega_i^h|_{F_{ij}}$, which have nonempty intersections with the simply connected support of ξ_k^{ij}, is bounded independently of mesh sizes and F_{ij}.

(A.2) W_{ij} and M_{ij} have the same dimension.

(A.3) There is a constant C such that

$$\|\phi\|_{L^2(F_{ij})} \le C \sup_{\psi \in M_{ij}} \frac{\int_{F_{ij}} \phi\psi\, ds}{\|\psi\|_{L^2(F_{ij})}} \quad \forall \phi \in W_{ij}.$$

(A.4) For $\mu \in H^{k-1/2}(F_{ij})$, there exists $\mu_h \in M_{ij}$ such that

$$\|\mu - \mu_h\|^2_{L^2(F_{ij})} \le C h_i^{2k-1} |\mu|^2_{H^{k-1/2}(F_{ij})},$$

where k is the order of finite elements in X_i.

The condition (A.4) implies that $1 \in M_{ij}$. In the following, we assume that the Lagrange multiplier space M_{ij} satisfies the above conditions; the standard Lagrange multiplier space in [2] and the Lagrange multipliers with dual basis in [9] are those examples.

ETNA
Kent State University
etna@mcs.kent.edu

In our FETI-DP formulation, we will use the mortar matching condition (2.4) as continuity constraints. These continuity constraints can further be written into

$$\sum_{i=1}^{N} B_i w_i = 0,$$

where $w_i = v_i|_{\partial\Omega_i}$. We note that the matrices B_i are not boolean matrices as in the original FETI (or FETI-DP) methods.

In the following, we will use the same notation for finite element functions and the corresponding vectors of nodal values. For example, w_i is used to denote a finite element function or the vector of nodal values of that function. The same applies to the notations for function spaces such as W_i, X, W, etc.

3. FETI-DP formulation.

3.1. FETI-DP operator. In this section, we formulate the FETI-DP operator for the problem (2.1) with the mortar matching condition as continuity constraints. For $3D$ elliptic problems, it was shown that using the primal variables at vertices is not enough to get the same condition number bound as $2D$ problems; see numerical results in [7, 8]. Hence, additional primal constraints are introduced to accelerate the convergence of the FETI-DP method.

For the $3D$ elliptic problems with conforming discretizations, Klawonn *et al.* [16] developed FETI-DP methods with various redundant constraints. They introduced edge average or face average constraints as primal constraints to achieve the same condition number bound as $2D$ elliptic problems. The continuity constraints on edges (faces) are that the averages of functions across a common edge (face) are the same. In [17], they extended the results to a case with face constraints only.

In the mortar discretizations, two sets of primal constraints are possible; the one that contains continuity constraints at vertices and the face average constraints, and the other one with only the face average constraints. Using the second set of the primal constraints we can generalize our algorithm and theory to the second generation of mortar discretizations [1] and to geometrically non-conforming partitions [10], since we can relax the continuity at subdomain vertices. In our work, we consider the first case with the vertex continuity and the face averages as primal constraints. We may impose the face average constraints by introducing additional Lagrange multipliers and then treat them as primal variables in the FETI-DP formulation; see [7, 8, 16]. In our FETI-DP formulation, we follow the change of basis formulation introduced in [15, 18] that leads to much easier analysis and more robust implementation; see [12, 13].

On each interface F_{ij}, for $w_{ij} = w_i|_{F_{ij}}$ (or $w_{ij} = w_j|_{F_{ij}}$) we consider a change of variables so that

$$w_{ij} = T_{F_{ij}} \begin{pmatrix} w_\Delta^{(ij)} \\ w_\Pi^{(ij)} \end{pmatrix},$$

where $T_{F_{ij}}$ retains unknowns at the boundary of F_{ij}, $w_\Pi^{(ij)}$ is the average of w_{ij} on F_{ij}, i.e.,

$$w_\Pi^{(ij)} = \frac{\int_{F_{ij}} w_{ij}\, ds}{\int_{F_{ij}} 1\, ds},$$

and the function $w_\Delta^{(ij)}$ has the average value zero on F_{ij}, i.e.,

$$\int_{F_{ij}} w_\Delta^{(ij)}\, ds = 0.$$

ETNA
Kent State University
etna@mcs.kent.edu

We note that $w_\Delta^{(ij)}$ is a function in the above equation and it can be represented using the change of basis given by the transform $T_{F_{ij}}$

After the transforms, we express the unknowns w_i into

$$w_i = \begin{pmatrix} w_\Delta^{(i)} \\ w_\Pi^{(i)} \end{pmatrix},$$

where Π stands for the unknowns of primal variables, i.e., the averages on faces and unknowns at vertices, and Δ stands for the remaining unknowns. These notations will be used throughout this paper.

We now consider a subspace $\widetilde{W}$ of W that satisfies the primal constraints

$$\widetilde{W} := \Big\{ w \in W \ : \ \int_{F_{ij}} (w_i - w_j)\, ds = 0 \quad \text{and } w \text{ is continuous at subdomain vertices} \Big\}. \tag{3.1}$$

Let W_Δ be a space of the vectors,

$$w_\Delta = \begin{pmatrix} w_\Delta^{(1)} \\ \vdots \\ w_\Delta^{(N)} \end{pmatrix},$$

and let W_Π be the space of primal unknowns w_Π. We then decompose the space $\widetilde{W}$ into the dual and the primal parts,

$$\widetilde{W} = W_\Delta \oplus W_\Pi.$$

We define

$$R_\Pi^{(i)} : W_\Pi \to W_\Pi|_{\Omega_i}$$

that restricts the primal unknowns to the local primal unknowns.

