Electronic Transactions on Numerical Analysis.
Volume 26, pp. 146-160, 2007.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

A BDDC ALGORITHM FOR FLOW IN POROUS MEDIA WITH A HYBRID FINITE ELEMENT DISCRETIZATION*

XUEMIN TU†

Abstract. The BDDC (balancing domain decomposition by constraints) methods have been applied successfully to solve the large sparse linear algebraic systems arising from conforming finite element discretizations of elliptic boundary value problems. In this paper, the scalar elliptic problems for flow in porous media are discretized by a hybrid finite element method which is equivalent to a nonconforming finite element method. The BDDC algorithm is extended to these problems which originate as saddle point problems. Edge/face average constraints are enforced across the interface and the same rate of convergence is obtained as in conforming cases. The condition number of the preconditioned system is estimated and numerical experiments are discussed.

Key words. BDDC, domain decomposition, saddle point problem, condition number, hybrid finite element method

AMS subject classifications. 65N30, 65N55, 65F10

1. Introduction.

Mixed formulations of elliptic problems (see [3]) have many applications, e.g., for flow in porous media, for which a good approximation to the velocity, which involves derivatives of the solution of the differential equations, is required. These discretizations lead to large, sparse, symmetric, indefinite linear systems.

In our recent paper [24], we extended the BDDC algorithm to this mixed formulation of elliptic problems. The BDDC algorithms are nonoverlapping domain decomposition methods, introduced by Dohrmann [6] and further analyzed in [15, 16], and are similar to the balancing Neumann-Neumann algorithms; see [14, 7]. However, the BDDC methods have different coarse components which are formed by a small number of continuity constraints enforced across the interface throughout the iterations. An important advantage of using such coarse problems is that the Schur complements and all other matrices that arise in the computation will be invertible.

In [24], the original saddle point problem is reduced to finding a correction pair which stays in the divergence free, benign subspace, as in [8, 17, 18, 19]. Then the BDDC method, with edge/face constraints, is applied to the reduced system. This method is similar to the BDDC algorithm proposed for the Stokes case in [12]. The analysis of this approach is focused on estimating the norm of an averaging operator. Several useful technical tools for the Raviart-Thomas finite elements, originally given in [29, 22, 28], are used and the algorithm converges at a rate similar to that of simple elliptic cases.

The hybrid finite element discretization is equivalent to a nonconforming finite element method. Two-level domain decomposition methods have been developed for a nonconforming approximation in [21, 20]. The condition number bounds are independent of the jumps in the coefficients of the original equations and grow only logarithmically with the number of degrees of freedom in each subdomain, a result which is the same as for a conforming case.

A non-overlapping domain decomposition algorithm for the hybrid formulation, called Method II, was proposed already in [10]. It is an unpreconditioned conjugate gradient method for certain interface variables. The rate of convergence is independent of the coefficients, but depends mildly on the number of degrees of freedom in the subdomains. Problems related

*Received June 26, 2005. Accepted for publication December 4, 2006. Recommended by S. Brenner. This work was supported in part by the US Department of Energy under Contracts DE-FG02-92ER25127 while the author was a student at Courant Institute of Mathematical Sciences, New York University.

†Department of Mathematics, University of California, Berkeley, 970 Evans Hall #3840, Berkeley, CA 94720 (xuemin@math.berkeley.edu).

to singular local Neumann problems arising in the preconditioners were also addressed in [10]. In addition, other non-overlapping domain decomposition methods were proposed with improved rates of convergence in [9] and [5].

A Balancing Neumann-Neumann (BNN) method was extended and analyzed in [4] for Method II of [10], see also [21] for a nonconforming case. A similar rate of convergence was obtained as for the conforming case. We will extend the BDDC algorithm to Method II of [10] in this paper. In contrast to [4], we need not solve any singular systems when using the BDDC algorithm.

The method proposed here differs from the one in [24]. In this paper, we reduce the original saddle point problem to a positive definite system for the pressure by introducing Lagrange multipliers on the interface of the subdomains and eliminating the velocity in each subdomain. Thus, we need not find a velocity that satisfies the divergence constraint at the beginning of the computation and then restrict the iterates to the divergence free, benign subspace. Our approach is quite similar to the work on the FETI-DP methods as described in [23, Chapter 6]. We use the BDDC preconditioner to solve the interface problem for a set of Lagrange multipliers, which can be interpreted as an approximation to the trace of the pressure. By enforcing a suitable set of constraints, we obtain a similar convergence rate as for a conforming finite element case. As in other studies of BDDC, our analysis will focus on the estimate of the norm of an averaging operator. However, we cannot use properties of the Raviart-Thomas finite elements directly since we work with Lagrange multipliers. The technical tools, originally given in [21, 20, 4], are needed to make a connection between the hybrid finite element method and a conforming finite element method.

The rest of the paper is organized as follows. The mixed formulation for the elliptic problem and its hybrid finite element discretization are described in Section 2. In Section 3, we reduce our problem to a symmetric positive definite interface problem. We introduce the BDDC preconditioner for the interface system in Section 4 and give some auxiliary results in Section 5. In Section 6, we provide an estimate of the condition number for the system with the BDDC preconditioner which is of the form $C\left(1+\log\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively. Finally, some computational results are presented in Section 7.

2. An elliptic problem and its discretization by hybrid finite elements.

We consider the following elliptic problem on a bounded polygonal domain Ω, in two or three dimensions, with a Dirichlet boundary condition:

$$\left\{\begin{array}{lll} -\nabla\cdot(\rho\nabla p)=f & \text{in} & \Omega, \\ p=g & \text{on} & \partial\Omega, \end{array}\right. \tag{2.1}$$

where ρ is a positive definite matrix function with entries in $L^\infty(\Omega)$ satisfying

$$\xi^T\rho(\mathbf{x})\xi \geq \alpha\|\xi\|^2, \quad \text{for a.e.} \quad \mathbf{x}\in\Omega,$$

for some positive constant α; $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Omega)$.

The equation (2.1) has a unique solution p. Without loss of generality, we assume that $g=0$. We use the Dirichlet boundary condition for convenience. The algorithm can also be extended to other boundary conditions.

