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ANALYSIS OF THE $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ POTENTIAL FORMULATION FOR THE EDDY CURRENT PROBLEM IN A BOUNDED DOMAIN*

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Abstract. The aim of this paper is to provide a mathematical analysis of the well-known $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation for the eddy current problem. The resulting variational problem is proved to be well posed and error estimates are settled for a numerical method based on standard nodal finite elements.

Key words. eddy currents, potential formulation, well-posedness, finite elements, error estimates

AMS subject classifications. 78M10, 65N30

1. Introduction.

The mathematical and numerical analysis of Maxwell's equations has experienced important developments in different areas of applied mathematics and engineering during the last thirty years. We refer the reader to the books by Bossavit [17], Monk [29] and Silvester and Ferrari [32], as a representative sampling of text books devoted to numerical solution of electromagnetic problems.

Among the numerical methods found in the literature to approximate Maxwell's equations, the finite element method is the most extended. See for instance [13] for a survey on this subject including a large list of references. Nowadays, it is the basis of several commercial codes such us Ansys, Femlab, Flux, Magnet, MSC/Emas, Opera, etc. We refer the reader to [33] for a description of most of these codes and further references.

The eddy current problem is obtained from Maxwell's equations by assuming that all fields are harmonic and the frequency is low enough as to neglect the electric displacement in Ampère's Law. Such a situation happens, for instance, in problems related to electric machines working at power frequencies and in non-destructive materials testing.

In most practical situations, it is necessary to solve the electromagnetic problem in a bounded domain which contains conducting and non-conducting material (dielectrics), the equations in these two parts being typically of different kind. Moreover, the treatment of multiply connected conductors or dielectrics in three-dimensional domains presents special difficulties. The choice of the unknowns in each subdomain is a crucial point for analysis of the problem in domains with a general topology.

An important number of formulations and finite element methods for solving the eddy current problem in three-dimensional bounded domains can be found in the literature. There is a group of papers devoted to solving the problem in terms of certain scalar and vector potentials [1, 2, 14, 15, 26, 30, 31] and another group using formulations in terms of the magnetic field [4, 6, 7, 9, 10, 12, 34] or the electric field [3, 11, 27].

A thorough mathematical analysis of the formulations in terms of the magnetic or the electric field has been recently performed. This is not the case, however, for formulations in terms of scalar and vector potentials. Indeed, in spite of the fact that the latter are the most frequently used in applications, there are only a very small number of papers dealing with their mathematical analysis. Among them, we mention a paper by Alonso et al. [5], where

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the well-posedness of some of these formulations is analyzed, and another one by Bíró and Valli [16] with the analysis of one such formulation in a general topological setting.

Different potentials have been used for the eddy current problem: a vector potential $\boldsymbol{A}$ for the magnetic field, a scalar potential V for the electric field in the conducting domain, a scalar magnetic potential ψ in dielectric domains, etc. A hierarchy of formulations involving these potentials has been discussed by Bíró and Preis [15]. In particular, they conclude that the so-called $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation, which involves all of them, is the most convenient in terms of computer cost. Numerical experiments illustrating the performance of this approach are also reported in this reference.

The aim of this paper is to provide a rigorous mathematical analysis of this formulation. Under rather general topological conditions, we prove that it leads to a well-posed problem, which can be numerically approximated by standard nodal finite elements. We also prove error estimates for the resulting numerical method. These estimates are valid as long as the three potentials are sufficiently smooth.

The smoothness of the scalar potentials V and ψ only relies on that of the original physical variables of the problem: the magnetic and the electric fields. However, the smoothness of the vector potential $\boldsymbol{A}$ also depends on the geometry of the domain chosen to define this non-physical variable. In principle this domain can be chosen freely, as far as it contains the conductors and the source currents. However, when it is chosen so that its connected components are either convex polyhedra or simply connected domains with smooth boundaries, the smoothness of $\boldsymbol{A}$ is mainly determined by the regularity of another physical variable: the magnetic induction field.

Because of this, we make such a choice for the domain of $\boldsymbol{A}$, which is not restrictive in practice. However, it is convenient to choose it as small as possible, because the magnetic field is written in terms of the more economical scalar potential ψ outside this domain. Thus, in the applications, the domain of $\boldsymbol{A}$ typically consists of a union of disjoint boxes, as small as possible, containing the current source and the conductors.

The outline of the paper is as follows: We introduce the eddy current problem and discuss the topological setting in Section 2. The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation is introduced in Section 3. The corresponding variational problem is obtained in Section 4, where we also prove its well-posedness. Finally, in Section 5, we prove error estimates for a standard finite element method to solve the problem numerically. We also discuss in this section the convenience of choosing a domain with convex connected components for the vector potential.

2. Eddy current problem. We consider a standard eddy current problem: to determine the electromagnetic fields induced in a three-dimensional conducting domain Ω_{C} by a given source current density $\boldsymbol{J}_{\mathrm{d}}$. We assume that the support of $\boldsymbol{J}_{\mathrm{d}}$ is compact and disjoint with Ω_{C}. The eddy current problem is in principle posed in the whole space. However, we restrict it to a bounded domain Ω containing both, Ω_{C} and the support of $\boldsymbol{J}_{\mathrm{d}}$, such that adequate boundary conditions can be imposed on its boundary. To this aim, we choose the geometry of Ω as simple as possible (e.g., simply connected with a connected boundary). See Fig. 2.1 for a two-dimensional sketch.

Let $\Omega_{\mathrm{C}} \subset \mathbb{R}^3$ be an open and bounded set with boundary Γ_{C}. Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with a connected boundary Γ, such that $\overline{\Omega}_{\mathrm{C}} \subset \Omega$. We suppose that both, Ω and Ω_{C}, are either Lipschitz polyhedra or domains with $\mathcal{C}^{1,1}$ boundaries. We denote by $\boldsymbol{n}$ and $\boldsymbol{n}_{\mathrm{C}}$ the outward unit normal vectors to Ω and Ω_{C}, respectively, and by $\Omega_{\mathrm{D}} := \Omega \setminus \overline{\Omega}_{\mathrm{C}}$ the subdomain of Ω occupied by dielectric material, which includes the support of the source current; see Fig. 2.1. We will use standard notation for Sobolev spaces and norms.