Let $S^{(i)}$ be the Schur complement matrix obtained from the bilinear form $a_i(\cdot,\cdot)$ in (2.2) and let $g^{(i)}$ be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i\, dx$. After the change of variables, the matrix $S^{(i)}$ and vector $g^{(i)}$ are written into

$$S^{(i)} = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & S_{\Delta\Pi}^{(i)} \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix}, \quad g^{(i)} = \begin{pmatrix} g_\Delta^{(i)} \\ g_\Pi^{(i)} \end{pmatrix}.$$

We recall the mortar matching condition

$$\sum_{i=1}^N B_i w_i = 0.$$

Since $w \in \widetilde{W}$ satisfies the face average constraints and $1 \in M_{ij}$, the above continuity constraints are redundant for $w \in \widetilde{W}$. We consider a subspace $\overline{M}_{ij}$ of M_{ij} that has one less basis than M_{ij}. We impose the mortar matching condition (2.4) with the Lagrange multiplier space $\overline{M}_{ij}$ instead of M_{ij} and obtain its matrix representation

$$\sum_{i=1}^N \overline{B}_i w_i = 0.$$

ETNA
Kent State University
etna@mcs.kent.edu

The above constraints are then non-redundant constraints for $w \in \widetilde{W}$. We rewrite it as

$$\sum_{i=1}^{N} (\overline{B}_{\Delta}^{(i)} w_{\Delta}^{(i)} + \overline{B}_{\Pi}^{(i)} w_{\Pi}^{(i)}) = 0. \tag{3.2}$$

Let

$$M_{\Delta} = \prod_{ij} \overline{M}_{ij}.$$

We then obtain the following mixed formulation of the problem (2.1) with the constraints (3.2):

Find $(w_{\Delta}, w_{\Pi}, \lambda) \in W_{\Delta} \times W_{\Pi} \times M_{\Delta}$ satisfying

$$\begin{aligned} S_{\Delta\Delta} w_{\Delta} + S_{\Delta\Pi} w_{\Pi} + B_{\Delta}^{t} \lambda &= g_{\Delta}, \\ S_{\Pi\Delta} w_{\Delta} + S_{\Pi\Pi} w_{\Pi} + B_{\Pi}^{t} \lambda &= g_{\Pi}, \\ B_{\Delta} w_{\Delta} + B_{\Pi} w_{\Pi} &= 0, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} S_{\Delta\Delta} &= \operatorname{diag}_{i=1,\cdots,N} \left(S_{\Delta\Delta}^{(i)} \right), \\ S_{\Delta\Pi} &= \begin{pmatrix} S_{\Delta\Pi}^{(1)} R_{\Pi}^{(1)} \\ \vdots \\ S_{\Delta\Pi}^{(N)} R_{\Pi}^{(N)} \end{pmatrix}, \\ S_{\Pi\Delta} &= S_{\Delta\Pi}^{t}, \\ S_{\Pi\Pi} &= \sum_{i=1}^{N} (R_{\Pi}^{(i)})^{t} S_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)}, \\ B_{\Delta} &= \left(\overline{B}_{\Delta}^{(1)}, \cdots, \overline{B}_{\Delta}^{(N)} \right), B_{\Pi} = \sum_{i=1}^{N} \overline{B}_{\Pi}^{(i)} R_{\Pi}^{(i)}, \\ g_{\Delta} &= \begin{pmatrix} g_{\Delta}^{(1)} \\ \vdots \\ g_{\Delta}^{(N)} \end{pmatrix}, \; g_{\Pi} = \sum_{i=1}^{N} (R_{\Pi}^{(i)})^{t} \; g_{\Pi}^{(i)}, \; w_{\Delta} = \begin{pmatrix} w_{\Delta}^{(1)} \\ \vdots \\ w_{\Delta}^{(N)} \end{pmatrix}. \end{aligned}$$

After eliminating w_{Δ} and w_{Π} from (3.3), we obtain

$$F_{DP} \lambda = d.$$

We note that

$$\langle F_{DP} \lambda, \lambda \rangle = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle \widetilde{S} w, w \rangle},$$

where

$$\begin{aligned} B &= \begin{pmatrix} B_{\Delta} & B_{\Pi} \end{pmatrix}, \\ \widetilde{S} &= \begin{pmatrix} S_{\Delta\Delta} & S_{\Delta\Pi} \\ S_{\Pi\Delta} & S_{\Pi\Pi} \end{pmatrix}, \quad w = \begin{pmatrix} w_{\Delta} \\ w_{\Pi} \end{pmatrix}. \end{aligned}$$

More precisely, we compute

$$F_{DP} = B\widetilde{S}^{-1}B^t = F_{\Delta\Delta} + F_{\Delta\Pi}F_{\Pi\Pi}^{-1}F_{\Pi\Delta},$$

where

$$\begin{aligned}
F_{\Delta\Delta} &= B_\Delta S_{\Delta\Delta}^{-1}B_\Delta^t = \sum_{i=1}^N \overline{B}_\Delta^{(i)}(S_{\Delta\Delta}^{(i)})^{-1}(\overline{B}_\Delta^{(i)})^t,\\
F_{\Delta\Pi} &= -(B_\Pi - B_\Delta S_{\Delta\Delta}^{-1}S_{\Delta\Pi}) = -\sum_{i=1}^N(\overline{B}_\Pi^{(i)} - \overline{B}_\Delta^{(i)}(S_{\Delta\Delta}^{(i)})^{-1}S_{\Delta\Pi}^{(i)})R_\Pi^{(i)},\\
F_{\Pi\Delta} &= F_{\Delta\Pi}^t,\\
F_{\Pi\Pi} &= \sum_{i=1}^N (R_\Pi^{(i)})^t(S_{\Pi\Pi}^{(i)} - S_{\Pi\Delta}^{(i)}(S_{\Delta\Delta}^{(i)})^{-1}S_{\Delta\Pi}^{(i)})R_\Pi^{(i)}.
\end{aligned}$$

From the above formula, we can see that the computation $F_{DP}\lambda$ can be done by applying matrix-vector multiplications in each subdomain except the term $F_{\Pi\Pi}^{-1}$.

3.2. Preconditioner. We derive a preconditioner from the similar idea to [11], in which a Neumann-Dirichlet preconditioner is built from a dual norm on the Lagrange multiplier space using a duality pairing between the Lagrange multiplier space and the finite element space on nonmortar sides. In the following, the idea is provided in more detail.