We assume that we are interested in computing $-\rho\nabla p$ directly as often required in flow in porous media. We therefore introduce the velocity $\mathbf{u}$:

$$\mathbf{u}=-\rho\nabla p.$$

We obtain the following system for the velocity $\mathbf{u}$ and the pressure p:

$$\left\{\begin{array}{lll} \mathbf{u} = -\rho\nabla p & \text{in} & \Omega, \\ \nabla\cdot\mathbf{u} = f & \text{in} & \Omega, \\ p = 0 & \text{in} & \partial\Omega. \end{array}\right. \tag{2.2}$$

Let $c(\mathbf{x}) = \rho(\mathbf{x})^{-1}$ and define a Hilbert space by

$$H(\mathsf{div},\Omega) = \{\mathbf{v}\in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \nabla\cdot\mathbf{v}\in L^2(\Omega)\},$$

with the norm

$$\|\mathbf{v}\|^2_{H(\mathsf{div},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}.$$

The weak form of (2.2) is as follows: find $\mathbf{u}\in H(\mathsf{div},\Omega)$ and $p\in L^2(\Omega)$ such that

$$\left\{\begin{array}{lcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & 0, & \forall\mathbf{v}\in H(\mathsf{div},\Omega), \\ b(\mathbf{u},q) & = & -\int_\Omega fq\,d\mathbf{x}, & \forall q\in L^2(\Omega), \end{array}\right.$$

where $a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}^T c(\mathbf{x})\mathbf{v}\,d\mathbf{x}$ and $b(\mathbf{u},q) = -\int_\Omega(\nabla\cdot\mathbf{u})q\,d\mathbf{x}$.

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i=1,\cdots,N$, and set $H=\max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles/tetrahedra and that the number of such triangles/tetrahedra forming an individual subdomain is uniformly bounded. We also assume $\rho(\mathbf{x})$, the coefficient of (2.1), is constant in each subdomain. We note that our algorithm is equally well defined for a more general situation; the assumptions just formulated make our analysis possible.

Let $\mathcal{T}$ be a triangulation of Ω and $\mathcal{T}(\Omega_i)$ is the restriction of this triangulation to Ω_i. Let $\widehat{\mathbf{W}}$ be the lowest order Raviart-Thomas finite element space defined in (2.3) (also see [3, Chapter III, 3]) and let Q be the space of piecewise constants, which are finite dimensional subspaces of $H(\mathsf{div},\Omega)$ and $L^2(\Omega)$, respectively. The pair $\widehat{\mathbf{W}}$ and Q satisfies a uniform inf-sup condition; see [3, Chapter IV, 1.2].

We have

$$\widehat{\mathbf{W}} = \{\mathbf{v}\in H(\mathsf{div},\Omega); \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x}\ \ \forall T\in\mathcal{T}\}, \tag{2.3}$$

where $\mathbf{a}_T\in\mathbb{R}^2$ or $\mathbb{R}^3$, $c_T\in\mathbb{R}$. We note that a piecewise polynomial $v\in H(\mathsf{div},\Omega)$ if and only if the continuity of the normal components of v across ∂T holds. We can specify the degrees of freedom $\mathbf{a}_T$ and c_T in each element T by the normal components of v on ∂T and take the midpoints of the edges/faces of T as the nodes.

The finite element discrete problem is: find $\mathbf{u}_h\in\widehat{\mathbf{W}}$ and $p_h\in Q$ such that

$$\left\{\begin{array}{lcll} a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) & = & 0, & \forall\mathbf{v}_h\in\widehat{\mathbf{W}}, \\ b(\mathbf{u}_h,q_h) & = & -\int_\Omega fq_h\,d\mathbf{x}, & \forall q_h\in Q. \end{array}\right. \tag{2.4}$$

Let $\widehat{\mathbf{W}}^{(i)}$ be the lowest order Raviart-Thomas finite element space on Ω_i, i.e.,

$$\widehat{\mathbf{W}}^{(i)} = \{\mathbf{v}\in H(\mathsf{div},\Omega_i); \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x}\ \ \forall T\in\mathcal{T}(\Omega_i)\},$$

where $\mathbf{a}_T\in\mathbb{R}^2$ or $\mathbb{R}^3$, $c_T\in\mathbb{R}$.

We also define $\mathbf{W}$ and $\mathbf{W}^{(\mathbf{i})}$ which are similar to $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{W}}^{(i)}$, respectively. However, they are not required to be the subspaces of $H(\mathsf{div},\Omega)$ or $H(\mathsf{div},\Omega_i)$. Equivalently, there are no continuity constraints on the normal components of the functions, i.e.,

$$\mathbf{W} = \{\mathbf{v}\in L^2(\Omega)^2 \text{ or } L^2(\Omega)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x}\ \ \forall T\in\mathcal{T}\},$$

ETNA
Kent State University
etna@mcs.kent.edu

and

$$\mathbf{W}^{(i)} = \{\mathbf{v} \in L^2(\Omega_i)^2 \text{ or } L^2(\Omega_i)^3; \mathbf{v}|_T = \mathbf{a}_T + c_T\mathbf{x} \;\; \forall T \in \mathcal{T}(\Omega_i)\},$$

where $\mathbf{a}_T \in \mathbb{R}^2$ or $\mathbb{R}^3$ and $c_T \in \mathbb{R}$.

We relax the continuity of the normal components on the element interface in $\mathbf{W}$ and $\mathbf{W}^{(i)}$. Instead, we will introduce Lagrange multipliers to enforce this continuity of the Raviart-Thomas space. The goal of introducing Lagrange multipliers is to make it possible to reduce the saddle point problem (2.4) to a positive definite problem by eliminating the velocity. Without Lagrange multipliers, we would have to eliminate the velocity globally on the whole computational domain which would be very expensive. After relaxing the continuity of the normal components of the velocity on each element interface, we can eliminate the velocity on each element. In practice, we can afford to eliminate the velocity in each subdomain. In fact, as in [10, 4], in an implementation, we only need to use inter-element Lagrange multiplier on the subdomain interfaces.

Let $\mathcal{F}$ denote the set of edges/faces in $\mathcal{T}$ and $\mathcal{F}^\partial$ be the subset of $\mathcal{F}$ which contains the edges/faces on $\partial\Omega$. Then the Lagrange multiplier space $\widehat{\Lambda}$ is the set of functions on $\mathcal{F} \setminus \mathcal{F}^\partial$ which take constant values on individual edges/faces of $\mathcal{F}$ and vanish on $\mathcal{F}^\partial$; see [3, Section V1.2].

We can then reformulate the mixed problem (2.4) as follows: find $(\mathbf{u}, p, \lambda) \in \mathbf{W} \times Q \times \widehat{\Lambda}$ such that for all $(\mathbf{v}, q, \mu) \in \mathbf{W} \times Q \times \widehat{\Lambda}$

$$\left\{ \begin{array}{lcl} \sum_{T\in\mathcal{T}} \left(\int_T \mathbf{u}^T c\mathbf{v} \, d\mathbf{x} - \int_T \nabla \cdot \mathbf{v} p \, d\mathbf{x} + \int_{\partial T} \lambda \mathbf{v} \cdot \mathbf{n}_T \, ds \right) & = & 0, \\ -\sum_{T\in\mathcal{T}} \int_T q \nabla \cdot \mathbf{u} \, d\mathbf{x} & = & -\int_\Omega f q \, d\mathbf{x}, \\ \sum_{T\in\mathcal{T}} \int_{\partial T} \mu \mathbf{u} \cdot \mathbf{n}_T \, ds & = & 0. \end{array} \right. \tag{2.5}$$

The additional function λ is naturally interpreted as an approximation to the trace of p on the boundary of the elements. A proof of the equivalence of (2.4) and (2.5) can be found in [1, 2].