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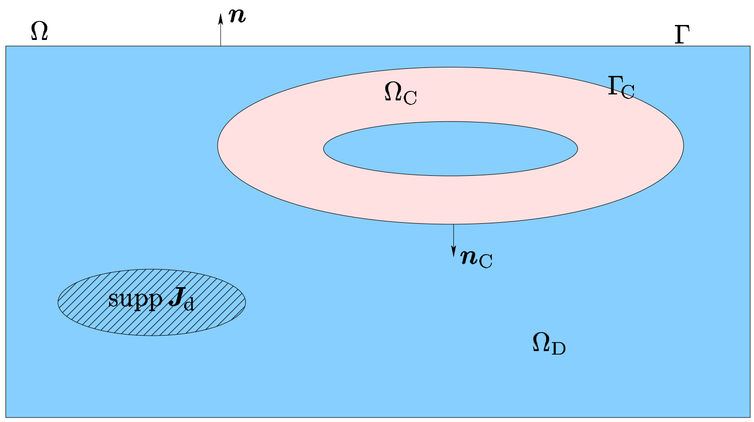

FIG. 2.1. *Two-dimensional sketch of the domain.*

The eddy current problem reads as follows:

Find $\boldsymbol{E} \in H(\mathbf{curl};\Omega_{\mathrm{C}})$ *and* $\boldsymbol{H} \in H(\mathbf{curl};\Omega)$ *such that:*

$$\mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} \qquad \textit{in } \Omega_{\mathrm{C}}, \tag{2.1}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \qquad \textit{in } \Omega_{\mathrm{C}}, \tag{2.2}$$

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}_{\mathrm{d}} \qquad \textit{in } \Omega_{\mathrm{D}}, \tag{2.3}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \qquad \textit{in } \Omega, \tag{2.4}$$

$$\boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \textit{on } \Gamma. \tag{2.5}$$

The unknowns $\boldsymbol{E}$ and $\boldsymbol{H}$ are the magnetic and electric fields, respectively. The magnetic permeability μ and the conductivity σ are bounded functions satisfying:

$$0 < \mu_{\min} \le \mu \le \mu_{\max} \qquad \text{in } \Omega,$$
$$0 < \sigma_{\min} \le \sigma \le \sigma_{\max} \qquad \text{in } \Omega_{\mathrm{C}}.$$

Let us remark that the magnetic field has to satisfy the following coupling conditions:

$$\boldsymbol{H}|_{\Omega_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} = \boldsymbol{H}|_{\Omega_{\mathrm{D}}} \times \boldsymbol{n}_{\mathrm{C}} \qquad \text{on } \Gamma_{\mathrm{C}},$$
$$(\mu\boldsymbol{H})|_{\Omega_{\mathrm{C}}} \cdot \boldsymbol{n}_{\mathrm{C}} = (\mu\boldsymbol{H})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_{\mathrm{C}} \qquad \text{on } \Gamma_{\mathrm{C}}.$$

In fact, the latter is a consequence of (2.4), whereas the former follows from the fact that $\boldsymbol{H}$ must belong to $H(\mathbf{curl};\Omega)$.

The data of the problem are the source current density $\boldsymbol{J}_{\mathrm{d}} \in L^2(\Omega)^3$, for which we assume

$$\operatorname{supp}\boldsymbol{J}_{\mathrm{d}} \subset \Omega_{\mathrm{D}} \qquad \text{and} \qquad \operatorname{div}\boldsymbol{J}_{\mathrm{d}} = 0 \quad \text{in } \Omega_{\mathrm{D}},$$

and the tangential trace of the magnetic field $\boldsymbol{f}_{\mathrm{d}}$. Precise assumptions on $\boldsymbol{f}_{\mathrm{d}}$ will be made in Section 4 below; they essentially mean that $\boldsymbol{f}_{\mathrm{d}}$ has to be the tangential trace of a curl-free vector field (recall that $\mathbf{curl}\,\boldsymbol{H}$ vanishes in the neighborhood of Γ).

Equations (2.1)–(2.5) are enough to determine $\boldsymbol{E}$ and $\boldsymbol{H}$ only if the topology of the conducting domain Ω_{C} is trivial. Otherwise, additional constraints must be imposed. To do this, we reduce our analysis to domains satisfying a standard topological assumption; see for instance Amrouche et al. [8]. We assume that there exists m_{D} connected open surfaces Σ_k (so called "cuts") contained in Ω_{D}, such that:

(i) each surface Σ_k is an open part of a smooth manifold,
(ii) the boundary of each Σ_k is contained in Γ_{C},
(iii) the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$,
(iv) the open set $\widehat{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_k \Sigma_k$ is pseudo-Lipschitz and simply connected.

Under this assumption, since Γ is connected, the space of *harmonic fields*

$$\begin{aligned}\mathscr{H}_\mu(\Gamma, \Gamma_{\mathrm{C}}) := \big\{ \boldsymbol{v} \in L^2(\Omega_{\mathrm{D}})^3 : \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}},\ \operatorname{div}(\mu \boldsymbol{v}) = 0 \text{ in } \Omega_{\mathrm{D}}, \\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \text{ and } \mu \boldsymbol{v} \cdot \boldsymbol{n}_{\mathrm{C}} = 0 \text{ on } \Gamma_{\mathrm{C}} \big\}\end{aligned}$$

satisfies $\dim \mathscr{H}_\mu(\Gamma, \Gamma_{\mathrm{C}}) = m_{\mathrm{D}}$; see, for instance Fernandez and Gilardi [24, Proposition 5.6]. A basis for this space is given by $\{\mathbf{grad}\, \varphi_j\}_{j=1}^{m_{\mathrm{D}}}$, where each $\varphi_j \in H^1_\Gamma(\Omega_{\mathrm{D}} \setminus \Sigma_j)$ is the solution of the following elliptic problem:

$$\begin{aligned} [\![\varphi_j]\!]_{\Sigma_j} &= 1, \\ \int_{\Omega_{\mathrm{D}} \setminus \Sigma_j} \mu\, \mathbf{grad}\, \varphi_j \cdot \mathbf{grad}\, \chi &= 0 \qquad \forall \chi \in H^1_\Gamma(\Omega_{\mathrm{D}}). \end{aligned}$$

In the expression above $[\![\cdot]\!]_{\Sigma_j}$ denotes the jump across Σ_j. Here and thereafter the subscript Γ in $H^1_\Gamma(\cdot)$ refers to function in $H^1(\cdot)$ with a vanishing trace on Γ.

Notice that although in principle $\mathbf{grad}\, \varphi_j \in L^2(\Omega_{\mathrm{D}} \setminus \Sigma_j)$, the last equation implies that $\mu\, \mathbf{grad}\, \varphi_j$ is a divergence-free function in the whole Ω_{D} (not only in $\Omega_{\mathrm{D}} \setminus \Sigma_j$). Moreover, since the jump $[\![\varphi_j]\!]_{\Sigma_j}$ is constant, $\mathbf{grad}\, \varphi_j$ has also a vanishing $\mathbf{curl}$ in the whole Ω_{D} (and not only in $\Omega_{\mathrm{D}} \setminus \Sigma_j$, again). Thus, $\mathbf{grad}\, \varphi_j \in \mathscr{H}_\mu(\Gamma, \Gamma_{\mathrm{C}})$.