We further decompose the space $\widetilde{W}$ into

$$\widetilde{W} = W_\Delta \oplus W_\Pi = W_{\Delta,n} \oplus W_{\Delta,m} \oplus W_\Pi, \tag{3.4}$$

where the subscript n stands for the space of vectors for the unknowns at the interior of nonmortar faces and the subscript m stands for the remaining unknowns. In other words, we split a vector $w_\Delta \in W_\Delta$ into

$$w_\Delta = \begin{pmatrix} w_{\Delta,n} \\ w_{\Delta,m} \end{pmatrix},$$

where $w_{\Delta,n}$ are unknowns at the interior of nonmortar faces and $w_{\Delta,m}$ are the remaining unknowns. We recall the mortar matching condition

$$B_\Delta w_\Delta + B_\Pi w_\Pi = 0.$$

It is then written into

$$B_{\Delta,n}w_{\Delta,n} + B_{\Delta,m}w_{\Delta,m} + B_\Pi w_\Pi = 0.$$

Here, the matrix $B_{\Delta,n}$ is square and invertible.

We will propose the Neumann-Dirichlet preconditioner of the form

$$\widehat{F}_{DP}^{-1} = BD\widetilde{S}DB^t,$$

where B and D are given by

$$B = \begin{pmatrix} B_{\Delta,n} & B_{\Delta,m} & B_\Pi \end{pmatrix}, \quad D = \begin{pmatrix} D_{nn} & & \\ & D_{mm} & \\ & & D_{\Pi\Pi} \end{pmatrix}.$$

The Neumann-Dirichlet preconditioner provides the weights

$$D_{nn} = \left(B_{\Delta,n}^t B_{\Delta,n}\right)^{-1}, \quad D_{mm} = 0, \quad D_{\Pi\Pi} = 0. \tag{3.5}$$

This preconditioner is originated from a dual norm on the Lagrange multiplier space M_Δ; see [11]. We recall the space $\widetilde{W}$ and $W_{\Delta,n}$ in (3.1) and (3.4). For $w \in \widetilde{W}$, we define a norm

$$\|w\|_{\widetilde{W}}^2 = \langle \widetilde{S} w, w \rangle.$$

Since a function $w_{\Delta,n} \in W_{\Delta,n}$ has the zero average on each face F_{ij} and has zero values at subdomain vertices, its zero extension $\widetilde{w}_{\Delta,n}$ to W satisfies the primal constraints, i.e., $\widetilde{w}_{\Delta,n} \in \widetilde{W}$. We may write

$$\widetilde{w}_{\Delta,n} = \begin{pmatrix} w_{\Delta,n} \\ 0 \\ 0 \end{pmatrix} \in \widetilde{W}.$$

We then define a norm for $w_{\Delta,n}$ by

$$\|w_{\Delta,n}\|_{W_{\Delta,n}}^2 = \langle \widetilde{S} \widetilde{w}_{\Delta,n}, \widetilde{w}_{\Delta,n} \rangle,$$

and a dual norm on the space M_Δ by

$$\|\lambda\|_{W'_{\Delta,n}} = \max_{w_{\Delta,n} \in W_{\Delta,n}} \frac{\langle B_{\Delta,n} w_{\Delta,n}, \lambda \rangle}{\|w_{\Delta,n}\|_{W_{\Delta,n}}}. \tag{3.6}$$

The Neumann-Dirichlet preconditioner $\widehat{F}_{DP}^{-1}$ is given by

$$\langle \widehat{F}_{DP} \lambda, \lambda \rangle = \|\lambda\|_{W'_{\Delta,n}}^2. \tag{3.7}$$

Similarly, the matrix F_{DP} can be obtained from a dual norm

$$\langle F_{DP} \lambda, \lambda \rangle = \|\lambda\|_{\widetilde{W}'}^2,$$

where the dual norm is given by

$$\|\lambda\|_{\widetilde{W}'}^2 = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_{\widetilde{W}}^2} = \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle \widetilde{S} w, w \rangle}.$$

The preconditioner is originated from the idea that these two dual norms will be sufficiently close so as to get $\widehat{F}_{DP}^{-1}$ as a good preconditioner for F_{DP}. The lower bound estimate can be done from

$$\|\lambda\|_{W'_{\Delta,n}}^2 = \max_{w_{\Delta,n} \in W_{\Delta,n}} \frac{\langle B \widetilde{w}_{\Delta,n}, \lambda \rangle^2}{\langle \widetilde{S} \widetilde{w}_{\Delta,n}, \widetilde{w}_{\Delta,n} \rangle} \leq \max_{w \in \widetilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_{\widetilde{W}}^2} = \|\lambda\|_{\widetilde{W}'}^2, \tag{3.8}$$

because $\widetilde{w}_{\Delta,n}$ is contained in $\widetilde{W}$. In the following section, we will provide an upper bound of the Neumann-Dirichlet preconditioner $\widehat{F}_{DP}^{-1}$.

From (3.6) and (3.7), we find the following form of the preconditioner

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} \begin{pmatrix} ((B_{\Delta,n}^{(i)})^t)^{-1} & 0 \end{pmatrix} S_{\Delta\Delta}^{(i)} \begin{pmatrix} (B_{\Delta,n}^{(i)})^{-1} \\ 0 \end{pmatrix}, \tag{3.9}$$

ETNA
Kent State University
etna@mcs.kent.edu

that provides the weights in (3.5). The computation $\widehat{F}_{DP}^{-1}\lambda$ can be done by solving a Neumann-Dirichlet problem in each subdomain, i.e.,

$$S_{\Delta\Delta}^{(i)} = (A_{\Delta\Delta}^{(i)} - A_{\Delta I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Delta}^{(i)}),$$

where

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix}.$$

Here $A^{(i)}$ is the stiffness matrix of the bilinear form $a_i(u,v)$ for $u,v \in X_i$ and the subscripts I, Π, and Δ stand for the subdomain interior unknowns, the unknowns for the primal variables, and the remaining unknowns, respectively.