Correspondingly, the matrix form of (2.5) is

$$\begin{bmatrix} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}. \tag{2.6}$$

3. The problem reduced to the subdomain interface. We denote the discrete space of nodal values of $Q \times \widehat{\Lambda}$ by $\widehat{\mathcal{P}}$. Here we use the center points of the elements as the nodes for Q and the midpoints of the element edges/faces as the nodes for $\widehat{\Lambda}$. We note that $\widehat{\mathcal{P}}$ has the natural interpretation as the space of values of the pressure p in the interior and on the edges/faces of the elements. By this definition, $\widehat{\mathcal{P}}$ is isomorphic to $Q \times \widehat{\Lambda}$; we can then write an element of $\widehat{\mathcal{P}}$ as $\hat{p} = [p, \lambda]$.

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_i \partial\Omega_{i,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$.

We can write the discrete pressure spaces $\widehat{\mathcal{P}}$ as

$$\widehat{\mathcal{P}} = Q \bigoplus \widehat{\Lambda}.$$

The space Q is a direct sum of subdomain interior pressure spaces $Q^{(i)}$, i.e.,

$$Q = \bigoplus_{i=1}^{N} Q^{(i)}.$$

ETNA
Kent State University
etna@mcs.kent.edu

The elements of $Q^{(i)}$ are the restrictions of the elements in Q to Ω_i.

We can further decompose $\widehat{\Lambda}$ into

$$\widehat{\Lambda} = \Lambda_I \bigoplus \widehat{\Lambda}_\Gamma,$$

where $\widehat{\Lambda}_\Gamma$ denotes the set of degrees of freedom associated with Γ and Λ_I is a direct sum of subdomain interior degrees of freedom, i.e,

$$\Lambda_I = \bigoplus_{i=1}^{N} \Lambda_I^{(i)}.$$

We denote the subdomain interface pressure space by $\Lambda_\Gamma^{(i)}$ and the associated product space by $\Lambda_\Gamma = \bigoplus_{i=1}^N \Lambda_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface pressure space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\Lambda_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ.

The global saddle point problem (2.6) is assembled from subdomain problems

$$\begin{bmatrix} A^{(i)} & {B_1^{(i)}}^T & {B_{2,I}^{(i)}}^T & {B_{2,\Gamma}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 & 0 \\ B_{2,I}^{(i)} & 0 & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \\ 0 \\ 0 \end{bmatrix}, \tag{3.1}$$

where $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_I^{(i)}, \lambda_\Gamma^{(i)}) \in (\mathbf{W}^{(i)}, Q^{(i)}, \Lambda_I^{(i)}, \Lambda_\Gamma^{(i)})$. We note that $A^{(i)}$ is block diagonal, with each block corresponding to an element $T \subset \mathcal{T}(\Omega_i)$.

As we mentioned before, in practice, for each subdomain Ω_i, we only need to use the inter-element multipliers on the interface of the subdomains. In that case, let $(\mathbf{u}^{(i)}, p^{(i)}, \lambda_\Gamma^{(i)}) \in (\widehat{\mathbf{W}}^{(i)}, Q^{(i)}, \Lambda_\Gamma^{(i)})$ and we obtain the following subdomain problems

$$\begin{bmatrix} \hat{A}^{(i)} & {B_1^{(i)}}^T & {B_{2,\Gamma}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \\ 0 \end{bmatrix}. \tag{3.2}$$

We note that we can obtain $\hat{A}^{(i)}$ from $A^{(i)}$ by assembling. Therefore $\hat{A}^{(i)}$ is no longer block diagonal by element.

We define the subdomain Schur complement $S_\Gamma^{(i)}$ by solving a Dirichlet problem in the variational form: find $\{\mathbf{u}_i, p_i\} \in \widehat{\mathbf{W}}^{(i)} \times Q^{(i)}$ such that

$$\begin{aligned} \int_{\Omega_i} \mathbf{u}_i^T c \mathbf{v}_i d\mathbf{x} - \int_{\Omega_i} \nabla \cdot \mathbf{v}_i d\mathbf{x} &= -\int_{\partial\Omega_i \setminus \partial\Omega} \lambda_\Gamma^{(i)} \mathbf{v}_i \cdot \mathbf{n} ds, \quad \forall\, \mathbf{v}_i \in \widehat{\mathbf{W}}^{(i)}, \\ \int_{\Omega_i} \nabla \cdot \mathbf{u}_i q_i = 0, &\quad \forall\, q_i \in Q^{(i)}, \end{aligned} \tag{3.3}$$

then set $S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = -B_{2,\Gamma}^{(i)} \mathbf{u}_i$. We note that these Dirichlet problems are always well posed and that $S_\Gamma^{(i)}$ is symmetric and positive definite. We also can write (3.3) in matrix form as: given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} \lambda_\Gamma^{(i)}$ such that

$$\begin{bmatrix} \hat{A}^{(i)} & {B_1^{(i)}}^T & {B_{2,\Gamma}^{(i)}}^T \\ B_1^{(i)} & 0 & 0 \\ B_{2,\Gamma}^{(i)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)} \\ p^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ -S_\Gamma^{(i)} \lambda_\Gamma^{(i)} \end{bmatrix}.$$

We note that, to obtain the subdomain Schur complement $S_\Gamma^{(i)}$ for $\lambda_\Gamma^{(i)}$ from (3.1), we can first eliminate the velocity $\mathbf{u}^{(i)}$ in each element and obtain a system for the $p^{(i)}$, $\lambda_I^{(i)}$, and $\lambda_\Gamma^{(i)}$. We then eliminate the remaining degrees of freedom interior to the subdomain, i.e., the $p^{(i)}$ and $\lambda_I^{(i)}$. However, from (3.2), we cannot eliminate the velocity $\mathbf{u}^{(i)}$ in each element since $\hat{A}^{(i)}$ is no longer block diagonal by element. But we still can eliminate the velocity $\mathbf{u}^{(i)}$ and the pressure $p^{(i)}$ locally in each subdomain.