To determine a unique solution of the eddy current problem (2.1)–(2.5), it is enough to add the following constraints (see Alonso et al. [5]):

$$\int_{\Omega_{\mathrm{D}}} i\omega\mu \boldsymbol{H} \cdot \mathbf{grad}\, \varphi_j + \int_{\Gamma_{\mathrm{C}}} (\boldsymbol{E} \times \boldsymbol{n}_{\mathrm{C}}) \cdot \mathbf{grad}\, \varphi_j = 0, \qquad j = 1, \ldots, m_{\mathrm{D}}. \tag{2.6}$$

Let us remark that the second integral above has a weak sense for $\boldsymbol{E} \in H(\mathbf{curl}; \Omega_{\mathrm{C}})$ and $\mathbf{grad}\, \varphi_j \in H(\mathbf{curl}; \Omega_{\mathrm{D}})$, as was shown by Buffa and Ciarlet [18, 19] for Lipschitz polyhedra and by Buffa et al. [20] for arbitrary Lipschitz domains; see Section 4 below for a precise definition.

3. The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation. In this section we recall a classical formulation of the eddy current problem in terms of three potentials, $\boldsymbol{A}$, V and ψ, which was introduced by Leonard and Rodger [28]. We refer to Bíró and Preis [15] for a detailed discussion, which includes numerical tests showing the efficiency of this approach.

First, we introduce a magnetic vector potential $\boldsymbol{A}$ defined in a subdomain Ω_A of Ω, which contains the conducting domain and the support of the source current. This subdomain does not need to be connected, but each of its connected components will be chosen either convex or simply connected with a smooth boundary. The reason for such a choice will be discussed in Section 5 below. On the other hand, for the sake of discretization, it is convenient to choose a polyhedral domain Ω_A; moreover, outside Ω_A, we will use a scalar potential, which will consequently require much less degrees of freedom for its discretization. Because of this, Ω_A will be chosen as small as possible, but with convex polyhedral connected components containing Ω_{C} and $\operatorname{supp} \boldsymbol{J}_{\mathrm{d}}$; see Fig. 3.1.

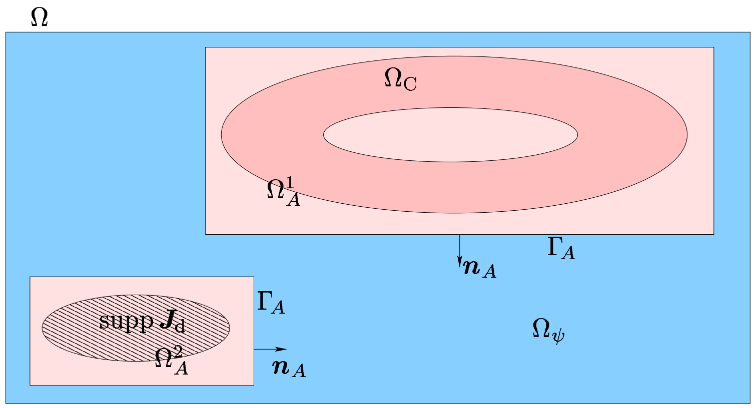

FIG. 3.1. *Two-dimensional sketch of the domains for the different potentials.*

Let $\Omega_A \subset \mathbb{R}^3$ be an open set satisfying

$$\overline{\Omega}_{\mathrm{C}} \cup \operatorname{supp} \boldsymbol{J}_{\mathrm{d}} \subset \Omega_A \qquad \text{and} \qquad \overline{\Omega}_A \subset \Omega. \tag{3.1}$$

We denote by Ω_A^j, $j = 1, \ldots, m_A$, the connected components of Ω_A. We assume that each Ω_A^j is either a convex polyhedron or a simply connected domain with a $\mathcal{C}^{1,1}$ boundary, and that $\overline{\Omega}_A^j$ are mutually disjoint. We denote by Γ_A the boundary of Ω_A and by $\boldsymbol{n}_A$ its outward unit normal vector; see Fig. 3.1.

As a consequence of [25, Theorem I.3.5.], equation (2.4) implies that there exist unique $\boldsymbol{A}_j \in H(\mathbf{curl}; \Omega_A^j)$ such that

$$\mu \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{A}_j \qquad \text{in } \Omega_A^j, \tag{3.2}$$

$$\operatorname{div} \boldsymbol{A}_j = 0 \qquad \text{in } \Omega_A^j, \tag{3.3}$$

$$\boldsymbol{A}_j \cdot \boldsymbol{n}_A = 0 \qquad \text{on } \partial\Omega_A^j. \tag{3.4}$$

Thus, if we define $\boldsymbol{A} : \Omega_A \longrightarrow \mathbb{C}$ by

$$\boldsymbol{A}|_{\Omega_A^j} := \boldsymbol{A}_j, \qquad j = 1, \ldots, m_A,$$

then $\boldsymbol{A}$ belongs to the space

$$\boldsymbol{\mathcal{X}} := H_0(\operatorname{div}; \Omega_A) \cap H(\mathbf{curl}; \Omega_A),$$

whose natural norm is

$$\|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{X}}} := \left(\|\boldsymbol{Z}\|_{0,\Omega_A}^2 + \|\operatorname{div} \boldsymbol{Z}\|_{0,\Omega_A}^2 + \|\mathbf{curl}\, \boldsymbol{Z}\|_{0,\Omega_A}^2 \right)^{\frac{1}{2}}.$$

Next, according to Bíró and Preis [15] (see also Bíró [14] and Bíró and Valli [16]) we introduce an electric scalar potential $V \in H^1(\Omega_{\mathrm{C}})$, such that

$$\boldsymbol{E} = -i\omega \boldsymbol{A} - i\omega\, \mathbf{grad}\, V \qquad \text{in } \Omega_{\mathrm{C}}. \tag{3.5}$$

Let us remark that (2.6) is a necessary condition for a global potential V to exist; see [5] and the formal argument at the end of this section. Notice that, from (2.1),

$$\operatorname{div}\left(-i\omega\sigma \boldsymbol{A} - i\omega\sigma\, \mathbf{grad}\, V\right) = 0 \qquad \text{in } \Omega_{\mathrm{C}}.$$

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Moreover, since $\boldsymbol{H} \in H(\mathbf{curl};\Omega)$, (2.1) and (2.3) also imply that

$$(i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V)\cdot\boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}.$$

These last two equations will be also collected in the potential formulation.

Equation (3.5) determines the electric potential V on each connected component of Ω_{C} up to an additive constant. Thus, if Ω_{C} has m_{C} connected components Ω_{C}^j, then the natural space for V is

$$\mathcal{M} := \prod_{j=1}^{m_{\mathrm{C}}} H^1(\Omega_{\mathrm{C}}^j)/\mathbb{C},$$

endowed with the norm $\|\mathbf{grad}\,V\|_{0,\Omega_{\mathrm{C}}}$.