When we compute

$$S_{\Delta\Delta}^{(i)} \begin{pmatrix} (B_{\Delta,n}^{t})^{-1} \\ 0 \end{pmatrix} \lambda,$$

we solve the problem

$$A_{II}^{(i)} u_I^{(i)} = A_{I\Delta}^{(i)} \begin{pmatrix} (B_{\Delta,n}^{(i)})^{-1}\lambda \\ 0 \end{pmatrix},$$

where Neumann boundary condition, $(B_{\Delta,n}^{(i)})^{-1}\lambda$, is given on the nonmortar faces and zero Dirichlet boundary condition is provided on the remaining part of the subdomain boundary.

4. Condition number bound estimation.

On the interface F_{ij}, we assume that Ω_i is the nonmortar side and Ω_j is the mortar side. We denote the mesh sizes in each subdomains Ω_i and Ω_j by h_i and h_j, respectively. We recall the space W_{ij} in (2.5).

DEFINITION 4.1. *We define a mortar projection $\pi_{ij} : L^2(F_{ij}) \to W_{ij}$ for $v \in L^2(F_{ij})$ by*

$$\int_{F_{ij}} (v - \pi_{ij}v)\lambda_{ij}\, ds \; = 0 \;\; \forall \lambda_{ij} \in M_{ij}.$$

For the space M_{ij} satisfying the conditions (A.1)-(A.4) (see Section 2.2), we can show that the mortar projection π_{ij} is continuous on the space $H_{00}^{1/2}(F_{ij})$ (see [9] or [24]);

$$\|\pi_{ij}v\|_{H_{00}^{1/2}(F_{ij})} \le C\|v\|_{H_{00}^{1/2}(F_{ij})} \;\; \forall v \in H_{00}^{1/2}(F_{ij}),$$

where C is a constant not depending on H_i and h_i. Moreover, the projection is continuous on the space $L^2(F_{ij})$.

LEMMA 4.2. *When Ω_i is the nonmortar side of the interface $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$, any function $w \in H^{1/2}(F_{ij})$ satisfies*

$$\|\pi_{ij}w\|_{H_{00}^{1/2}(F_{ij})}^2 \le C\left(1 + log\frac{H_i}{h_i}\right)^2 \|w\|_{H^{1/2}(F_{ij})}^2,$$

where H_i/h_i denotes the number of elements across the nonmortar subdomain Ω_i.

ETNA
Kent State University
etna@mcs.kent.edu

Proof. Let Qw be the L^2-projection onto the finite element space $W_i|_{F_{ij}}$, that is the restriction of W_i to the interface F_{ij}. It then satisfies, see [3, Chapter II],

$$\|w - Qw\|^2_{L^2(F_{ij})} \le Ch_i \|w\|^2_{H^{1/2}(F_{ij})}, \quad \|Qw\|^2_{H^{1/2}(F_{ij})} \le C\|w\|^2_{H^{1/2}(F_{ij})}. \tag{4.1}$$

For the function Qw, we denote by $I^h_{F_{ij}}(Qw)$ and $I^h_{\partial F_{ij}}(Qw)$ the nodal interpolants of Qw to the space $W_i|_{F_{ij}}$ that vanish at the nodes on ∂F_{ij} and at the nodes interior to F_{ij}, respectively. We then decompose Qw into

$$Qw = I^h_{F_{ij}}(Qw) + I^h_{\partial F_{ij}}(Qw).$$

We now consider

$$\|\pi_{ij}(w)\|^2_{H^{1/2}_{00}(F_{ij})} \le 2\|\pi_{ij}(w - Qw)\|^2_{H^{1/2}_{00}(F_{ij})} + 2\|\pi_{ij}(Qw)\|^2_{H^{1/2}_{00}(F_{ij})}.$$

Using an inverse inequality, the continuity of π_{ij} in the L^2-norm, and (4.1), the first term is estimated by

$$\|\pi_{ij}(w - Qw)\|^2_{H^{1/2}_{00}(F_{ij})} \le Ch_i^{-1}\|\pi_{ij}(w - Qw)\|^2_{L^2(F_{ij})} \le C\|w\|^2_{H^{1/2}(F_{ij})}. \tag{4.2}$$

We consider the second term,

$$\begin{aligned}
\|\pi_{ij}(Qw)\|^2_{H^{1/2}_{00}(F_{ij})} &\le 2\|\pi_{ij}(I^h_{F_{ij}}(Qw))\|^2_{H^{1/2}_{00}(F_{ij})} + 2\|\pi_{ij}((I^h_{\partial F_{ij}}(Qw))\|^2_{H^{1/2}_{00}(F_{ij})} \\
&\le C\left(1 + \log\frac{H_i}{h_i}\right)^2 \|Qw\|^2_{H^{1/2}(F_{ij})} + Ch_i^{-1}\|\pi_{ij}((I^h_{\partial F_{ij}}(Qw))\|^2_{L^2(F_{ij})} \\
&\le C\left(1 + \log\frac{H_i}{h_i}\right)^2 \|w\|^2_{H^{1/2}(F_{ij})} + Ch_i^{-1}\|I^h_{\partial F_{ij}}(Qw)\|^2_{L^2(F_{ij})} \\
&\le C\left(1 + \log\frac{H_i}{h_i}\right)^2 \|w\|^2_{H^{1/2}(F_{ij})} + C\|I^h_{\partial F_{ij}}(Qw)\|^2_{L^2(\partial F_{ij})} \\
&\le C\left(1 + \log\frac{H_i}{h_i}\right)^2 \|w\|^2_{H^{1/2}(F_{ij})}. \qquad (4.3)
\end{aligned}$$

Here we have used a face lemma [21, Lemma 4.24], an inverse inequality, (4.1), the continuity of π_{ij} in $L^2(F_{ij})$, and an edge lemma [21, Lemma 4.17]. From (4.2) and (4.3), we obtain the desired estimate. □

LEMMA 4.3. *For $w = (w_1, \cdots, w_N) \in \widetilde{W}$, we have*

$$\|\pi_{ij}(w_i - w_j)\|^2_{H^{1/2}(F_{ij})} \le C\left(1 + \log\frac{H_i}{h_i}\right)^2 \left(|w_i|^2_{H^{1/2}(\partial\Omega_i)} + |w_j|^2_{H^{1/2}(\partial\Omega_j)}\right),$$

where Ω_i is the nonmortar side of the interface F_{ij}.