We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ, i.e.,

$$S_\Gamma = \begin{bmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{bmatrix}.$$

Given the definition of $S_\Gamma^{(i)}$, the subdomain problem (3.2) corresponds to the subdomain interface problem

$$S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = g_\Gamma^{(i)}, \quad i = 1, 2, ..., N,$$

where

$$g_\Gamma^{(i)} = - \left[B_{2,\Gamma}^{(i)} \; 0 \right] \begin{bmatrix} \hat{A}^{(i)} & B_1^{(i)^T} \\ B_1^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h^{(i)} \end{bmatrix}.$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widehat{S}_\Gamma \lambda_\Gamma = g_\Gamma, \tag{3.4}$$

where $g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} g_\Gamma^{(i)}$ and

$$\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}.$$

Thus, $\widehat{S}_\Gamma$ is a symmetric, positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (3.4) with a preconditioned conjugate gradient method.

4. The BDDC preconditioner and a change of variables. In Section 3, the original saddle point problem (2.6) was reduced to a positive definite problem (3.4) for the interface pressure (or Lagrange multipliers). It has a similar algebraic structure as the system which the original BDDC method was derived for. The difference is that the degrees of freedom in our case are the midpoints of the element edges/faces on the subdomain interface. It is nature to use the edge/face average primal constraints.

We introduce a partially assembled interface pressure space $\widetilde{\mathbf{\Lambda}}_\Gamma$ by

$$\widetilde{\mathbf{\Lambda}}_\Gamma = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \mathbf{\Lambda}_\Delta = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \left(\prod_{i=1}^N \mathbf{\Lambda}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{\Lambda}}_\Pi$ is the coarse level, primal interface pressure space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the subdomain

ETNA
Kent State University
etna@mcs.kent.edu

edges/faces for two/three dimensions, respectively. We change the variables so that the degree of freedom of each primal constraint is explicit; see the last part of this section and also [13, 11]. The dual space $\mathbf{\Lambda}_\Delta$ is the direct sum of the $\mathbf{\Lambda}_\Delta^{(i)}$, which are spanned by the remaining interface pressure degrees of freedom with a zero average over each edge/face. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints on the pressure across the interface but retain all primal continuity constraints, which makes all the linear systems nonsingular. This is the main difference from the BNN method in [4], where we encounter singular local problems.

We introduce two extension operators

$$\widehat{\mathbf{\Lambda}}_\Gamma \xrightarrow{\widetilde{R}_\Gamma} \widetilde{\mathbf{\Lambda}}_\Gamma \xrightarrow{\overline{R}_\Gamma} \mathbf{\Lambda}_\Gamma .$$

We also define a positive scaling factor $\delta_i^\dagger(\mathbf{x})$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(\mathbf{x}) = \frac{\rho_i^\gamma(\mathbf{x})}{\sum_{j \in \mathcal{N}_\mathbf{x}} \rho_j^\gamma(\mathbf{x})}, \quad \mathbf{x} \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_\mathbf{x}$ is the set of indices j of the subdomains such that $\mathbf{x} \in \partial\Omega_j$. We note that $\delta_i^\dagger(\mathbf{x})$ is constant on each edge/face, since we have assumed that the $\rho_i(\mathbf{x})$ is constant in each subdomain, and the nodes on each edge/face are shared by the same subdomains. Multiplying each entry of $\widetilde{R}_\Gamma$, which is for a dual variable in Λ_Δ and corresponds to a node $\mathbf{x} \in \partial\Omega_i$, by $\delta_i^\dagger(x)$ gives us $\widetilde{R}_{D,\Gamma}$.

The interface Schur complement $\widetilde{S}_\Gamma$ is partially assembled from subdomain Schur complements $S_\Gamma^{(i)}$, i.e.,

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma .$$

We can also obtain $\widehat{S}_\Gamma$, introduced in (3.4), from $\widetilde{S}_\Gamma$ by assembling the dual interface pressure part on the subdomain interface, i.e.,

$$\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma .$$

The BDDC preconditioner for solving the global interface problem (3.4) is

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} .$$

see [6, 15, 16, 13, 12] for the details.

The preconditioned BDDC algorithm is then of the form: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$M^{-1}\widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma .$$

This preconditioned operator is the product of two symmetric positive definite matrices and we can use the preconditioned conjugate gradient method.

Finally, we provide some details about changing variables to make the primal variables (edge/face average) explicit. We follow the notations in [13].

We denote the nodal degrees of freedom in a given edge by $(u_1, \cdots, u_m, \cdots, u_l)$. We can choose any node on this edge as m. The other unknowns are denoted by u_I. We are given a linear system written as

$$\begin{bmatrix} A_{II} & A_{1I}^T & \cdots & A_{mI}^T & \cdots & A_{lI}^T \\ A_{1I} & a_{11} & \cdots & a_{1m} & \cdots & a_{1l} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{mI} & a_{m1} & \cdots & a_{mm} & \cdots & a_{ml} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{lI} & a_{l1} & \cdots & a_{lm} & \cdots & a_{ll} \end{bmatrix} \begin{bmatrix} u_I \\ u_1 \\ \vdots \\ u_m \\ \vdots \\ u_l \end{bmatrix} = \begin{bmatrix} f_I \\ f_1 \\ \vdots \\ f_m \\ \vdots \\ f_l \end{bmatrix} . \tag{4.1}$$

ETNA
Kent State University
etna@mcs.kent.edu

Define a $l \times l$ sparse matrix T_E as follows:

$$\begin{bmatrix} 1 & & 1 & & \\ & \ddots & \vdots & & \\ -1 & \cdots & 1 & \cdots & -1 \\ & & \vdots & \ddots & \\ & & 1 & & 1 \end{bmatrix},$$

and let $(u_1, \cdots, u_m, \cdots, u_l)^T = T_E(\hat{u}_1, \cdots, \hat{u}_m, \cdots, \hat{u}_l)^T$. We have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \\ \vdots \\ u_l \end{bmatrix} = T_E \begin{bmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_m \\ \vdots \\ \hat{u}_l \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \hat{u}_m + \begin{bmatrix} \hat{u}_1 \\ \vdots \\ -\hat{u}_1 - \cdots - \hat{u}_{m-1} - \hat{u}_{m+1} - \cdots - \hat{u}_l \\ \vdots \\ \hat{u}_l \end{bmatrix}.$$

Here, the columns of T_E are the new basis of the edge variables and $(\hat{u}_1, \cdots, \hat{u}_m, \cdots, \hat{u}_l)$ are the new coordinates in the new basis. The basis function corresponding to $\hat{u}_m$ is constant on the edge and $\hat{u}_m$ represents the edge average. The others have zero edge average.