Finally, a magnetic scalar potential ψ is defined in

$$\Omega_\psi := \Omega \setminus \overline{\Omega}_A$$

(see Fig. 3.1). To do this, notice that since Ω_A is a disjoint union of convex sets with $\overline{\Omega}_A \subset \Omega$ and Ω is simply connected, it turns out that Ω_ψ is simply connected too. Therefore, from (2.3) and (3.1) we know that there exists $\psi \in H^1(\Omega_\psi)$ (unique up to an additive constant) such that

$$\boldsymbol{H} = \omega\,\mathbf{grad}\,\psi \qquad \text{in } \Omega_\psi.$$

Thus, we are lead to the following formulation of problem (2.1)–(2.6) in terms of the potentials $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$, $V \in \mathcal{M}$ and $\psi \in H^1(\Omega_\psi)/\mathbb{C}$:

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) + i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{C}}, \tag{3.6}$$

$$\operatorname{div}(-i\omega\sigma\boldsymbol{A} - i\omega\sigma\,\mathbf{grad}\,V) = 0 \qquad \text{in } \Omega_{\mathrm{C}}, \tag{3.7}$$

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) = \boldsymbol{J}_{\mathrm{d}} \qquad \text{in } \Omega_A \setminus \overline{\Omega}_{\mathrm{C}}, \tag{3.8}$$

$$\left.\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\right|_{\Omega_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} - \left.\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\right|_{\Omega_A\setminus\overline{\Omega}_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{C}}, \tag{3.9}$$

$$\operatorname{div}(\mu\,\mathbf{grad}\,\psi) = 0 \qquad \text{in } \Omega_\psi, \tag{3.10}$$

$$\operatorname{div}\boldsymbol{A} = 0 \qquad \text{in } \Omega_A, \tag{3.11}$$

$$\boldsymbol{A}\cdot\boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{3.12}$$

$$\omega\,\mathbf{grad}\,\psi \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \text{on } \Gamma, \tag{3.13}$$

$$\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A - \omega\,\mathbf{grad}\,\psi\cdot\boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{3.14}$$

$$\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_A - \omega\,\mathbf{grad}\,\psi\times\boldsymbol{n}_A = \mathbf{0} \qquad \text{on } \Gamma_A, \tag{3.15}$$

$$(i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V)\cdot\boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}. \tag{3.16}$$

Let us remark that (3.9) and (3.15) are consequences of the fact that $\boldsymbol{H} \in H(\mathbf{curl};\Omega)$, whereas (3.14) follows from the fact that $\mu\boldsymbol{H} \in H(\operatorname{div};\Omega)$, which in its turn is a consequence of (2.4)

To end this section we show that any solution of the above equations yields a solution of the eddy current problem (2.1)–(2.6). In fact, let $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)/\mathbb{C}$ satisfying (3.6)–(3.16). Let

$$\boldsymbol{H} := \begin{cases} \dfrac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A} & \text{in } \Omega_A, \\ \omega\,\mathbf{grad}\,\psi & \text{in } \Omega_\psi, \end{cases} \tag{3.17}$$

and $\boldsymbol{E}$ be defined by (3.5). It is immediate to show that $\boldsymbol{H} \in H(\mathbf{curl};\Omega)$, $\boldsymbol{E} \in H(\mathbf{curl};\Omega_{\mathrm{C}})$, and they satisfy (2.1)–(2.5). There only remains to prove that (2.6) also holds true. To do this, notice that (2.4) implies that there exists a vector potential $\boldsymbol{B} \in H(\mathbf{curl};\Omega)$ such that

$$\mu\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{B} \qquad \text{in } \Omega. \tag{3.18}$$

Taking into account that the sets $\overline{\Omega}_A^j$ are simply connected and mutually disjoint, from (3.17) there follows that there exists $\xi \in H^1(\Omega_A)$ such that

$$\boldsymbol{A} = \boldsymbol{B} + \mathbf{grad}\,\xi \qquad \text{in } \Omega_A.$$

Consequently, if we define $\widetilde{V} := V + \xi|_{\Omega_{\mathrm{C}}}$, we obtain from (3.5) that

$$\boldsymbol{E} = -i\omega(\boldsymbol{B} + \mathbf{grad}\,\widetilde{V}) \qquad \text{in } \Omega_{\mathrm{C}}. \tag{3.19}$$

Equations (3.18) and (3.19) fall in the framework analyzed by Alonso et al. [5, Section 6 (ii)], where it is shown that (2.6) holds true. This can be formally verified by using (3.17), the fact that $\mathbf{grad}\,\varphi_j \in \mathscr{H}_\mu(\Gamma, \Gamma_{\mathrm{C}})$ and (3.19), as follows:

$$\begin{aligned}
\int_{\Omega_{\mathrm{D}}} i\omega\mu\boldsymbol{H}\cdot\mathbf{grad}\,\varphi_j &= \int_{\Omega_{\mathrm{D}}} i\omega\,\mathbf{curl}\,\boldsymbol{B}\cdot\mathbf{grad}\,\varphi_j \\
&= i\omega\int_{\Gamma_{\mathrm{C}}} (\boldsymbol{B}\times\boldsymbol{n}_{\mathrm{C}})\cdot\mathbf{grad}\,\varphi_j \\
&= -\int_{\Gamma_{\mathrm{C}}} (\boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}})\cdot\mathbf{grad}\,\varphi_j - i\omega\int_{\Gamma_{\mathrm{C}}} (\mathbf{grad}\,\widetilde{V}\times\boldsymbol{n}_{\mathrm{C}})\cdot\mathbf{grad}\,\varphi_j \\
&= -\int_{\Gamma_{\mathrm{C}}} (\boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}})\cdot\mathbf{grad}\,\varphi_j,
\end{aligned}$$

where for the last equality we have used that

$$\begin{aligned}
\int_{\Gamma_{\mathrm{C}}} (\mathbf{grad}\,\widetilde{V}\times\boldsymbol{n}_{\mathrm{C}})\cdot\mathbf{grad}\,\varphi_j = \int_{\Omega_{\mathrm{D}}} &\mathbf{grad}\,\widetilde{V}^*\cdot\mathbf{curl}\,(\mathbf{grad}\,\varphi_j) \\
&- \int_{\Omega_{\mathrm{D}}} \mathbf{curl}(\mathbf{grad}\,\widetilde{V}^*)\cdot\mathbf{grad}\,\varphi_j = 0,
\end{aligned}$$

with $\widetilde{V}^* \in H^1(\Omega)$ being an extension of $\widetilde{V}$ to the whole Ω.

4. Variational formulation. Existence and uniqueness of solution.

The aim of this section is to give a variational formulation of problem (3.6)–(3.16) and to prove its well-posedness.