Proof. Since w satisfies the primal constraints, w_i and w_j have the same average value c_{ij} on F_{ij}. We then obtain the resulting bound from Lemma 4.2 and a Poincaré inequality by replacing w_i and w_j by $w_i - c_{ij}$ and $w_j - c_{ij}$, respectively. □

REMARK 4.4. The face average constraints are important in applying a Poincaré inequality to the above analysis, while the continuity at vertices are not necessary. Since the continuity constraints at the vertices can be relaxed, it is possible to extend the theory and algorithm to the second generation of mortar discretizations and to geometrically non-conforming subdomain partitions.

To obtain a condition number bound that does not depend on mesh sizes and the coefficients, we need the following assumptions.

ASSUMPTION 4.5. *On a common interface F_{ij}, we choose the subdomain Ω_i with smaller ρ_i as the nonmortar side and the subdomain Ω_j with larger ρ_j as the mortar side.*

The Assumption 4.5 is conventionally used in the analysis of the mortar methods; see [22, Remark 3.1].

We now estimate the upper bound of the FETI-DP algorithm.

LEMMA 4.6. *Under Assumption 4.5, for $\lambda \in M_\Delta$, we have*

$$\langle F_{DP}\lambda, \lambda\rangle \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \langle \widehat{F}_{DP}\lambda, \lambda\rangle,$$

where the constant C does not depend on h_i, H_i, and ρ_i.

Proof. We note that

$$\langle F_{DP}\lambda, \lambda\rangle = \|\lambda\|^2_{\widetilde{W}'} = \max_{w\in\widetilde{W}} \frac{\langle Bw, \lambda\rangle^2}{\langle \widetilde{S}w, w\rangle}, \tag{4.4}$$

$$\langle \widehat{F}_{DP}\lambda, \lambda\rangle = \|\lambda\|^2_{W'_{\Delta,n}} = \max_{w_{\Delta,n}\in W_{\Delta,n}} \frac{\langle B_{\Delta,n} w_{\Delta,n}, \lambda\rangle^2}{\|w_{\Delta,n}\|^2_{W_{\Delta,n}}}. \tag{4.5}$$

From the definitions of B and π_{ij}, we have

$$\langle Bw, \lambda\rangle^2 = \left(\sum_{ij} \int_{F_{ij}} \pi_{ij}(w_i - w_j)\lambda_{ij}\, ds \right)^2.$$

Let

$$z_{ij} = \pi_{ij}(w_i - w_j).$$

From $1 \in M_{ij}$ and the definition of π_{ij}, we have

$$\int_{F_{ij}} z_{ij}\, ds = \int_{F_{ij}} (w_i - w_j)\, ds = 0.$$

Since z_{ij} has zero average on F_{ij} and has zero values on ∂F_{ij}, after the transform introduced in Section 3.1 we may write

$$\langle Bw, \lambda\rangle = \sum_{ij} \int_{F_{ij}} \pi_{ij}(w_i - w_j)\lambda_{ij}\, ds = \langle B_{\Delta,n} z_{\Delta,n}, \lambda\rangle^2, \tag{4.6}$$

where $z_{\Delta,n} = z_{ij}$ on F_{ij} and $z_{\Delta,n} \in W_{\Delta,n}$. We note that $z_{\Delta,n}$can be a function or a vector of unknowns. From the above relation (4.6) and (4.5), we get

$$\langle Bw, \lambda\rangle^2 \leq \langle \widehat{F}_{DP}\lambda, \lambda\rangle \|z_{\Delta,n}\|^2_{W_{\Delta,n}}. \tag{4.7}$$

We will show that

$$\|z_{\Delta,n}\|^2_{W_{\Delta,n}} \leq C \max_i \left\{ \left(1 + \log\frac{H_i}{h_i}\right)^2 \right\} \langle \widetilde{S}w, w\rangle. \tag{4.8}$$

The desired bound then follows from (4.4) and (4.7).

ETNA
Kent State University
etna@mcs.kent.edu

Let $\widetilde{z}_{\Delta,n}$ be the zero extension of $z_{\Delta,n}$ to the all interfaces, i.e., into the space W. It is easy to see that $\widetilde{z}_{\Delta,n} \in \widetilde{W}$. Let $z_i = \widetilde{z}_{\Delta,n}|_{\partial\Omega_i}$, the restriction of $\widetilde{z}_{\Delta,n}$ to the subdomain Ω_i. We then obtain

$$\begin{aligned}
\|z_{\Delta,n}\|^2_{W_{\Delta,n}} &= \langle \widetilde{S}\widetilde{z}_{\Delta,n}, \widetilde{z}_{\Delta,n}\rangle \\
&= \sum_i \langle S^{(i)} z_i, z_i\rangle \\
&\le C \sum_i \rho_i |z_i|^2_{H^{1/2}(\partial\Omega_i)} \\
&\le C \sum_i \sum_j \rho_i \|z_{ij}\|^2_{H^{1/2}_{00}(F_{ij})} \\
&= C \sum_i \sum_j \rho_i \|\pi_{ij}(w_i - w_j)\|^2_{H^{1/2}_{00}(F_{ij})} \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \left(\sum_{ij} \left(\langle S^{(i)} w_i, w_i\rangle + \frac{\rho_i}{\rho_j} \langle S^{(j)} w_j, w_j\rangle \right) \right).
\end{aligned}$$

Here we used the well-known inequality

$$\rho_i |z_i|^2_{H^{1/2}(\partial\Omega_i)} \le \langle S^{(i)} z_i, z_i\rangle \le C\rho_i |z_i|^2_{H^{1/2}(\partial\Omega_i)},$$

the relation in (2.3), and Lemma 4.3, and Ω_i is the nonmortar side of the interface F_{ij}. From Assumption 4.5, $\rho_i/\rho_j \le 1$ so that we have shown (4.8) with the constant C independent of h_i, H_i, and ρ_i.□

REMARK 4.7. The above analysis can be applied to two dimensional problems with edge average constraints only as primal constraints.