Let

$$T = \begin{bmatrix} I & \\ & T_E \end{bmatrix}.$$

Then, (4.1) can be transformed into

$$T^T \begin{bmatrix} A_{II} & A_{1I}^T & \cdots & A_{mI}^T & \cdots & A_{lI} \\ A_{1I} & a_{11} & \cdots & a_{1m} & \cdots & a_{1l} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{mI} & a_{m1} & \cdots & a_{mm} & \cdots & a_{ml} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{lI} & a_{l1} & \cdots & a_{lm} & \cdots & a_{ll} \end{bmatrix} T \begin{bmatrix} u_I \\ \hat{u}_1 \\ \vdots \\ \hat{u}_m \\ \vdots \\ \hat{u}_l \end{bmatrix} = T^T \begin{bmatrix} f_I \\ f_1 \\ \vdots \\ f_m \\ \vdots \\ f_l \end{bmatrix}.$$

Since the changing variables for each edge is a local procedure, we can do this transformation edge by edge in each subdomain.

5. Some auxiliary results.

In this section, we will collect a number of results which are needed in our theory.

In order to connect our hybrid finite element discretization to a conforming finite element method, we need to introduce a new mesh on each subdomain. The idea follows [20, 21, 4]. In order to be complete and for the readers unfamiliar with these technical tools, we give the construction of the new mesh, the definitions of two important maps, and some useful lemmas, which were originally given in [4, 20, 21].

Given an element $\tau \in \mathcal{T}$, let $\hat{\tau}$ be a subtriangulation of τ which includes the vertices of τ and the nodal points in τ for the degrees of the freedom of $Q \times \Lambda$. We then obtain a quasi-uniform sub-triangulation $\widehat{\mathcal{T}}$. We partition the vertices in the new mesh $\widehat{\mathcal{T}}$ into two sets. The nodes in $\mathcal{T}$ are called primary and the rest are called secondary. We say that two vertices in the triangulation $\widehat{\mathcal{T}}$ are adjacent if there is an edge of $\widehat{\mathcal{T}}$ between them.

Let $U_h(\Omega)$ be the continuous piecewise linear finite element function space with respect to the new triangulation $\widehat{\mathcal{T}}$. For a subdomain Ω_i, $U_h(\Omega_i)$ and $U_h(\partial\Omega_i)$ are defined by restrictions:

$$U_h(\Omega_i) = \{u|_{\Omega_i}: \ u \in U_h(\Omega)\}, \qquad U_h(\partial\Omega_i) = \{u|_{\partial\Omega_i}: \ u \in U_h(\Omega)\}.$$

Define a mapping $I_h^{\Omega_i}$ from any function ϕ defined at the primary vertices in Ω_i to $U_h(\Omega_i)$ by

$$I_h^{\Omega_i}\phi(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}), \ \text{if } \mathbf{x} \text{ is a primary vertex;} \\ \\ \text{the average of all adjacent primary vertices on } \partial\Omega_i, \\ \ \text{if } \mathbf{x} \text{ is a secondary vertex on } \partial\Omega_i; \\ \\ \text{the average of all adjacent primary vertices,} \\ \ \text{if } \mathbf{x} \text{ is a secondary vertex in the interior of } \Omega_i; \\ \\ \text{the linear interpolation of the vertex values,} \\ \ \text{if } \mathbf{x} \text{ is not a vertex of } \widehat{\mathcal{T}}. \end{cases}$$

We note that $I_h^{\Omega_i}$ defines a map from $Q(\Omega_i) \times \Lambda(\Omega_i)$ to $U_h(\Omega_i)$ and also a map from $U_h(\Omega_i)$ to $U_h(\Omega_i)$.

Let $I_h^{\partial\Omega_i}$ be the mapping from a function ϕ, defined at the primary vertices on $\partial\Omega_i$, to $U_h(\partial\Omega_i)$ and defined by $I_h^{\partial\Omega_i}\phi = (I_h^{\Omega_i}\hat{p})|_{\partial\Omega_i}$, where $\hat{p}$ is any functions in $Q(\Omega_i) \times \Lambda(\Omega_i)$ such that $\hat{p}|_{\partial\Omega_i} = \phi$. The map is well defined since the boundary values of $I_h^{\Omega_i}\hat{p}$ only depend on the boundary values of $\hat{p}$.

Let

$$\widetilde{U}_h(\Omega_i) = \{\psi = I_h^{\Omega_i}\phi, \ \ \phi \in U_h(\Omega_i)\} \quad \text{and} \quad \widetilde{U}_h(\partial\Omega_i) = \{\psi|_{\partial\Omega}, \ \ \psi \in \widetilde{U}_h(\Omega_i)\}.$$

We list some useful lemmas from [4].

LEMMA 5.1. *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$|I_h^{\Omega_i}\phi|_{H^1(\Omega_i)} \le C|\phi|_{H^1(\Omega_i)}, \quad \forall\phi \in U_h(\Omega_i),$$

$$\|I_h^{\Omega_i}\phi\|_{L^2(\Omega_i)} \le C\|\phi\|_{L^2(\Omega_i)}, \quad \forall\phi \in U_h(\Omega_i).$$

Proof: See [4, Lemma 6.1]. □

LEMMA 5.2. *For $\hat{\phi} \in \widetilde{U}_h(\partial\Omega_i)$, there exist two positive constants C_1 and C_2, independent of h and $|\Omega_i|$, such that*

$$C_1\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \le \inf_{\phi\in\widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i}=\hat{\phi}} \|\phi\|_{H^1(\Omega_i)} \le C_2\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)},$$

$$C_1|\hat{\phi}|_{H^{1/2}(\partial\Omega_i)} \le \inf_{\phi\in\widetilde{U}_h(\Omega_i)\phi|_{\partial\Omega_i}=\hat{\phi}} |\phi|_{H^1(\Omega_i)} \le C_2|\hat{\phi}|_{H^{1/2}(\partial\Omega_i)}.$$

Proof: See [4, Lemma 6.2]. □

LEMMA 5.3. *There exists a constant $C > 0$ independent of h and $|\Omega_i|$ such that*

$$\|I_h^{\partial\Omega_i}\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \leq C\|\hat{\phi}\|_{H^{1/2}(\partial\Omega_i)} \qquad \forall\, \hat{\phi} \in U_h(\partial\Omega_i).$$

Proof: See [4, Lemma 6.3]. □

LEMMA 5.4. *There exist positive constants C_1 and C_2 independent of H, h, and the coefficient of (2.1), such that for all $\lambda_i \in \Lambda_\Gamma^{(i)}$,*

$$\rho_i C_1 |I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)} \leq |\lambda_i|^2_{S_\Gamma^{(i)}} \leq \rho_i C_2 |I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)}.$$

Proof: See [4, Theorem 6.5]. □

We define the interface averaging operator E_D, by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T,$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the boundary points of the subdomain.