First, we recall some results settled in [20] for Lipschitz domains. We write these results for Ω_A, as will be used in the sequel. The tangential trace operator $\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u}|_{\Gamma_A}\times\boldsymbol{n}_A$ is a bounded linear operator from $H(\mathbf{curl};\Omega_A)$ onto $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$. The tangential projection $\pi_\tau(\boldsymbol{v}) := \boldsymbol{n}_A\times(\boldsymbol{v}|_{\Gamma_A}\times\boldsymbol{n}_A)$ is a bounded linear operator from $H(\mathbf{curl};\Omega_A)$ onto

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$H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma_A)$. Thus, the duality pairing between $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$ and $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma_A)$ is well defined by

$$\langle\gamma_\tau(\boldsymbol{u}),\pi_\tau(\boldsymbol{v})\rangle_{\Gamma_A} := \int_{\Omega_A} \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v} - \int_{\Omega_A} \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} \qquad \forall \boldsymbol{u},\boldsymbol{v}\in H(\mathbf{curl};\Omega_A).$$

For any $\boldsymbol{w}\in H(\mathbf{curl};\Omega_\psi)$, its tangential trace on Γ_A also belongs to $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$ and, consequently, $\langle\boldsymbol{w}\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{v})\rangle_{\Gamma_A}$ is also well defined.

To obtain a variational formulation of problem (3.6)–(3.16), notice that by virtue of (3.6), (3.8) and (3.9) we have that $\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\in H(\mathbf{curl};\Omega_A)$, and for all $\boldsymbol{Z}\in\boldsymbol{\mathcal{X}}$

$$\int_{\Omega_A}\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\cdot\bar{\boldsymbol{Z}} = -i\omega\int_{\Omega_{\mathrm{C}}}\sigma\left(\boldsymbol{A}+\mathbf{grad}\,V\right)\cdot\bar{\boldsymbol{Z}} + \int_{\Omega_A}\boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}}.$$

Integrating by parts the left-hand side above and using (3.11) and (3.15), there follows

$$\begin{aligned}\int_{\Omega_A}\frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}}+(\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right]+i\omega\int_{\Omega_{\mathrm{C}}}\sigma\boldsymbol{A}\cdot\bar{\boldsymbol{Z}}\\ +i\omega\int_{\Omega_{\mathrm{C}}}\sigma\,\mathbf{grad}\,V\cdot\bar{\boldsymbol{Z}}-\omega\,\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{Z})\rangle_{\Gamma_A}=\int_{\Omega_A}\boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}}.\end{aligned}\tag{4.1}$$

On the other hand, from (3.7), by integrating by parts and using (3.16) we have for all $U\in H^1(\Omega_{\mathrm{C}})$

$$i\omega\int_{\Omega_{\mathrm{C}}}\sigma\boldsymbol{A}\cdot\mathbf{grad}\,\bar{U}+i\omega\int_{\Omega_{\mathrm{C}}}\sigma\,\mathbf{grad}\,V\cdot\mathbf{grad}\,\bar{U}=0.\tag{4.2}$$

Finally, for any $\varphi\in H^1_\Gamma(\Omega_\psi)$, from (3.10), by integrating by parts and using (3.14), we obtain

$$\omega\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi}+\int_{\Gamma_A}\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A\,\bar{\varphi}=0,$$

where the last integral must be understood as the duality pairing between $H^{-\frac{1}{2}}(\Gamma_A)$ and $H^{\frac{1}{2}}(\Gamma_A)$. Now, let $\varphi^*\in H^1(\Omega)$ be an extension of φ to the whole Ω. Hence,

$$\int_{\Gamma_A}\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A\,\bar{\varphi}=\int_{\Omega_A}\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{grad}\,\bar{\varphi}^*=\left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}.$$

Therefore, we obtain

$$\omega\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi}+\left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}=0.\tag{4.3}$$

Equations (4.1)–(4.3) together with the essential condition (3.13), provide the following

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variational formulation of problem (3.6)–(3.16):

Find $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$, $V \in \mathcal{M}$ *and* $\psi \in H^1(\Omega_\psi)/\mathbb{C}$ *such that:*

$$\omega\, \mathbf{grad}\, \psi \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \text{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \tag{4.4}$$

$$\begin{aligned} &\int_{\Omega_A} \frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + i\omega\int_{\Omega_{\mathrm{C}}}\sigma\boldsymbol{A}\cdot\bar{\boldsymbol{Z}} \\ &\qquad + i\omega\int_{\Omega_{\mathrm{C}}}\sigma\,\mathbf{grad}\,V\cdot\bar{\boldsymbol{Z}} - \omega\left\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{Z})\right\rangle_{\Gamma_A} = \int_{\Omega_A}\boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}} \\ &\hspace{25em}\forall \boldsymbol{Z}\in\boldsymbol{\mathcal{X}}, \end{aligned} \tag{4.5}$$

$$i\omega\int_{\Omega_{\mathrm{C}}}\sigma\boldsymbol{A}\cdot\mathbf{grad}\,\bar{U} + i\omega\int_{\Omega_{\mathrm{C}}}\sigma\,\mathbf{grad}\,V\cdot\mathbf{grad}\,\bar{U} = 0 \qquad \forall U\in\mathcal{M}, \tag{4.6}$$

$$\omega\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} + \left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A} = 0 \qquad \forall\varphi\in H^1_\Gamma(\Omega_\psi). \tag{4.7}$$

Our next goal is to prove that this variational problem has a unique solution. For this purpose, first of all notice that (4.4) can be satisfied only if $\boldsymbol{f}_{\mathrm{d}}$ is the tangential trace on Γ of a gradient. Thus, this additional hypothesis turns out necessary for the problem to have a solution. So, we make the following assumption:

$$\exists\eta\in H^1(\Omega_\psi): \qquad \boldsymbol{f}_{\mathrm{d}} = \mathbf{grad}\,\eta\times\boldsymbol{n} \qquad \text{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma). \tag{4.8}$$

Now, let $\mathscr{A}$ be the bilinear form defined on $\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1(\Omega_\psi)/\mathbb{C}$ by

$$\begin{aligned} &\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{Z},U,\varphi)) \\ &\quad := \int_{\Omega_A}\frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + \omega^2\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} \\ &\qquad + i\omega\int_{\Omega_{\mathrm{C}}}\sigma\left(\boldsymbol{A}+\mathbf{grad}\,V\right)\cdot\left(\bar{\boldsymbol{Z}}+\mathbf{grad}\,\bar{U}\right) \\ &\qquad - \omega\left\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{Z})\right\rangle_{\Gamma_A} + \omega\left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}. \end{aligned}$$

Clearly, (4.4)–(4.7) can be equivalently written as follows:

Find $(\boldsymbol{A},V,\psi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1(\Omega_\psi)/\mathbb{C}$ *such that:*

$$\omega\,\mathbf{grad}\,\psi\times\boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \text{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \tag{4.9}$$

$$\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{Z},U,\varphi)) = \int_{\Omega_A}\boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}} \qquad \forall(Z,U,\varphi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1_\Gamma(\Omega_\psi). \tag{4.10}$$

THEOREM 4.1. *Under assumption* (4.8)*, the variational problem* (4.9)–(4.10) *has a unique solution.*

Proof. It is enough to show that $\mathscr{A}$ is elliptic, since, in such a case, the theorem follows from Lax-Milgram's Lemma.