From the lower bound estimate in (3.8) and the upper bound estimate in Lemma 4.6, we then have the condition number bound of the FETI-DP algorithm.

THEOREM 4.8. *Under Assumption 4.5, the condition number bound holds for the FETI-DP algorithm,*

$$\kappa(\widehat{F}^{-1}_{DP} F_{DP}) \le C \max_i \left\{ \left(1 + log \frac{H_i}{h_i}\right)^2 \right\},$$

where the constant C does not depend on h_i, H_i, and ρ_i.

5. Numerical results. In this section, we provide numerical tests for the proposed FETI-DP algorithm. We consider the following model problem:

$$\begin{aligned}
-\nabla \cdot (\rho(x,y,z)\nabla u) &= f \ \text{ in } \ \Omega, \\
u &= 0 \ \text{ on } \ \partial\Omega,
\end{aligned}$$

where $\Omega = (0,1)^3$ is the unit cube, the exact solution is

$$u(x,y,z) = \frac{1}{\rho(x,y,z)64\pi^2} \sin(8\pi x)\sin(8\pi y)\sin(8\pi z).$$

We divide the domain Ω into $N \times N \times N$ uniform cubical subdomains with side length $H = 1/N$. Each subdomain Ω_i is discretized by conforming trilinear finite elements with uniform cubes of the size h_i. The mesh size h_i can be different to different subdomains.

TABLE 5.1
The number of CG iterations (Iter), corresponding condition numbers (Cond), and the broken H^1-norm error, $(\sum_i \|u - u^h\|^2_{H^1(\Omega_i)})^{1/2}$, of the FETI-DP algorithm for the model problem with the constant coefficient $\rho(x,y,z) = 1$; left four columns (the subdomain problem size $4n$ increases with the fixed number of subdomains $N^3 = 2^3$), right three columns (the number of subdomains N^3 increases with the fixed local problem size $4n = 8$).

$4n$	Iter	Cond	H^1-error	N^3	Iter	Cond
4	14	6.1185	1.099819e-02	2^3	14	6.1185
16	16	8.8967	5.576953e-03	4^3	18	7.3615
24	18	10.9198	3.706825e-03	8^3	18	7.5818
32	19	11.7914	2.773728e-03			

The corresponding Lagrange multiplier is given by the tensor product of two dimensional multipliers considered in [23]. Even though the theory provided in the previous section was developed for tetrahedral finite elements, it extends to the trilinear finite elements without difficulty.

To see the performance of the preconditioner, we perform two types of experiments; the case when the coefficient $\rho(x,y,z) = 1$ and the case when the coefficient $\rho(x,y,z)$ is a positive constant ρ_i in each subdomain Ω_i, they can be discontinuous across the interface. We solve the FETI-DP equation using the conjugate gradient method with the Neumann-Dirichlet preconditioner $\widehat{F}_{DP}^{-1}$ in (3.9). The conjugate gradient iteration is performed up to the relative residual norm reduced by 10^{-6}.

We first consider the model problem with $\rho(x,y,z) = 1$. All subdomains have a uniform triangulation of cubical elements with the mesh size $H/(4n)$, where H is the diameter of the subdomains. In Table 5.1, the number of CG iterations, condition numbers, and the errors in the broken H^1-norm are shown as the increase of the local problem size with a fixed number of subdomains $N^3 (= 2^3)$ and as the increase of the number of subdomains N^3 with a fixed local problem size $4n (= 8)$. From the result, we observe the $\log^2$-growth of the condition number in terms of the local problem size (see Figure 5.3), the optimal convergence of errors in the broken H^1-norm, and a scalability in terms of the number of subdomains.

We now consider the cases when $\rho(x,y,z)$ is discontinuous across the subdomain interface. In our example, we consider four values of $\rho(x,y,z)$, 1, 10, 250, and 1000, and distribute these values to the subdomain partition. In Figure 5.1, we present a cube that is divided into eight uniform cubical subdomains. For the eight subdomains we distribute $\rho(x,y,z)$ as in the right of Figure 5.1. When we introduce more number of subdomains, for an example $N^3 = 4^3$, we put the eight cubical partition in a periodic pattern and obtain the coefficient distribution. In the following, we will present the results varying the choices of the mortar and the nonmortar sides and varying mesh sizes depending on the coefficient distribution.

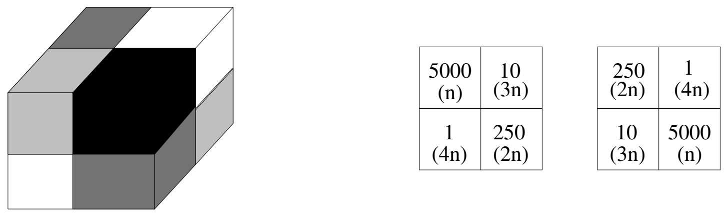

FIG. 5.1. *A subdomain partition (left) and the values $\rho(x,y,z)$ (right) of four subdomains on the bottom and four subdomains on the top, respectively.*

ETNA
Kent State University
etna@mcs.kent.edu

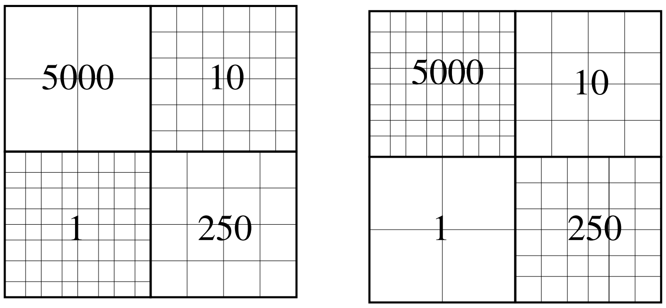

FIG. 5.2. *An example of non-uniform meshes depending on the coefficients (left: the smaller mesh sizes for the smaller coefficients, right: the reversed selection) when $H/(4n)(n = 2)$ is the smallest mesh size.*