The interface averaging operator E_D satisfies the following bound:

LEMMA 5.5.

$$|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C\left(1 + \log\frac{H}{h}\right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma},$$

for any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, where C is a positive constant independent of H, h, and the coefficient of (2.1),

Proof: Given any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, we have

$$\begin{aligned} |E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} &\leq 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + |\lambda_\Gamma - E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}\right) \\ &\leq 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + |\overline{R}_\Gamma\left(\lambda_\Gamma - E_D\lambda_\Gamma\right)|^2_{S_\Gamma}\right) \\ &= 2\left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + \sum_{i=1}^{N} |\left(\lambda_\Gamma - E_D\lambda_\Gamma\right)_i|^2_{S_\Gamma^{(i)}}\right), \end{aligned} \tag{5.1}$$

where $\left(\lambda_\Gamma - E_D\lambda_\Gamma\right)_i$ is the restriction of $\lambda_\Gamma - E_D\lambda_\Gamma$ to the subdomain Ω_i. Also let λ_i be the restriction of λ_Γ to the subdomain Ω_i and set

$$v_i(\mathbf{x}) := (\lambda_\Gamma - E_D\lambda_\Gamma)_i(\mathbf{x}) = \sum_{j\in\mathcal{N}_\mathbf{x}} \delta_j^\dagger(\lambda_i(\mathbf{x}) - \lambda_j(\mathbf{x})), \quad \mathbf{x} \in \partial\Omega^i \cap \Gamma. \tag{5.2}$$

Here $\mathcal{N}_\mathbf{x}$ is the set of indices of the subdomains that have $\mathbf{x}$ on their boundaries. Since a fine edge/face only belongs to exactly two subdomains, we have, for an edge/face $\mathcal{F}^{ij} \subset \partial\Omega_i$ that is also shared by Ω_j,

$$v_i = \delta_j^\dagger\lambda_i - \delta_j^\dagger\lambda_j, \text{ on } \mathcal{F}^{ij}. \tag{5.3}$$

We note that the simple inequality

$$\rho_i {\delta_j^\dagger}^2 \leq \min(\rho_i, \rho_j),$$

holds for $\gamma \in [1/2, \infty)$.

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/faces. Let $\zeta_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$, i.e., the function that is identically one on $\mathcal{F}$ and zero on $\partial\Omega_i\backslash\mathcal{F}$. We clearly have

$$\sum_{\mathcal{F}\subset\partial\Omega_i}\zeta_{\mathcal{F}}(\mathbf{x})=1,\qquad \text{almost everywhere on } \partial\Omega_i\backslash\partial\Omega.$$

We also need the partition of unity functions associated with the edges/faces for a function in the space $U_h(\Omega_i)$, denoted by $\vartheta_{\mathcal{F}}$, which is defined in [23, Lemma 4.23].

We have

$$|v_i|^2_{S_\Gamma^{(i)}}\le C\sum_{\mathcal{F}^{ij}\subset\partial\Omega_i}|\zeta_{\mathcal{F}^{ij}}v_i|^2_{S_\Gamma^{(i)}}. \tag{5.4}$$

By Lemma 5.4, with $\overline{\lambda}_{i,\mathcal{F}^{ij}}$ the average over $\mathcal{F}^{ij}$,

$$\begin{aligned}
|\zeta_{\mathcal{F}^{ij}}v_i|^2_{S_\Gamma^{(i)}} &\le C_2\rho_i|I_h^{\partial\Omega_i}(\zeta_{\mathcal{F}^{ij}}v_i)|^2_{H^{1/2}(\partial\Omega_i)}\\
&= C_2\rho_i|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}\delta_j^\dagger(\lambda_i-\lambda_j)\right)|^2_{H^{1/2}(\partial\Omega_i)}\\
&= C_2\rho_i{\delta_j^\dagger}^2|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_i-\lambda_j)\right)|^2_{H^{1/2}(\partial\Omega_i)}\\
&\le 2C_2\rho_i{\delta_j^\dagger}^2\left(|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_i-\overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)}\right.\\
&\quad\left.+\ |I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_j-\overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)}\right).
\end{aligned}\tag{5.5}$$

We estimate these two terms in (5.5) separately.

The first term is estimated as follows:

$$\begin{aligned}
\rho_i{\delta_j^\dagger}^2|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_i-\overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} &\le \rho_i|I_h^{\partial\Omega_i}\left(\vartheta_{\mathcal{F}^{ij}}I_h^{\partial\Omega_i}(\lambda_i-\overline{\lambda}_{i,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)}\\
&\le \rho_i\|\vartheta_{\mathcal{F}^{ij}}I_h^{\partial\Omega_i}(\lambda_i-\overline{\lambda}_{i,\mathcal{F}^{ij}})\|^2_{H^{1/2}(\partial\Omega_i)}\\
&\le \rho_i\|\vartheta_{\mathcal{F}^{ij}}(I_h^{\partial\Omega_i}\lambda_i-(\overline{I_h^{\partial\Omega_i}\lambda_i})_{\mathcal{F}^{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
&\le C\rho_i\left(1+\log\frac{H}{h}\right)^2|I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)},
\end{aligned}\tag{5.6}$$

where we use (5.3) and the definition of $I_h^{\partial\Omega_i}$ for the first inequality. Using Lemma 5.3, we obtain the second inequality. We use $I_h^{\partial\Omega_i}(\overline{\lambda}_{i,\mathcal{F}^{ij}})=(\overline{I_h^{\partial\Omega_i}\lambda_i})_{\mathcal{F}^{ij}}$ and [23, Lemma 4.26] for the penultimate and final inequalities.

For the second term in (5.5), similarly as for the first term, we have,

$$\begin{aligned}
\rho_i{\delta_j^\dagger}^2|I_h^{\partial\Omega_i}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_j-\overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)} &\le \rho_j|I_h^{\partial\Omega_i}\left(\vartheta_{\mathcal{F}^{ij}}I_h^{\partial\Omega_j}(\lambda_j-\overline{\lambda}_{j,\mathcal{F}^{ij}})\right)|^2_{H^{1/2}(\partial\Omega_i)}\\
&\le \rho_j\|\vartheta_{\mathcal{F}^{ij}}I_h^{\partial\Omega_j}(\lambda_j-\overline{\lambda}_{j,\mathcal{F}^{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
&\le \rho_j\|\vartheta_{\mathcal{F}^{ij}}(I_h^{\partial\Omega_j}\lambda_j-(\overline{I_h^{\partial\Omega_j}\lambda_j})_{\mathcal{F}^{ij}})\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
&\le C\rho_j\left(1+\log\frac{H}{h}\right)^2|I_h^{\partial\Omega_j}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)},
\end{aligned}\tag{5.7}$$

where we use (5.3) and the definition of $I_h^{\partial\Omega_i}$ and $I_h^{\partial\Omega_j}$ for the first inequality. Using Lemma 5.3, we obtain the second inequality. We use $I_h^{\partial\Omega_j}(\overline{\lambda}_{j,\mathcal{F}^{ij}})=(\overline{I_h^{\partial\Omega_j}\lambda_j})_{\mathcal{F}^{ij}}$ and [23, Lemma 4.26] for the penultimate and final inequalities.