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To prove the ellipticity, for $(\boldsymbol{Z}, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)/\mathbb{C}$ we write

$$\begin{aligned}\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi)) = &\int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{Z}|^2 + |\mathrm{div}\, \boldsymbol{Z}|^2 \right) + \omega^2 \int_{\Omega_\psi} \mu\, |\mathbf{grad}\, \varphi|^2 \\ &+ i\omega \left\{ \int_{\Omega_C} \sigma \left(|\boldsymbol{Z}|^2 + |\mathbf{grad}\, U|^2 \right) + 2 \int_{\Omega_C} \sigma\, \mathrm{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \right. \\ &\left. + 2\, \mathrm{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A} \right\}.\end{aligned}$$

Thus,

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 = (a + \omega^2 b)^2 + \omega^2 (c + 2d)^2,$$

where

$$\begin{aligned} a &:= \int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{Z}|^2 + |\mathrm{div}\, \boldsymbol{Z}|^2 \right), & b &:= \int_{\Omega_\psi} \mu\, |\mathbf{grad}\, \varphi|^2, \\ c &:= \int_{\Omega_C} \sigma \left(|\boldsymbol{Z}|^2 + |\mathbf{grad}\, U|^2 \right), & d &:= e + f, \end{aligned}$$

with

$$e := \int_{\Omega_C} \sigma\, \mathrm{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \qquad \text{and} \qquad f := \mathrm{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A}.$$

Next, we proceed as in [16] and use the elementary inequality

$$(c + 2d)^2 \geq \rho c^2 - 8\rho d^2 \qquad \forall c, d \in \mathbb{R},\ \forall \rho \in (0, 1/2],$$

to obtain

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 \geq a^2 + \omega^4 b^2 + \omega^2 (\rho c^2 - 8\rho d^2) \qquad \forall \rho \in (0, 1/2].$$

Now, since[1]

$$a \geq \frac{K}{\mu_{\max}} \|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{X}}}^2 \qquad \text{and} \qquad b \geq \mu_{\min} \|\mathbf{grad}\, \varphi\|_{0,\Omega_\psi}^2,$$

with $K > 0$ independent of $\boldsymbol{Z}$, we have

$$\begin{aligned}|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 \geq &\frac{K^2}{\mu_{\max}^2} \|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{X}}}^4 + \omega^4 \mu_{\min}^2 \|\mathbf{grad}\, \varphi\|_{0,\Omega_\psi}^4 \\ &+ \omega^2 \rho \left(\int_{\Omega_C} \sigma\, |\mathbf{grad}\, U|^2 \right)^2 - 16\omega^2 \rho (e^2 + f^2).\end{aligned}$$

To estimate the last term in the right-hand side above, notice first that, for all $\varepsilon > 0$,

$$e^2 \leq \left(\int_{\Omega_C} |\sigma\, \mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}| \right)^2 \leq \frac{\varepsilon}{2} \left(\int_{\Omega_C} \sigma\, |\mathbf{grad}\, U|^2 \right)^2 + \frac{1}{2\varepsilon} \left(\int_{\Omega_C} \sigma\, |\boldsymbol{Z}|^2 \right)^2.$$

[1] For the first inequality, see, for instance, [25, Lemma I.3.6].

On the other hand, $\exists C > 0$ independent of φ and $\boldsymbol{Z}$ such that

$$f^2 \le \|\mathbf{grad}\,\bar\varphi \times \boldsymbol{n}_A\|^2_{H^{-\frac12}(\mathrm{div}_\Gamma;\Gamma_A)} \left\|\pi_\tau(\bar{\boldsymbol{Z}})\right\|^2_{H^{-\frac12}(\mathrm{curl}_\Gamma;\Gamma_A)} \le C\left(\|\mathbf{grad}\,\varphi\|^4_{0,\Omega_\psi} + \|\boldsymbol{Z}\|^4_{\boldsymbol{\mathcal{X}}}\right).$$

Therefore, by combining the last three inequalities and taking ε and ρ small enough, we obtain that $\exists\alpha > 0$ such that, $\forall(\boldsymbol{Z}, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)/\mathbb{C}$,

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 \ge \alpha\left(\|\boldsymbol{Z}\|^4_{\boldsymbol{\mathcal{X}}} + \|\mathbf{grad}\,U\|^4_{0,\Omega_{\mathrm{C}}} + \|\mathbf{grad}\,\varphi\|^4_{0,\psi}\right),$$

which allows us to conclude the ellipticity of $\mathscr{A}$. ◻

To end this section, we prove that the unique solution of the variational problem (4.9)–(4.10) is actually a solution of the strong form of the problem given by equations (3.6)–(3.16).

THEOREM 4.2. *The solution* $(\boldsymbol{A}, V, \psi)$ *of* (4.9)–(4.10) *satisfies* (3.6)–(3.16).

Proof. First, let $\xi \in H^1(\Omega_A)$ be a solution of the compatible Neumann problem $\Delta\xi = \mathrm{div}\,\boldsymbol{A}$ in Ω_A, $\partial\xi/\partial\boldsymbol{n}_A = 0$ on Γ_A. By testing (4.5) with $\boldsymbol{Z} = \mathbf{grad}\,\xi \in \boldsymbol{\mathcal{X}}$, we obtain (3.11) by using (4.6) (since $\xi|_{\Omega_{\mathrm{C}}} \in \mathcal{M}$) and $\langle\mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\mathbf{grad}\,\xi)\rangle_{\Gamma_A} = 0$ (which is a consequence of the definition of the duality pairing).

Second, by testing (4.5)–(4.7) with smooth functions supported in adequate domains and proceeding in the standard way, it is easy to verify equations (3.6)–(3.10), (3.14) and (3.16). Since (3.12) is imposed in the definition of the space $\boldsymbol{\mathcal{X}}$ and (3.13) coincides with (4.9), there only remains to prove (3.15) in $H^{-\frac12}(\mathrm{div}_\Gamma;\Gamma_A)$; namely, that for all $\boldsymbol{\zeta} \in H(\mathbf{curl};\Omega_A)$,

$$\left\langle \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta})\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta})\rangle_{\Gamma_A} = 0. \tag{4.11}$$

To do this, notice first that by substituting (3.11) in (4.5), integrating by parts and having into account (3.6) and (3.8), we obtain

$$\left\langle \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{Z})\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{Z})\rangle_{\Gamma_A} = 0 \qquad \forall\boldsymbol{Z} \in \boldsymbol{\mathcal{X}}.$$