TABLE 5.2
The number of CG iterations (Iter), corresponding condition numbers (Cond), and the broken H^1-norm error, $(\sum_i \|u - u^h\|^2_{H^1(\Omega_i)})^{1/2}$, of the FETI-DP algorithm as the decrease of the largest mesh size $H/(4n)$ for the model problem with a discontinuous coefficient $\rho(x, y, z)$ and the subdomain partition $N^2(N = 2)$; left (uniform meshes), center (non-uniform meshes), right (reversed non-uniform meshes).

	uniform			nonuniform			reversed nonuniform		
$4n$	Iter	Cond	H^1-error	Iter	Cond	H^1-error	Iter	Cond	H^1-error
8	12	4.39	5.5265e-03	12	4.15	5.5494e-03	10	2.96	7.8789e-03
16	14	5.74	2.8026e-03	14	5.31	2.8130e-03	13	4.22	1.0320e-02
24	15	6.61	1.8626e-03	14	6.06	1.8698e-03	14	5.06	7.3698e-03
32	16	7.29	1.3937e-03	15	6.66	1.3991e-03	15	5.69	5.5064e-03

Table 5.2 presents the performance of the algorithm varying the mesh sizes. We have used the uniform mesh size $H/(4n)$ for all subdomains, the non-uniform mesh sizes, from H/n to $H/(4n)$, depending on the coefficients. In addition we have selected the subdomain with smaller $\rho(x, y, z)$ as the nonmortar. For the non-uniform case, we selected smaller mesh sizes for smaller coefficients, see the right in Figure 5.1, with $H/(4n)$ the smallest. Such non-uniform meshes satisfy the optimal ratio of meshes $h_i/h_j \simeq (\rho_i/\rho_j)^{1/4}$ that was observed in [24] by applying an appropriate adaptivity strategy when the right hand side $f(x)$ of the model problem does not reflect the jump in the coefficient distribution. In addition, we have tested the case when the selection for the mesh size is reversed, see Figure 5.2. For the reversed case, the nonmortar side has the larger mesh size.

The results in Table 5.2 present that the condition numbers seem to be not affected by the selection of mesh sizes. All three give the $\log^2$-growth of the condition number bound with respect to the local problem size, see Figure 5.3. We can see that the errors are almost the same for the uniform and the non-uniform cases. However the reversed non-uniform case gives larger errors than the previous two cases at the same discretization level. We can conclude that using nonuniform mesh sizes, that are finer for smaller coefficients, is more practical for the model problem with discontinuous coefficients considering the presented errors and the number of iterations.

In Table 5.3, we present the performance varying the selection of mortar and nonmortar sides. Here we have used smaller mesh sizes for smaller coefficients. The first case is when the subdomain with smaller $\rho(x, y, z)$ is the nonmortar side. This selection satisfies Assumption 4.5. We also consider the reversed selection to see how it affects the performance. The reversed selection shows poor performance. The errors are also larger compared to the regu-

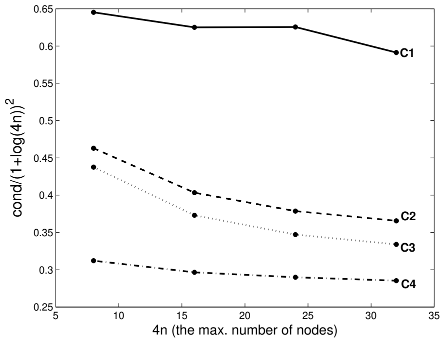

FIG. 5.3. *Plot of* $(cond/(1+log(4n))^2)$ *as the increase of the maximum number of nodes* $4n$*,* $C1$ *: constant coefficient* $\rho(x,y,z)=1$*,* $C2$ *: discontinuous* $\rho(x,y,z)$*, uniform meshes,* $C3$ *: discontinuous* $\rho(x,y,z)$*, non-uniform meshes,* $C4$ *: discontinuous* $\rho(x,y,z)$*, reversed non-uniform meshes*

TABLE 5.3
The number of CG iterations (Iter), corresponding condition numbers (Cond), and the broken H^1*-norm error,* $(\sum_i \|u-u^h\|^2_{H^1(\Omega_i)})^{1/2}$*, of the FETI-DP algorithm as the decrease of the largest mesh size* $H/(4n)$ *for the model problem with a discontinuous coefficient* $\rho(x,y,z)$ *and the subdomain partition* $N^2 (N=2)$*; regular selection (the subdomain with the smaller* $\rho(x,y,x)$ *is the nonmortar), reversed selection (the opposite).*

	regular selection			reversed selection		
$4n$	Iter	Cond	H^1-error	Iter	Cond	H^1-error
8	12	4.15	5.5494e-03	31	73.67	6.2726e-03
16	14	5.31	2.8130e-03	102	1.79e+03	3.7076e-03
24	14	6.06	1.8698e-03	147	2.14e+03	2.3687e-03
32	15	6.66	1.3991e-03	162	2.41e+03	1.8248e-03

lar selection at the same discretization level. We can conclude that the selection of nonmortar sides is important in obtaining a good performance of the Neumann-Dirichlet preconditioner while the selection of mesh sizes is not. A good mortar approximation can also be obtained from the regular selection.

In Table 5.4, we present the scalability of the algorithm as the increase of the number of subdomains when the local problem size is fixed. The number of iterations and the estimated condition numbers are shown. We perform tests varying the type of meshes; uniform meshes, non-uniform meshes, and reversed non-uniform meshes. In addition we perform tests varying the selection of the nonmortar sides; the regular selection and the reversed selection. The results also show that the performance depends on the selection of the nonmortar side and seem to be not affected by the type of meshes. Except the reversed selection of the nonmortar side, all gives a good scalability in terms of the number of subdomains.