ETNA
Kent State University
etna@mcs.kent.edu

Combining (5.6), (5.7), (5.5), and (5.4), we have

$$
\begin{aligned}
|v_i|^2_{S^{(i)}_\Gamma} &\leq CC_2 \left(1+\log\frac{H}{h}\right)^2 \left(\rho_i |I_h^{\partial\Omega_i}\lambda_i|^2_{H^{1/2}(\partial\Omega_i)} + \rho_j |I_h^{\partial\Omega_j}\lambda_j|^2_{H^{1/2}(\partial\Omega_j)}\right) \\
&\leq C\frac{C_2}{C_1}\left(1+\log\frac{H}{h}\right)^2 \left(|\lambda_i|^2_{S^{(i)}_\Gamma} + |\lambda_j|^2_{S^{(j)}_\Gamma}\right),
\end{aligned} \tag{5.8}
$$

where we use Lemma 5.4 again for the final inequality.

Using (5.1), (5.2), and (5.8), we obtain $|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C\left(1+\log\frac{H}{h}\right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma}$. □

6. Condition number estimate for BDDC preconditioner.

We are now ready to formulate and prove our main result; it follows as in the proof of [12, Theorem 1] using Lemma 5.5. Also see the proof of [16, Theorem 25], [27, Lemma 4.6], [25, Lemma 4.7], and [26, Theorem 2.8].

THEOREM 6.1. *The condition number of the preconditioned operator $M^{-1}\widehat{S}_\Gamma$ is bounded by $C(1+\log\frac{H}{h})^2$, where C is a constant which is independent of h, H, and the coefficients ρ of (2.1).*

Proof: It is enough to prove that, for any $\lambda_\Gamma \in \widehat{\mathbf{\Lambda}}_\Gamma$,

$$
\lambda_\Gamma^T M\lambda_\Gamma \leq \lambda_\Gamma^T \widehat{S}_\Gamma\lambda_\Gamma \leq C\left(1+\log(H/h)\right)^2 \lambda_\Gamma^T M\lambda_\Gamma,
$$

Lower bound: Let

$$
w_\Gamma = \left(\widetilde{R}^T_{D,\Gamma}\widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}\right)^{-1}\lambda_\Gamma \in \widehat{\mathbf{\Lambda}}_\Gamma. \tag{6.1}
$$

Using the properties $\widetilde{R}^T_\Gamma\widetilde{R}_{D,\Gamma} = \widetilde{R}^T_{D,\Gamma}\widetilde{R}_\Gamma = I$ and (6.1), we have

$$
\begin{aligned}
\lambda_\Gamma^T M\lambda_\Gamma &= \lambda_\Gamma^T\left(\widetilde{R}^T_{D,\Gamma}\widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}\right)^{-1}\lambda_\Gamma = \lambda_\Gamma^T w_\Gamma \\
&= \lambda_\Gamma^T\widetilde{R}^T_\Gamma\widetilde{S}_\Gamma\widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}w_\Gamma = < \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}w_\Gamma >_{\widetilde{S}_\Gamma} \\
&\leq < \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{R}_\Gamma\lambda_\Gamma >^{1/2}_{\widetilde{S}_\Gamma} < \widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}w_\Gamma, \widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}w_\Gamma >^{1/2}_{\widetilde{S}_\Gamma} \\
&= \left(\lambda_\Gamma^T\widetilde{R}^T_\Gamma\widetilde{S}_\Gamma\widetilde{R}_\Gamma\lambda_\Gamma\right)^{1/2}\left(w_\Gamma^T\widetilde{R}^T_{D,\Gamma}\widetilde{S}^{-1}_\Gamma\widetilde{S}_\Gamma\widetilde{S}^{-1}_\Gamma\widetilde{R}_{D,\Gamma}w_\Gamma\right)^{1/2} \\
&= \left(\lambda_\Gamma^T\widetilde{R}^T_\Gamma\widetilde{S}_\Gamma\widetilde{R}_\Gamma\lambda_\Gamma\right)^{1/2}\left(\lambda_\Gamma^T M\lambda_\Gamma\right)^{1/2}.
\end{aligned}
$$

We obtain

$$
\lambda_\Gamma^T M\lambda_\Gamma \leq \lambda_\Gamma^T\widehat{S}_\Gamma\lambda_\Gamma,
$$

by canceling a common factor and squaring.

Upper bound: Using the definition of w_Γ in (6.1), the Cauchy-Schwarz inequality, and

ETNA
Kent State University
etna@mcs.kent.edu

Lemma 5.5, we obtain the upper bound:

$$\begin{aligned}
\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma &= \lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \\
&= < \widetilde{R}_\Gamma \lambda_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma} \\
&\leq < \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{R}_\Gamma \lambda_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} < E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} \\
&\leq C < \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{R}_\Gamma \lambda_\Gamma >_{\widetilde{S}_\Gamma}^{1/2} (1 + \log(H/h)) \, |\widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma|_{\widetilde{S}_\Gamma} \\
&= C\,(1 + \log(H/h)) \left(\lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \lambda_\Gamma \right)^{1/2} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right)^{1/2} \\
&= C\,(1 + \log(H/h)) \left(\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma \right)^{1/2} \left(\lambda_\Gamma^T M \lambda_\Gamma \right)^{1/2}.
\end{aligned}$$

Thus, $\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma \leq C\,(1 + \log(H/h))^2\, \lambda_\Gamma^T M \lambda_\Gamma$. $\square$

7. Numerical experiments. We have applied our BDDC algorithms to the model problem (2.1), where $\Omega = [0, 1]^2$. We decompose the unit square into $\sqrt{N} \times \sqrt{N}$ subdomains with the sidelength $H = 1/\sqrt{N}$. Equation (2.1) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure, respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

TABLE 7.1
Condition number estimates and iteration counts for the operator with the BDDC preconditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ *and* $\rho \equiv 1$.

Number of Subdomains	Iterations	Condition number
4×4	7	2.53
8×8	10	3.01
12×12	10	3.06
16×16	10	3.06
20×20	10	3.06

In the first set of experiments, we take the coefficient $\rho \equiv 1$. Table 7.1 gives the iteration counts and the estimates of the condition numbers, with a change of the number of subdomains. We find that the condition numbers are independent of the number of subdomains. Table 7.2 gives results with a change of the size of the subdomain problems.