Next, for $\boldsymbol{\zeta} \in H(\mathbf{curl};\Omega_A)$, let φ be a solution of the following auxiliary problem:

$$\varphi \in H^1(\Omega_A)/\mathbb{C}: \qquad \int_{\Omega_A} \mathbf{grad}\,\varphi \cdot \mathbf{grad}\,\bar\chi = \int_{\Omega_A} \boldsymbol{\zeta} \cdot \mathbf{grad}\,\bar\chi \qquad \forall\chi \in H^1(\Omega_A)/\mathbb{C}.$$

Hence, $\mathrm{div}(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi) = 0$ in Ω_A and $(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi)\cdot\boldsymbol{n}_A = 0$ on Γ_A. Consequently, $\boldsymbol{Z} := \boldsymbol{\zeta} - \mathbf{grad}\,\varphi \in \boldsymbol{\mathcal{X}}$, and using it as a test function in the equation above, we obtain

$$\left\langle \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi)\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi)\rangle_{\Gamma_A} = 0.$$

Now, from (3.6) and (3.8), we have

$$\begin{aligned}\left\langle \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\mathbf{grad}\,\varphi)\right\rangle_{\Gamma_A} &= \int_{\Omega_A} \mathbf{curl}\left(\frac{1}{\mu}\mathbf{curl}\,\boldsymbol{A}\right)\cdot\mathbf{grad}\,\bar\varphi \\ &= -\int_{\Omega_{\mathrm{C}}} (i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V)\cdot\mathbf{grad}\,\bar\varphi \\ &\quad + \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}}\cdot\mathbf{grad}\,\bar\varphi \\ &= 0,\end{aligned}$$

where, for the last step, we have used integration by parts, (3.7), (3.16), the assumption that $\boldsymbol{J}_{\mathrm{d}}$ is divergence-free and (3.1).

Thus, using again that $\langle \mathbf{grad}\, \psi \times \boldsymbol{n}_A, \pi_\tau(\mathbf{grad}\, \varphi)\rangle_{\Gamma_A}$ vanishes, (4.11) follows from the last two equations, and we conclude the proof. $\square$

5. Numerical approximation. In this section we describe and analyze a finite element method to approximate the solution of problem (4.9)–(4.10). To do this, first notice that (4.9) implies that the surface gradient of ψ can be written as follows:

$$\nabla_\Gamma \psi := \boldsymbol{n} \times (\nabla \psi \times \boldsymbol{n}) = \frac{1}{\omega}\, \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}}.$$

Therefore, if we take an arbitrary but fixed point $\boldsymbol{x}_0 \in \Gamma$ and if the data $\boldsymbol{f}_{\mathrm{d}}$ is sufficiently smooth (for instance, it is enough that $\boldsymbol{f}_{\mathrm{d}} \in H^{\frac{1}{2}+\delta}(\Gamma)^3$ with $\delta > 0$), then we can compute in advance the values of ψ on Γ as follows:

$$\psi(\boldsymbol{x}) = \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \nabla_\Gamma \psi \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})} = \frac{1}{\omega} \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}} \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})},$$

where $\boldsymbol{\alpha}(\boldsymbol{x})$ is any simple curve lying on Γ and joining $\boldsymbol{x}_0$ with $\boldsymbol{x}$, and $\boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})}$ is its unit tangent vector. Notice that the computed value of $\psi(\boldsymbol{x})$ is independent of the particular curve $\boldsymbol{\alpha}(\boldsymbol{x})$. Thus, if we define

$$g_{\mathrm{d}}(\boldsymbol{x}) := \frac{1}{\omega} \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}} \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})}, \tag{5.1}$$

then problem (4.9)–(4.10) is equivalent to the following one:

Find $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)$ *such that:*

$$\psi = g_{\mathrm{d}} \qquad \text{on } \Gamma, \tag{5.2}$$

$$\mathscr{A}((\boldsymbol{A}, V, \psi), (\boldsymbol{Z}, U, \varphi)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}} \cdot \bar{\boldsymbol{Z}} \qquad \forall (Z, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1_\Gamma(\Omega_\psi). \tag{5.3}$$

To obtain a discrete formulation of this problem, we further assume that all the domains are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}$ be a family of tetrahedral meshes of Ω such that, for each mesh, all the elements $T \in \mathcal{T}_h$ are completely included in one of the three subdomains $\overline{\Omega}_A$, $\overline{\Omega}_{\mathrm{C}}$ or $\overline{\Omega}_\psi$.

Consider the following finite element spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_h &:= \left\{ \boldsymbol{Z}_h \in \boldsymbol{\mathcal{X}} :\ \boldsymbol{Z}_h|_T \in \mathbb{P}_m^3\ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_A \right\}, \\
\mathcal{M}_h &:= \left\{ U_h \in \mathcal{M} :\ U_h|_T \in \mathbb{P}_m\ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_{\mathrm{C}} \right\}, \\
\mathcal{Q}_h &:= \left\{ \varphi_h \in H^1(\Omega_\psi) :\ \varphi_h|_T \in \mathbb{P}_m\ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_\psi \right\}, \\
\mathcal{Q}_{\Gamma,h} &:= \left\{ \varphi_h \in \mathcal{Q}_h :\ \varphi_h|_\Gamma = 0 \right\},
\end{aligned}$$

where $\mathbb{P}_m$, $m \geq 1$, is the set of polynomials of degree not greater than m.

For the boundary condition, we choose the following discrete approximation of g_{d}:

$$g_{\mathrm{d},h} := \Pi_h^\Gamma g_{\mathrm{d}}, \tag{5.4}$$

where Π_h^Γ is the Lagrange interpolant on the triangular mesh on Γ which consists of the faces of tetrahedra of $T \in \mathcal{T}_h$ lying on Γ, that we denote $\mathcal{T}_h^\Gamma$. Notice that the definition of

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$g_{\mathrm{d},h}$ makes sense because g_{d}, as defined by (5.1), is continuous. Let us remark that $g_{\mathrm{d},h}$ is completely determined by its values at the vertices of the triangulation $\mathcal{T}_h^\Gamma$, which can be conveniently computed from the data $\boldsymbol{f}_{\mathrm{d}}$ by means of (5.1), with $\boldsymbol{\alpha}(\boldsymbol{x})$ being a curve formed by edges of $\mathcal{T}_h^\Gamma$.