In conclusion, the FETI-DP algorithm works efficiently for elliptic problems with jump coefficients when the selection of nonmortar sides satisfies Assumption 4.5. It even gives smaller condition numbers and the number of iterations than the case with $\rho(x,y,z)=1$, see Figure 5.3.

Acknowledgments. The author is grateful to Professor Chang-Ock Lee at KAIST in South Korea for the valuable discussions and also thanks to the anonymous referees for the helpful comments.

TABLE 5.4
The number of CG iterations (Iter) and corresponding condition numbers (Cond) of the FETI-DP algorithm as the increase of the number of subdomains for the model problem with a discontinuous coefficient $\rho(x, y, z)$; top (varying the type of meshes), bottom (varying the selection of the nonmortar).

	uniform		nonuniform		reversed nonuniform	
N^3	Iter	Cond	Iter	Cond	Iter	Cond
2^3	14	6.11	12	4.15	10	2.96
4^3	14	5.63	14	5.03	13	4.00
8^3	15	5.73	14	5.10	13	4.04

	regular selection		reversed selection	
N^3	Iter	Cond	Iter	Cond
2^3	12	4.15	31	73.67
4^3	14	5.03	52	92.26
8^3	14	5.10	56	91.60

REFERENCES

[1] F. B. BELGACEM, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999), pp. 173–197.
[2] F. B. BELGACEM AND Y. MADAY, *The mortar element method for three dimensional finite elements*, Math. Model. Numer. Anal., 31 (1997), pp. 289–302.
[3] D. BRAESS, *Finite Elements*, Cambridge University Press, Cambridge, 1997.
[4] N. DOKEVA, M. DRYJA, AND W. PROSKUROWSKI, *A FETI-DP preconditioner with a special scaling for mortar discretization of elliptic problems with discontinuous coefficients*, SIAM J. Numer. Anal., 44 (2006), pp. 283–299.
[5] M. DRYJA AND O. B. WIDLUND, *A FETI-DP method for a mortar discretization of elliptic problems*, in Recent Developments in Domain Decomposition Methods (Zürich, 2001), Lect. Notes Comput. Sci. Eng., Vol. 23, Springer, Berlin, 2002, pp. 41–52.
[6] ———, *A generalized FETI-DP method for a mortar discretization of elliptic problems*, in Domain Decomposition Methods in Science and Engineering (Cocoyoc, Mexico, 2002), UNAM, Mexico City, 2003, pp. 27–38.
[7] C. FARHAT, M. LESOINNE, P. LETALLEC, K. PIERSON, AND D. RIXEN, *FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method*, Internat. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.
[8] C. FARHAT, M. LESOINNE, AND K. PIERSON, *A scalable dual-primal domain decomposition method*, Numer. Linear Algebra Appl., 7 (2000), pp. 687–714.
[9] C. KIM, R. D. LAZAROV, J. E. PASCIAK, AND P. S. VASSILEVSKI, *Multiplier spaces for the mortar finite element method in three dimensions*, SIAM J. Numer. Anal., 39 (2001), pp. 519–538.
[10] H. H. KIM, M. DRYJA, AND O. B. WIDLUND, *A BDDC algorithm for problems with mortar discretization*, Technical Report 873, Department of Computer Science, Courant Institute, New York University, 2005.
[11] H. H. KIM AND C.-O. LEE, *A preconditioner for the FETI-DP formulation with mortar methods in two dimensions*, SIAM J. Numer. Anal., 42 (2005), pp. 2159–2175.
[12] A. KLAWONN AND O. RHEINBACH, *A parallel implementation of Dual-Primal FETI methods for three dimensional linear elasticity using a transformation of basis*, SIAM J. Sci. Comput., 28 (2006), pp. 1886–1906.
[13] ———, *Robust FETI-DP methods for heterogeneous three dimensional elasticity problems*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1400–1414.
[14] A. KLAWONN, O. RHEINBACH, AND O. B. WIDLUND, *Some computational results for Dual-Primal FETI methods for elliptic problems in 3D*, in The 15th International Conference in Domain Decomposition Methods, Berlin, 2003.
[15] A. KLAWONN AND O. B. WIDLUND, *Dual-Primal FETI methods for linear elasticity*, Comm. Pure Appl. Math., 59 (2006), pp. 1523–1572.
[16] A. KLAWONN, O. B. WIDLUND, AND M. DRYJA, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., 40 (2002), pp. 159–179.
[17] ———, *Dual-primal FETI methods with face constraints*, in Recent Developments in Domain Decomposition Methods (Zürich, 2001), Lect. Notes Comput. Sci. Eng., Vol. 23, Springer, Berlin, 2002, pp. 27–40.
[18] J. LI AND O. WIDLUND, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg.,

66 (2006), pp. 250–271.

[19] D. STEFANICA, *A numerical study of FETI algorithms for mortar finite element methods*, SIAM J. Sci. Comput., 23 (2001), pp. 1135–1160.

[20] ———, *FETI and FETI-DP methods for spectral and mortar spectral elements: a performance comparison*, in Proceedings of the Fifth International Conference on Spectral and High Order Methods (ICOSAHOM-01) (Uppsala), vol. 17, 2002, pp. 629–638.

[21] A. TOSELLI AND O. WIDLUND, *Domain Decomposition Methods—Algorithms and Theory*, Springer Series in Computational Mathematics, Vol. 34, Springer, Berlin, 2005.

[22] B. I. WOHLMUTH, *A residual based error estimator for mortar finite element discretizations*, Numer. Math., 84 (1999), pp. 143–171.

[23] ———, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal., 38 (2000), pp. 989–1012.

[24] ———, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Lect. Notes Comput. Sci. Eng., Vol. 17, Springer, Berlin, 2001.

[25] J. XU AND J. ZOU, *Some nonoverlapping domain decomposition methods*, SIAM Rev., 40 (1998), pp. 857–914.