TABLE 7.2
Condition number estimates and iteration counts for the operator with the BDDC preconditioner with a change of the size of the subdomain problems. 8×8 *subdomains and* $\rho \equiv 1$.

$\frac{H}{h}$	Iterations	Condition number
4	8	2.23
8	10	3.01
12	11	3.54
16	11	3.95
20	11	4.29

TABLE 7.3

Condition number estimates and iteration counts for the operator with the BDDC preconditioner with a change of the number of subdomains. $\frac{H}{h} = 8$ *and* ρ *in a checkerboard pattern.*

	$\rho = 1$ or $\rho = 100$		$\rho = 1$ or $\rho = 10000$	
Number of Subdomains	Iterations	Condition number	Iterations	Condition number
4×4	8	2.98	7	2.99
8×8	10	2.97	10	2.99
12×12	11	2.98	10	2.99
16×16	11	2.98	10	2.99
20×20	10	2.98	10	2.99

TABLE 7.4

Condition number estimates and iteration counts for the operator with the BDDC preconditioner with a change of the size of the subdomain problems. 8×8 *subdomains and* ρ *in a checkerboard pattern.*

	$\rho = 1$ or $\rho = 100$		$\rho = 1$ or $\rho = 10000$	
$\frac{H}{h}$	Iterations	Condition number	Iterations	Condition number
4	9	2.19	9	2.20
8	10	2.97	10	2.99
12	11	3.51	11	3.52
16	12	3.92	11	3.94
20	13	4.26	11	4.27

In the second set of experiments, we take the coefficient $\rho = 1$ in half the subdomains and $\rho = 100$ or $\rho = 10000$ in the neighboring subdomains in a checkerboard pattern. Table 7.3 gives the iteration counts and condition number estimates with a change of the number of subdomains. We find that the condition numbers are independent of the number of subdomains. Table 7.4 gives results with a change of the size of the subdomain problems.

Acknowledgment. The author is grateful to Professor Olof Widlund for suggesting this problem and for all the help and time he has devoted to her work.

REFERENCES

[1] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: implementation, post-processing and error estimates*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 7–32.
[2] S. C. BRENNER, *A multigrid algorithm for the lowest-order Raviart-Thomas mixed triangular finite element method*, SIAM J. Numer. Anal., 29 (1992), pp. 647–678.
[3] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element*, Springer Series in Computational Mathematics, Vol. 15, Springer, Berlin-Heidelberg-New York, 1991.
[4] L. C. COWSAR, J. MANDEL, AND M. F. WHEELER, *Balancing domain decomposition for mixed finite elements*, Math. Comp., 64 (1995), pp. 989–1015.
[5] L. C. COWSAR AND M. F. WHEELER, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow, 1990), R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1991, pp. 358–372.
[6] C. R. DOHRMANN, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci Comput., 25 (2003), pp. 246–258.
[7] M. DRYJA AND O. B. WIDLUND, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math., 48 (1995), pp. 121–155.
[8] R. E. EWING AND J. WANG, *Analysis of the Schwarz algorithm for mixed finite element methods*, RAIRO Modél. Math. Anal. Numér., 26 (1992), pp. 739–756.

[9] R. GLOWINSKI, W. A. KINTON, AND M. F. WHEELER, *Acceleration of domain decomposition algorithms for mixed finite elements by multi-level methods*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989), T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1990, pp. 263–289.

[10] R. GLOWINSKI AND M. F. WHEELER, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987), R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, PA, 1988, pp. 144–172.

[11] A. KLAWONN AND O. B. WIDLUND, *Dual-primal FETI methods for linear elasticity*, Comm. Pure Appl. Math., 59 (2006), pp. 1523–1572.

[12] J. LI AND O. B. WIDLUND, *BDDC algorithms for incompressible Stokes equations*, SIAM J. Numer. Anal., 44 (2006), pp. 2432–2455.

[13] ———, *FETI–DP, BDDC, and Block Cholesky Methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.

[14] J. MANDEL, *Balancing domain decomposition*, Comm. Numer. Meth. Engrg., 9 (1993), pp. 233–241.

[15] J. MANDEL AND C. R. DOHRMANN, *Convergence of a balancing domain decomposition by constraints and energy minimization*, Numer. Linear Algebra Appl., 10 (2003), pp. 639–659.

[16] J. MANDEL, C. R. DOHRMANN, AND R. TEZAUR, *An algebraic theory for primal and dual substructuring methods by constraints*, Appl. Numer. Math., 54 (2005), pp. 167–193.

[17] T. P. MATHEW, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretizations of Elliptic Problems*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, September 1989. TR-463, Department of Computer Science, Courant Institute.

[18] ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and numerical results*, Numer. Math., 65 (1993), pp. 445–468.

[19] ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Theory*, Numer. Math., 65 (1993), pp. 469–492.

[20] M. V. SARKIS, *Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Confoming and Non-Conforming Elements*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, September 1994. TR-671, Department of Computer Science, Courant Institute.

[21] ———, *Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements*, Numer. Math., 77 (1997), pp. 383–406.

[22] A. TOSELLI, *Domain Decomposition Methods for Vector Field Problems*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, May 1999. TR-785, Department of Computer Science, Courant Institute.

[23] A. TOSELLI AND O. B. WIDLUND, *Domain Decomposition Methods - Algorithms and Theory*, Springer Series in Computational Mathematics, Vol. 34, Springer, Berlin-Heidelberg-New York, 2005.

[24] X. TU, *A BDDC algorithm for a mixed formulation of flows in porous media*, Electron. Trans. Numer. Anal., 20 (2005), pp. 164–179.
http://etna.math.kent.edu/vol.20.2005/pp164-179.dir/pp164-179.html.

[25] ———, *Three-level BDDC in three dimensions*, SIAM J. Sci. Comput., to appear.

[26] ———, *BDDC Domain Decomposition Algorithms: Methods with Three Levels and for Flow in Porous Media*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, January 2006. TR2005-879, Department of Computer Science, Courant Institute.

[27] ———, *Three-level BDDC in two dimensions*, Internat. J. Numer. Methods Engrg., 69 (2007), pp. 33–59.

[28] B. I. WOHLMUTH, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Lecture Notes in Computational Science and Engineering, Vol. 17, Springer, New York, 2001.

[29] B. I. WOHLMUTH, A. TOSELLI, AND O. B. WIDLUND, *Iterative substructuring method for Raviart-Thomas vector fields in three dimensions*, SIAM J. Numer. Anal., 37 (2000), pp. 1657–1676.