Thus, we are lead to the following discrete problem:

Find $(\boldsymbol{A}_h, V_h, \psi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_h$ *such that:*

$$\psi_h = g_{\mathrm{d},h} \qquad \text{on } \Gamma, \tag{5.5}$$

$$\mathscr{A}((\boldsymbol{A}_h, V_h, \psi_h), (\boldsymbol{Z}_h, U_h, \varphi_h)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}} \cdot \bar{\boldsymbol{Z}}_h \qquad \forall (\boldsymbol{Z}_h, U_h, \varphi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}. \tag{5.6}$$

The existence and uniqueness of the solution of this discrete problem is again an immediate consequence of the ellipticity of $\mathscr{A}$, proved in the proof of Theorem 4.2, and Lax-Milgram's Lemma. Moreover, if the solution of the continuous problem is smooth enough, the standard finite element error analysis techniques yield the following result:

THEOREM 5.1. *Let* $g_{\mathrm{d}} \in \mathcal{C}(\Gamma)$ *and* $g_{\mathrm{d},h}$ *be defined by* (5.4). *Let* $(\boldsymbol{A}, V, \psi)$ *and* $(\boldsymbol{A}_h, V_h, \psi_h)$ *be the solutions of problems* (5.2)–(5.3) *and* (5.5)–(5.6), *respectively.*

If $\boldsymbol{A} \in H^{1+s}(\Omega_A)^3$, $V \in H^{1+s}(\Omega_{\mathrm{C}})$ *and* $\psi \in H^{1+s}(\Omega_\psi)$ *with* $s > 0$, *then there exists a strictly positive constant* C, *independent of* h, $\boldsymbol{A}$, V *and* ψ, *such that*

$$\begin{aligned} \|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{X}}} + \|\mathbf{grad}\,(V - V_h)\|_{0,\Omega_{\mathrm{C}}} + \|\mathbf{grad}\,(\psi - \psi_h)\|_{0,\Omega_\psi} \\ \leq C h^r \left(\|\boldsymbol{A}\|_{1+s,\Omega_A} + \|V\|_{1+s,\Omega_{\mathrm{C}}} + \|\psi\|_{1+s,\Omega_\psi} \right), \end{aligned}$$

with $r := \min\{m, s\}$.

Proof. Let Π_h be the Lagrange interpolant on Q_h. Since

$$(\Pi_h \psi)|_\Gamma = \Pi_h^\Gamma \psi = \Pi_h^\Gamma g_{\mathrm{d}} = g_{\mathrm{d},h} = \psi_h|_\Gamma,$$

we have that $\psi_h - \Pi_h \psi \in Q_{\Gamma,h}$. Therefore, the theorem is a direct consequence of the ellipticity of $\mathscr{A}$, Cea's lemma and the approximation properties of the Lagrange interpolant; see, for instance, Ciarlet [21]. ☐

To end the paper we discuss the convenience of choosing the domain Ω_A of the vector potential so that its connected components be convex polyhedra. For simplicity, we take Ω_A connected in what follows, but all the statements hold true for each of its connected components. So let Ω_A be simply connected with a connected boundary.

According to [25, Theorem I.3.4], since $\operatorname{div}(\mu \boldsymbol{H}) = 0$ in Ω, there exists $\boldsymbol{\Phi} \in H^1(\Omega)^3$ satisfying:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{\Phi} &= \mu \boldsymbol{H} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\Phi} &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, according to Remark I.3.12 of the same reference, if $\mu \boldsymbol{H} \in H^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi} \in H^{1+p}(\Omega)^3$.

Therefore, by virtue of (3.2)–(3.4), there holds:

$$\begin{aligned} \mathbf{curl}(\boldsymbol{A} - \boldsymbol{\Phi}) &= \mathbf{0} && \text{in } \Omega_A, \\ \operatorname{div}(\boldsymbol{A} - \boldsymbol{\Phi}) &= 0 && \text{in } \Omega_A, \\ (\boldsymbol{A} - \boldsymbol{\Phi}) \cdot \boldsymbol{n}_A &= -\boldsymbol{\Phi} \cdot \boldsymbol{n}_A && \text{on } \Gamma_A. \end{aligned}$$

The first equation above and the simple-connectedness of Ω_A implies that there exists a unique $\chi \in H^1(\Omega_A)/\mathbb{C}$ such that $\boldsymbol{A} - \boldsymbol{\Phi} = \mathbf{grad}\,\chi$ in Ω_A, whereas the remaining equations imply that χ is the solution of the following compatible Neumann problem:

$$\Delta\chi = 0 \qquad \text{in } \Omega_A,$$
$$\frac{\partial\chi}{\partial\boldsymbol{n}_A} = -\boldsymbol{\Phi}\cdot\boldsymbol{n}_A \qquad \text{on } \Gamma_A.$$

The Neumann data of this problem will be in general smooth on each polygonal face F of Γ_A, since Γ_A is an arbitrary polyhedral surface within the dielectric domain. In fact, if $\mu\boldsymbol{H} \in H^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi}|_F \cdot \boldsymbol{n}_A \in H^{\frac{1}{2}+p}(F)$ for all faces F.

Therefore, if Ω_A is a convex polyhedron, then there exists $q > 0$ such that $\chi \in H^{2+q}(\Omega_A)$; see [23]. Consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\,\chi \in H^{1+s}(\Omega_A)^3,$$

with $s := \min\{p, q\} > 0$. Conversely, if Ω_A were a non-convex polyhedron, then, in general, $\chi \notin H^2(\Omega_A)$ and, consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\,\chi \notin H^1(\Omega_A)^3.$$

In such a case, Theorem 5.1 would become meaningless.

Moreover, $\boldsymbol{\mathcal{Y}} := \left\{\boldsymbol{Z} \in H^1(\Omega_A)^3 : \ \boldsymbol{Z}\cdot\boldsymbol{n}_A = 0 \text{ on } \Gamma_A\right\}$ is a closed subspace of $\boldsymbol{\mathcal{X}}$; see [22]. When Ω_A is a polyhedron, it is well-known that $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}}$ if and only if Ω_A is convex; see [25, Theorem I.3.9] and [22].

The finite element space $\boldsymbol{\mathcal{X}}_h$ is clearly a subspace of $\boldsymbol{\mathcal{Y}}$. Therefore, when Ω_A is a convex polyhedron, it makes sense to approximate $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$ by finite elements from $\boldsymbol{\mathcal{X}}_h$.

Instead, if Ω_A were not convex, then there would be no hope of approximating $\boldsymbol{A}$ by finite elements from $\boldsymbol{\mathcal{X}}_h$. Indeed, as stated above, in general $\boldsymbol{A} \notin H^1(\Omega_A)^3$ in such a case. Hence, $\boldsymbol{A}$ would not belong to the closed set $\boldsymbol{\mathcal{Y}}$ containing the finite element spaces $\boldsymbol{\mathcal{X}}_h$ for all meshes. So, there could not exist $\boldsymbol{A}_h$ such that $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{X}}} \to 0$ as h goes to zero.

6. Conclusions. We have proved that the $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation of the eddy current problem is well posed and that its discretization by standard nodal finite elements leads to an optimal-order numerical approximation. This gives mathematical support to the well-known efficiency of this approach in applications.

However, for the convergence of the numerical method, the connected components of the domain of the vector potential $\boldsymbol{A}$ must be chosen as convex polyhedra. Since this domain can be chosen freely (as far as it contains the conductors and the source current), this is not a severe restriction in practice.

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